

ON A CLASS OF ANTIPODAL DISTANCE-REGULAR
GRAPHS OF DIAMETER 4A.A. MAKNEV , M.P. GOLUDYATNIKOV , AND K.S. EFIMOV *Communicated by P.P. PETROV***Abstract:** Antipodal distance-regular graph with intersection array $\{-\mu^2 + u^2 + 2\mu, k - 1, \mu, 1; 1, \mu, k - 1, k\}$ has folded graph with parameters $(1/2(\mu^2 - u^2 - 3\mu + u)(\mu^2 - u^2 - 3\mu - u)/\mu, k, 0, 2\mu)$.For fixed μ infinite series are only in the cases $\mu = 1, 2, 3$ with corresponding arrays

$$\{u^2 + 1, u^2, 1, 1; 1, 1, u^2, u^2 + 1\},$$
$$\{u^2, u^2 - 1, 2, 1; 1, 2, u^2 - 1, u^2\} \text{ and}$$
$$\{u^2 - 3, u^2 - 4, 3, 1; 1, 3, u^2 - 4, u^2 - 3\}.$$

The article studies graphs with such arrays.

Keywords: distance regular graph, strongly-regular graph, antipodal graph, triple intersection numbers.

1 Introduction

We consider undirected graphs without loops and multiple edges. Given a vertex a in a graph Γ , we denote by $\Gamma_i(a)$ the subgraph induced by Γ on the set of all vertices, that are at a distance i from a . We put $[a] = \Gamma_1(a)$, $a^\perp = \{a\} \cup [a]$.

For given vertices a, b in the graph Γ , we denote by $d(a, b)$ a distance between a and b . The number of vertices in $[a] \cap [b]$ is denoted by $\mu(a, b)$ (by

МАХНЕВ, А.А., ГОЛУБЯТНИКОВ, М.П., ЕФИМОВ, К.С., ANTIPODAL GRAPHS Γ OF DIAMETER 4 FOR WHICH $\Gamma_{3,4}$ IS A STRONGLY REGULAR GRAPH WITH $\mu = 4, 6$.

© 2024 МАХНЕВ А.А., ГОЛУБЯТНИКОВ М.П., ЕФИМОВ К.С..

The work is supported by RFBR (Grant No. 06-01-00332).

Received January, 1, 2024, Published December, 31, 2024.

$\lambda(a, b)$), if $d(a, b) = 2$ (if a and b are adjacent) in Γ . Further, the subgraph induced by $[a] \cap [b]$ is called μ -subgraph (λ -subgraph). Let Γ be a graph of diameter d , $i \in \{2, 3, \dots, d\}$. The graph Γ_i has the same set of vertices, and the vertices u, w are adjacent in Γ_i if $d_\Gamma(u, w) = i$.

If the vertices u, w are at distance i in Γ , then by $b_i(u, w)$ ($c_i(u, w)$) we denote the number of vertices in the intersection $\Gamma_{i+1}(u)$ ($\Gamma_{i-1}(u)$) with $[w]$. A graph of diameter d is called distance-regular with intersection array $\{b_0, \dots, b_{d-1}; c_1, \dots, c_d\}$, if the values $b_i(u, w)$ and $c_i(u, w)$ do not depend on the choice of the vertices u, w at distance i . We put $a_i = k_i - b_i - c_i$. Note that for a distance-regular graph b_0 is the degree of the graph, $c_1 = 1$. Further, by $p_{ij}^l(x, y)$ we denote the number of vertices in the subgraph $\Gamma_i(x) \cap \Gamma_j(y)$ for vertices x, y located at distance l in the graph Γ . In a distance regular graph, the numbers $p_{ij}^l(x, y)$ do not depend on the choice of vertices x, y , are denoted by p_{ij}^l and are called the intersection numbers of the graph Γ [1].

We consider an antipodal distance-regular graph Γ for which $\Delta = \Gamma_{1,2}$ is a strongly regular graph. Then $\Delta = \Gamma_{1,2}$ is a strongly regular graph without triangles, the antipodality index is equals 2, $k(\Delta) = \mu(r + 1) + r^2$, where $r, s = -(\mu + r)$ are the nonprincipal eigenvalues of the graph Δ (Makhnev and Paduchikh in [3]).

An antipodal distance-regular graph with intersection array $\{-\mu^2 + u^2 + 2\mu, k-1, \mu, 1; 1, \mu, k-1, k\}$ has an antipodal quotient with parameters $(1/2(\mu^2 - u^2 - 3\mu + u)(\mu^2 - u^2 - 3\mu - u)/\mu, k, 0, 2\mu)$.

For a fixed μ , infinite series occur only in the cases $\mu = 1, 2, 3$ with corresponding intersection arrays:

- (1) $\{u^2 + 1, u^2, 1, 1; 1, 1, u^2, u^2 + 1\}$,
- (2) $\{u^2, u^2 - 1, 2, 1; 1, 2, u^2 - 1, u^2\}$ and
- (3) $\{u^2 - 3, u^2 - 4, 3, 1; 1, 3, u^2 - 4, u^2 - 3\}$.

In this paper, the graphs Γ with the shown intersection arrays are studied. As we see, all these graphs have diameter 4. Earlier authors studied distance regular graphs of diameter 3 (Makhnev, Golubyatnikov and Guo in [4]) and (Makhnev, Guo and Efimov in [5])

Graphs Γ with intersection arrays $\{k, k - \mu, (r - 1)\mu, 1; 1, \mu, k - 1, k\}$, where $k = t(t^2 + 3t + 1)$ and $r\mu = t(t + 1)$, are considered Brouwer, Cohen and Neumaier (1989). In case $r = 2, \mu = 1$ we have $k = 5$ and Γ is the Wells graph.

In this article we prove

Theorem 1. *If the graph Γ with intersection array $\{u^2+1, u^2, 1, 1; 1, 1, u^2, u^2+1\}$ exists, then $u = 2$ and Γ is the Wells graph.*

Theorem 2. *If a distance-regular graph Γ with intersection array $\{u^2, u^2 - 1, 2, 1; 1, 2, u^2 - 1, u^2\}$ exists, then $u = 2$ and Γ is a 4-cube.*

Corollary 1. *Distance-regular graphs with intersection arrays $\{25, 24, 2, 1; 1, 2, 24, 25\}$, $\{36, 35, 2, 1; 1, 2, 35, 36\}$, $\{49, 48, 2, 1; 1, 2, 48, 49\}$, $\{81, 80, 2, 1; 1, 2, 80, 81\}$ do not exist.*

Theorem 3. *Distance-regular graph with intersection array $\{u^2 - 3, u^2 - 4, 3, 1; 1, 3, u^2 - 4, u^2 - 3\}$ does not exist.*

Corollary 2. *The antipodal 2-covering of the Higman-Sims graph does not exist.*

The proofs of the theorems use the triple intersection numbers (Coolsaet, Jurishich in [2]).

Let Γ be a distance-regular graph of diameter d . If u_1, u_2, u_3 are the vertices of the graph Γ , r_1, r_2, r_3 are non-negative integers not greater than d , then $\left\{ \begin{smallmatrix} u_1 u_2 u_3 \\ r_1 r_2 r_3 \end{smallmatrix} \right\}$ – set of vertices $w \in \Gamma$ such that $d(w, u_i) = r_i$, $\left[\begin{smallmatrix} u_1 u_2 u_3 \\ r_1 r_2 r_3 \end{smallmatrix} \right] = \left| \left\{ \begin{smallmatrix} u_1 u_2 u_3 \\ r_1 r_2 r_3 \end{smallmatrix} \right\} \right|$. The numbers $\left[\begin{smallmatrix} u_1 u_2 u_3 \\ r_1 r_2 r_3 \end{smallmatrix} \right]$ are called triple intersection numbers. For a fixed triple of vertices u_1, u_2, u_3 instead of $\left[\begin{smallmatrix} u_1 u_2 u_3 \\ r_1 r_2 r_3 \end{smallmatrix} \right]$ we will write $[r_1 r_2 r_3]$. Let u, v, w be the vertices of the graph Γ , $W = d(u, v)$, $U = d(v, w)$, $V = d(u, w)$. Since there is exactly one vertex $x = u$ such that $d(x, u) = 0$, then the number $[0jh]$ is 0 or 1. Hence $[0jh] = \delta_{jW} \delta_{hV}$. Similarly, $[i0h] = \delta_{iW} \delta_{hU}$ and $[ij0] = \delta_{iU} \delta_{jV}$.

Another set of equations can be obtained by fixing the distance between two vertices from $\{u, v, w\}$. Counting the number of vertices located at all possible distances from the third, we get:

$$\sum_{l=1}^d [ljh] = p_{jh}^U - [0jh], \sum_{l=1}^d [ilh] = p_{ih}^V - [i0h], \sum_{l=1}^d [ijl] = p_{ij}^W - [ij0](+)$$

At the same time, some triplets disappear. For $|i - j| > W$ or $i + j < W$ we have $p_{ij}^W = 0$, therefore $[ijh] = 0$ for all $h \in \{0, \dots, d\}$. Put $S_{ijh}(u, v, w) = \sum_{r,s,t=0}^d Q_{ri} Q_{sj} Q_{th} \left[\begin{smallmatrix} uvw \\ rst \end{smallmatrix} \right]$. If Krein's parameter $q_{ij}^h = 0$, then $S_{ijh}(u, v, w) = 0$.

2 Properties of a graph with intersection array

$$\{u^2 + 1, u^2, 1, 1; 1, 1, u^2, u^2 + 1\}$$

In this section, Γ is a distance-regular graph with intersection array $\{u^2 + 1, u^2, 1, 1; 1, 1, u^2, u^2 + 1\}$. The antipodal quotient of Γ is a strongly regular graph with parameters $(1/2(u^2 + 2 - u)(u^2 + 2 + u), u^2 + 1, 0, 2)$. The graph $\Gamma_{3,4}$ is strongly regular with the same parameters.

Further, Γ has $1 + u^2 + 1 + (u^2 + 1)u^2 + u^2 + 1 + 1 = u^4 + 3u^2 + 4$ vertices,

a spectrum

$$\begin{aligned}
& u^2 + 1, \text{ of miltiple: } 1, \\
& u - 1, \text{ of miltiple: } \frac{1}{4} (u^2 + u + 2)(u^2 + 1), \\
& \sqrt{u^2 + 1}, \text{ of miltiple: } \frac{1}{4} (u^2 + u + 2)(u^2 - u + 2), \\
& -\sqrt{u^2 + 1}, \text{ of miltiple: } \frac{1}{4} (u^2 + u + 2)(u^2 - u + 2), \\
& -u - 1, \text{ of miltiple: } \frac{1}{4} (u^2 - u + 2)(u^2 + 1),
\end{aligned}$$

and the dual eigenvalue matrix

$$Q = \begin{pmatrix} 1 & f & h & h & g \\ 1 & \frac{1}{4}u^3 + \frac{1}{4}u - \frac{1}{2} & \frac{u^4+3u^2+4}{4\sqrt{u^2+1}} & -\frac{u^4+3u^2+4}{4\sqrt{u^2+1}} & -\frac{1}{4}u^3 - \frac{1}{4}u - \frac{1}{2} \\ 1 & -\frac{u^2+u+2}{2u} & 0 & 0 & \frac{u^2-u+2}{2u} \\ 1 & \frac{1}{4}u^3 + \frac{1}{4}u - \frac{1}{2} & -\frac{u^4+3u^2+4}{4\sqrt{u^2+1}} & \frac{u^4+3u^2+4}{4\sqrt{u^2+1}} & -\frac{1}{4}u^3 - \frac{1}{4}u - \frac{1}{2} \\ 1 & f & -h & -h & g \end{pmatrix},$$

where

$$\begin{aligned}
f &= \frac{1}{4}u^4 + \frac{1}{4}u^3 + \frac{3}{4}u^2 + \frac{1}{4}u + \frac{1}{2}, \\
g &= \frac{1}{4}u^4 - \frac{1}{4}u^3 + \frac{3}{4}u^2 - \frac{1}{4}u + \frac{1}{2}, \\
h &= \frac{1}{4}u^4 + \frac{3}{4}u^2 + 1.
\end{aligned}$$

Lemma 1. *The non-zero intersection numbers of the graph Γ are:*

$$p_{11}^2 = p_{13}^2 = p_{13}^4 = p_{24}^2 = p_{33}^2 = p_{334}^1 = 1, p_{12}^2 = p_{23}^2 = u^2 - 1, p_{12}^1 = p_{12}^3 = p_{23}^1 = p_{23}^3 = u^2, p_{13}^4 = u^2 + 1, p_{22}^1 = p_{22}^2 = p_{22}^3 = u^4 - u^2, p_{22}^4 = u^4 + u^2.$$

Proof. Direct calculations. \square

Let u, v, w be the vertices of the graph Γ , $\{ijl\} = \left\{ \begin{smallmatrix} uvw \\ ijl \end{smallmatrix} \right\}$, $[ijl] = \left[\begin{smallmatrix} uvw \\ ijl \end{smallmatrix} \right]$. We put $\Sigma = \Gamma_2(u)$, $\Lambda = \Sigma_2$. Then Λ is a regular graph of degree $p_{22}^2 = u^4 - u^2$ on $k_2 = u^4 + u^2$ vertices.

Lemma 2. *Let $d(u, v) = d(u, w) = 2, d(v, w) = 1$. Then the triple intersection numbers are:*

$$\begin{aligned}
[112] &= [121] = 1, [122] = u^2 - 3, [123] = [132] = 1; \\
[212] &= [221] = u^2 - 2, [222] = u^4 - 3u^2 + 4, [223] = [232] = u^2 - 2, \\
[234] &= [243] = 1; \\
[312] &= [321] = 1, [322] = u^2 - 3, [323] = [332] = 1; \\
[422] &= 1.
\end{aligned}$$

Proof. Simplification of formulas (+). \square

Lemma 3. *Let $d(u, v) = d(u, w) = 2, d(v, w) = 3$. Then the triple intersection numbers are:*

$$\begin{aligned} [112] &= [121] = 1, [122] = u^2 - 3, [123] = [132] = 1; \\ [212] &= u^2 - 2, [214] = [241] = 1, [221] = u^2 - 2, [222] = u^4 - 3u^2 + 4, \\ [223] &= [232] = u^2 - 2, [241] = 1; \\ [312] &= [321] = 1, [322] = u^2 - 3, [323] = [332] = 1; \\ [422] &= 1. \end{aligned}$$

Proof. Simplification of formulas (+). □

Lemma 4. *Let $d(u, v) = d(u, w) = 2, d(v, w) = 4$. Then the triple intersection numbers are:*

$$\begin{aligned} [113] &= [131] = 1, [122] = u^2 - 1; \\ [213] &= [213] = u^2 - 1, [222] = u^4 - u^2, [231] = u^2 - 1; \\ [313] &= [331] = 1, [322] = u^2 - 1; \\ [422] &= 1. \end{aligned}$$

Proof. Simplification of formulas (+). □

In view of Lemmas 2-4, for the number of edges d between $\Lambda(v)$ and $\Lambda - (\{v\} \cup \Lambda(v))$ in the graph Λ following equalities are true $d = (u^2 - 1)(u^4 - 3u^2 + 4) + u^2(u^2 - 1) = u^6 - 3u^4 + 6u^2 - 4$.

On the other hand, $d = (u^4 - u^2)(u^4 - u^2 - 1 - \lambda)$, where λ is the mean value of the parameter $\lambda(\Lambda)$. That's why $u^4 - u^2 - 1 - \lambda = (u^6 - 3u^4 + 6u^2 - 4)/(u^4 - u^2)$ and $\lambda = u^4 - u^2 - 1 - (u^6 - 3u^4 + 6u^2 - 4)/(u^4 - u^2)$.

Lemma 5. *Let $d(u, v) = d(u, w) = d(v, w) = 2$. Then the triple intersection numbers are:*

$$\begin{aligned} [111] &= r_1, [112] = -r_1 - r_2 + 1, [113] = r_2, [121] = -r_1 - r_3 + 1, [122] = \\ &r_1^2 + r_1 + r_2 + r_3 + r_4 - 3, [123] = -r_2 - r_4 + 1, [131] = r_3, [132] = -r_3 - r_4 + 1, \\ [133] &= r_4; \\ [211] &= -r_1 - r_4 + 1, [212] = [221] = r_1^2 + r_1 + r_2 + r_3 + r_4 - 3, [213] = \\ &-r_2 - r_3 + 1, [222] = r_1^4 - 3r_1^2 - 2r_1 - 2r_2 - 2r_3 - 2r_4 + 4, [223] = [232] = \\ &r_1^2 + r_1 + r_2 + r_3 + r_4 - 3, [224] = [242] = 1, [233] = -r_1 - r_4 + 1; \\ [311] &= r_4, [312] = -r_3 - r_4 + 1, [313] = r_3, [321] = -r_2 - r_4 + 1, [322] = \\ &r_1^2 + r_1 + r_2 + r_3 + r_4 - 3, [323] = -r_1 - r_3 + 1, [331] = r_2, [332] = -r_1 - r_2 + 1, \\ [333] &= r_1; \\ [422] &= 1, \\ &\text{where } r_1 + r_2, r_1 + r_3, r_2 + r_4, r_3 + r_4, r_1 + r_4, r_2 + r_3 \leq 1. \end{aligned}$$

Proof. Simplification of formulas (+). □

By Lemma 5, we have $[222] \leq 4$. On the other hand, $\lambda = u^4 - u^2 - 1 - (u^6 - 3u^4 + 6u^2 - 4)/(u^4 - u^2)$, so $u = 2$.

Theorem 1 is proved.

3 Properties of a graph with intersection array

$$\{u^2, u^2 - 1, 2, 1; 1, 2, u^2 - 1, u^2\}$$

In this section, Γ is a distance-regular graph with intersection array $\{u^2, u^2 - 1, 2, 1; 1, 2, u^2 - 1, u^2\}$. For this graph $\Gamma_{3,4}$ is strongly regular with a nonprincipal positive eigenvalue r . For $u = r + 1$ we consider intersection array $\{(r + 1)^2, r^2 + 2r, 2, 1; 1, 2, r^2 + 2r, (r + 1)^2\}$ with $r > 1$. Then Γ has $1 + (r + 1)^2 + (r^2 + 2r)(r + 1)^2/2 + (r + 1)^2 + 1$ vertices,

a spectrum

$$\begin{aligned} & r^2 + 2r + 1, \text{ of multiple } 1, \\ & r + 1, \text{ of multiple } \frac{1}{8} (r^2 + 3r + 4)(r^2 + r + 2), \\ & r - 1, \text{ of multiple } \frac{1}{8} (r^2 + 3r + 4)(r + 2)(r + 1), \\ & -(r + 1), \text{ of multiple } \frac{1}{8} (r^2 + 3r + 4)(r^2 + r + 2), \\ & -(r + 3), \text{ of multiple } \frac{1}{8} (r^2 + r + 2)(r + 1)r, \end{aligned}$$

and the dual eigenvalue matrix

$$\begin{pmatrix} 1 & f & g & f & h \\ 1 & \frac{f}{r+1} & \frac{g}{r+1} & -\frac{f}{r+1} & -\frac{(r^2+r+2)(r+3)r}{8(r+1)} \\ 1 & 0 & -\frac{r^2+3r+4}{2(r+1)} & 0 & \frac{r^2+r+2}{2(r+1)} \\ 1 & -\frac{f}{r+1} & \frac{g}{r+1} & \frac{f}{r+1} & -\frac{(r^2+r+2)(r+3)r}{8(r+1)} \\ 1 & -f & g & -f & h \end{pmatrix},$$

where

$$\begin{aligned} f &= \frac{1}{8} (r^2 + 3r + 4)(r^2 + r + 2), \\ g &= \frac{1}{8} (r^2 + 3r + 4)(r + 2)(r + 1), \\ h &= \frac{1}{8} (r^2 + r + 2)(r + 1)r. \end{aligned}$$

Lemma 6. *The intersection numbers are:*

$$\begin{aligned} & p_{11}^1 = 0, p_{21}^1 = r^2 + 2r, p_{32}^1 = r^2 + 2r, p_{22}^1 = r^4/2 + 2r^3 + r^2/2 - 3r, p_{33}^1 = 0, \\ & p_{34}^1 = 1; \\ & p_{11}^2 = 2, p_{12}^2 = r^2 + 2r - 3, p_{13}^2 = 2, p_{22}^2 = r^4/2 + 2r^3 + r^2/2 - 3r + 4, \\ & p_{23}^2 = r^2 + 2r - 3, p_{24}^2 = 1, p_{33}^2 = 2; \\ & p_{12}^3 = r^2 + 2r, p_{13}^3 = 0, p_{14}^3 = 1, p_{22}^3 = r^4/2 + 2r^3 + r^2/2 - 3r, p_{23}^3 = r^2 + 2r, \\ & p_{33}^3 = 0; \\ & p_{13}^4 = r^2 + 2r + 1, p_{22}^4 = r^4/2 + 2r^3 + 5r^2/2 + r. \end{aligned}$$

Proof. Direct calculations. \square

We fix vertices u, v, w of the graph Γ and put $\{ijh\} = \left\{ \begin{matrix} uvw \\ ijh \end{matrix} \right\}$, $[ijh] = \left[\begin{matrix} uvw \\ ijh \end{matrix} \right]$.

Let $\Delta = \Gamma_2(u)$, $\Lambda = \Delta_2$. Then Λ is a regular graph of degree $p_{22}^2 = r^4/2 + 2r^3 + r^2/2 - 3r + 4$ on $k_2 = (r^2 + 2r)(r^2 + 2r + 1)/2 = r^4/2 + 2r^3 + 5r^2/2 + r$ vertices.

Lemma 7. *Let $d(u, v) = d(u, w) = 2, d(v, w) = 1$. Then the triple intersection numbers are:*

$$\begin{aligned} [112] &= [121] = 2, [122] = r^2 + 2r - 7, [123] = [132] = 2; \\ [212] &= [221] = r^2 + 2r - 4, [222] = r^4/2 + 2r^3 - 3r^2/2 - 7r + 12, [223] = \\ [232] &= r^2 + 2r - 4, [234] = [243] = 1; \\ [312] &= [321] = 2, [322] = r^2 + 2r - 7, [323] = [332] = 2; \\ [422] &= 1. \end{aligned}$$

Proof. Simplification of formulas (+). □

By Lemma 7 we have $[222] = r^4/2 + 2r^3 - 3r^2/2 - 7r + 12$.

Lemma 8. *Let $d(u, v) = d(u, w) = 2, d(v, w) = 3$. Then the triple intersection numbers are:*

$$\begin{aligned} [112] &= [121] = 2, [122] = r^2 + 2r - 7, [123] = [132] = 2; \\ [212] &= [221] = r^2 + 2r - 4, [214] = [241] = 1, [222] = r^4/2 + 2r^3 - 3r^2/2 - \\ 7r + 12, [223] &= [232] = r^2 + 2r - 4, [214] = [241] = 1; \\ [312] &= [321] = 2, [322] = r^2 + 2r - 7, [323] = [332] = 2; \\ [422] &= 1. \end{aligned}$$

Proof. Simplification of formulas (+). □

By Lemma 8 we have $[222] = r^4/2 + 2r^3 - 3r^2/2 - 7r + 12$.

Lemma 9. *Let $d(u, v) = d(u, w) = 2, d(v, w) = 4$. Then the triple intersection numbers are:*

$$\begin{aligned} [113] &= [131] = 2, [122] = r^2 + 2r - 3; \\ [213] &= [231] = r^2 + 2r - 3, [222] = r^4/2 + 2r^3 + r^2/2 - 3r + 4; \\ [313] &= [331] = 2, [322] = r^2 + 2r - 3; \\ [422] &= 1. \end{aligned}$$

Proof. Simplification of formulas (+). □

By Lemma 9 we have $[222] = r^4/2 + 2r^3 + r^2/2 - 3r + 4$.

Recall that $p_{12}^2 = r^2 + 2r - 3$, $p_{22}^2 = r^4/2 + 2r^3 + r^2/2 - 3r + 4$, $p_{23}^2 = r^2 + 2r - 3$, $p_{24}^2 = 1$. Let v, w be vertices from Λ . Then the number d of edges between $\Lambda(w)$ and $\Lambda - (\{w\} \cup \Lambda(w))$ satisfies $d = (r^2 + 2r - 3)(r^4 + 4r^3 - 3r^2 - 14r + 24) + r^4/2 + 2r^3 + r^2/2 - 3r + 4$.

On the other hand, $d = (r^4/2 + 2r^3 + r^2/2 - 3r + 4)(r^4/2 + 2r^3 + r^2/2 - 3r + 3 - \lambda)$, where λ is the mean value of the parameter $\lambda(\Lambda)$. That's why $r^4/2 + 2r^3 + r^2/2 - 3r + 3 - \lambda = (r^2 + 2r - 3)(r^4 + 4r^3 - 3r^2 - 14r + 24)/(r^4/2 + 2r^3 + r^2/2 - 3r + 4) + 1$ and $\lambda = r^4/2 + 2r^3 + r^2/2 - 3r + 2 - (r^2 + 2r - 3)(r^4 + 4r^3 - 3r^2 - 14r + 24)/(r^4/2 + 2r^3 + r^2/2 - 3r + 4)$.

Lemma 10. *Let $d(u, v) = d(u, w) = d(v, w) = 2$. Then the triple intersection numbers are:*

$$\begin{aligned} [111] &= r_7, [112] = -r_6 - r_7 + 2, [113] = r_6, [121] = -r_8 - r_7 + 2, \\ [122] &= r_5 + r_6 + r_7 + r_8 + r^2 + 2r - 7, [123] = -r_5 - r_6 + 2, [131] = r_8, \\ [132] &= -r_5 - r_8 + 2, [133] = r_5; \end{aligned}$$

$$\begin{aligned} [211] &= -r_5 - r_7 + 2, [212] = [221] = r_5 + r_6 + r_7 + r_8 + r^2 + 2r - 7, \\ [213] &= [231] = -r_6 - r_8 + 2, [222] = -2r_5 - 2r_6 - 2r_7 - 2r_8 + r^4/2 + 2r^3 - \\ &3r^2/2 - 7r + 16, [223] = [232] = r_5 + r_6 + r_7 + r_8 + r^2 + 2r - 7, [224] = [242] = 1, \\ [233] &= -r_5 - r_7 + 2; \end{aligned}$$

$$\begin{aligned} [311] &= r_5, [312] = -r_5 - r_8 + 2, [313] = r_8, [321] = -r_5 - r_6 + 2, \\ [322] &= r_5 + r_6 + r_7 + r_8 + r^2 + 2r - 7, [323] = -r_7 - r_8 + 2, [331] = r_6, \\ [332] &= -r_7 - r_6 + 2, [333] = r_7; \end{aligned}$$

$$[422] = 1,$$

where $r_6 + r_7, r_8 + r_7, r_5 + r_6, r_5 + r_8, r_5 + r_7, r_8 + r_6 \leq 2$.

Proof. Simplification of formulas (+). □

By Lemma 10 we have $r^4/2 + 2r^3 - 3r^2/2 - 7r + 8 \leq [222] \leq r^4/2 + 2r^3 - 3r^2/2 - 7r + 16$.

Let $d(u, v) = 2$.

Count the number e_1 of pairs of vertices (s, t) at a distance 1, where $s \in \left\{ \begin{smallmatrix} uv \\ 21 \end{smallmatrix} \right\}$ and $t \in \left\{ \begin{smallmatrix} uv \\ 22 \end{smallmatrix} \right\}$. On the one hand, by Lemma 7 we have $[221] = r^2 + 2r - 4$, so $e_1 = (r^2 + 2r - 3)(r^2 + 2r - 4)$. On the other hand, by Lemma 10 we have $[211] = -r_5 - r_7 + 2$ and $e_1 = -\sum_i (r_5^i + r_7^i) + r^4 + 4r^3 + r^2 - 6r + 8 = (r^2 + 2r - 3)(r^2 + 2r - 4) = r^4 + 4r^3 - 3r^2 - 14r + 12$. Thus, $\sum_i (r_5^i + r_7^i) = 4r^2 + 8r - 4$ and $\sum_i (r_5^i + r_7^i)/(r^4/2 + 2r^3 + r^2/2 - 3r + 4) = (4r^2 + 8r - 4)/(r^4/2 + 2r^3 + r^2/2 - 3r + 4)$.

Count the number e_3 of pairs of vertices (s, t) at a distance 3, where $s \in \left\{ \begin{smallmatrix} uv \\ 21 \end{smallmatrix} \right\}$ and $t \in \left\{ \begin{smallmatrix} uv \\ 22 \end{smallmatrix} \right\}$. On the one hand, by Lemma 7 we have $[223] = r^2 + 2r - 4$, so $e_1 = (r^2 + 2r - 3)(r^2 + 2r - 4)$. On the other hand, by Lemma 10 we have $[213] = -r_6 - r_8 + 2$ and $e_1 = -\sum_i (r_6^i + r_8^i) + r^4 + 4r^3 + r^2 - 6r + 8 = (r^2 + 2r - 3)(r^2 + 2r - 4) = r^4 + 4r^3 - 3r^2 - 14r + 12$. Thus, $\sum_i (r_6^i + r_8^i) = 4r^2 + 8r - 4$ and $\sum_i (r_6^i + r_8^i)/(r^4/2 + 2r^3 + r^2/2 - 3r + 4) = (4r^2 + 8r - 4)/(r^4/2 + 2r^3 + r^2/2 - 3r + 4)$.

Now $\lambda = \sum_i [222]^i/(r^4/2 + 2r^3 + r^2/2 - 3r + 4) = -2\sum_i (r_5^i + r_6^i + r_7^i + r_8^i)/(r^4/2 + 2r^3 + r^2/2 - 3r + 4) + r^4/2 + 2r^3 - 3r^2/2 - 7r + 16$. Hence, $\lambda = r^4/2 + 2r^3 - 3r^2/2 - 7r + 16 - 4(4r^2 + 8r - 4)/(r^4/2 + 2r^3 + r^2/2 - 3r + 4)$. A contradiction with the fact, that $\lambda = r^4/2 + 2r^3 + r^2/2 - 3r + 2 - (r^2 + 2r - 3)(r^4 + 4r^3 - 3r^2 - 14r + 24)/(r^4/2 + 2r^3 + r^2/2 - 3r + 4)$ and $(r^2 + 2r - 7)(r^4 + 4r^3 + r^2 - 6r + 8) = (r^2 + 2r - 3)(r^4 + 4r^3 - 3r^2 - 14r + 24)$.

Theorem 2 is proved.

4 Properties of a graph with intersection array

$$\{u^2 - 3, u^2 - 4, 3, 1; 1, 3, u^2 - 4, u^2 - 3\}$$

In this section Γ is a distance-regular graph with intersection array $\{u^2 - 3, u^2 - 4, 3, 1; 1, 3, u^2 - 4, u^2 - 3\}$. Then Γ has $1 + (u^2 - 3) + (u^2 - 3)(u^2 - 4)/3 + (u^2 - 3) + 1 = (u^4 - u^2)/3$ vertices,

a spectrum

$$\begin{aligned} &u^2 - 3, \text{ of multiple: } 1, \\ &u - 3, \text{ of multiple: } \frac{1}{12} (u^2 - 3)(u + 2)(u + 1), \\ &\sqrt{u^2 - 3}, \text{ of multiple: } \frac{1}{12} (u + 1)(u - 1)u^2, \\ &-\sqrt{u^2 - 3}, \text{ of multiple: } \frac{1}{12} (u + 1)(u - 1)u^2, \\ &-u - 3, \text{ of multiple: } \frac{1}{12} (u^2 - 3)(u - 1)(u - 2), \end{aligned}$$

and the dual eigenvalue matrix

$$Q = \begin{pmatrix} 1 & f & h & h & g \\ 1 & \frac{1}{4}u^3 + \frac{1}{4}u - \frac{1}{2} & \frac{u^4+3u^2+4}{4\sqrt{u^2+1}} & -\frac{u^4+3u^2+4}{4\sqrt{u^2+1}} & -\frac{1}{4}u^3 - \frac{1}{4}u - \frac{1}{2} \\ 1 & -\frac{u^2+u+2}{2u} & 0 & 0 & \frac{u^2-u+2}{2u} \\ 1 & \frac{1}{4}u^3 + \frac{1}{4}u - \frac{1}{2} & -\frac{u^4+3u^2+4}{4\sqrt{u^2+1}} & \frac{u^4+3u^2+4}{4\sqrt{u^2+1}} & -\frac{1}{4}u^3 - \frac{1}{4}u - \frac{1}{2} \\ 1 & f & -h & -h & g \end{pmatrix},$$

where

$$\begin{aligned} f &= \frac{1}{4}u^4 + \frac{1}{4}u^3 + \frac{3}{4}u^2 + \frac{1}{4}u + \frac{1}{2}, \\ g &= \frac{1}{4}u^4 - \frac{1}{4}u^3 + \frac{3}{4}u^2 - \frac{1}{4}u + \frac{1}{2}, \\ h &= \frac{1}{4}u^4 + \frac{3}{4}u^2 + 1. \end{aligned}$$

Lemma 11. *The non-zero intersection numbers of the graph Γ are:*

$$\begin{aligned} p_{14}^3 &= p_{24}^2 = p_{34}^1 = 1, \quad p_{11}^2 = p_{13}^2 = p_{33}^2 = 3, \quad p_{12}^2 = p_{23}^2 = u^2 - 9, \quad p_{12}^1 = \\ p_{12}^3 &= p_{23}^1 = p_{23}^3 = u^2 - 4, \quad p_{13}^4 = u^2 - 3, \quad p_{22}^1 = p_{22}^3 = (u^4 - 13u^2)/3 + 12, \\ p_{22}^2 &= (u^4 - 13u^2)/3 + 20. \quad p_{22}^4 = (u^4 - 7u^2)/3 + 4. \end{aligned}$$

Proof. Direct calculations. \square

Let u, v, w are vertices of the graph Γ , $\{ijl\} = \left\{ \begin{smallmatrix} uvw \\ ijl \end{smallmatrix} \right\}$, $[ijl] = \left[\begin{smallmatrix} uvw \\ ijl \end{smallmatrix} \right]$. We put $\Sigma = \Gamma_2(u)$, $\Lambda = \Sigma_2$. Then Λ is a regular graph of degree $p_{22}^2 = (u^4 - 13u^2)/3 + 20$ on $k_2 = (u^2 - 3)(u^2 - 4)/3 = (u^4 - 7u^2 + 12)/3$ vertices.

Lemma 12. *Let $d(u, v) = d(u, w) = 2, d(v, w) = 1$. Then the triple intersection numbers are:*

$$[112] = [121] = 3, \quad [122] = u^2 - 15, \quad [123] = [132] = 3;$$

$$\begin{aligned}
[212] &= [221] = u^2 - 10, [222] = (u^4 - 19u^2)/3 + 40, [223] = [232] = u^2 - 10, \\
[234] &= [243] = 1; \\
[312] &= [321] = 3, [322] = u^2 - 15, [323] = [332] = 3; \\
[422] &= 1.
\end{aligned}$$

Proof. Simplification of formulas (+). □

Lemma 13. *Let $d(u, v) = d(u, w) = 2, d(v, w) = 3$. Then the triple intersection numbers are:*

$$\begin{aligned}
[112] &= [121] = 3, [122] = u^2 - 15, [123] = [132] = 3; \\
[212] &= [221] = u^2 - 10, [214] = [241] = 1, [222] = (u^4 - 19u^2)/3 + 40, \\
[223] &= [232] = u^2 - 10; \\
[312] &= [321] = 3, [322] = u^2 - 15, [323] = [332] = 3; \\
[422] &= 1.
\end{aligned}$$

Proof. Simplification of formulas (+). □

Lemma 14. *Let $d(u, v) = d(u, w) = 2, d(v, w) = 4$. Then the triple intersection numbers are:*

$$\begin{aligned}
[113] &= [131] = 3, [122] = u^2 - 9; \\
[213] &= [231] = u^2 - 9, [222] = (u^4 - 13u^2)/3 + 20; \\
[313] &= [331] = 3, [322] = u^2 - 9; \\
[422] &= 1.
\end{aligned}$$

Proof. Simplification of formulas (+). □

By Lemmas 12–14 for the number of edges d between $\Lambda(v)$ and $\Lambda - (\{v\} \cup \Lambda(v))$ in the graph Λ the equalities $d = 2(u^2 - 9)((u^4 - 19u^2)/3 + 40) + (u^4 - 13u^2)/3 + 20 = (2u^6 - 55u^4 + 329u^2 + 180)/3$ are true.

From the other hand, $d = ((u^4 - 13u^2 + 60)/3)((u^4 - 13u^2 + 57)/3 - \lambda)$, where λ is an average value of parameter $\lambda(\Lambda)$. Therefore, $(u^4 - 13u^2 + 57)/3 - \lambda = (2u^6 - 55u^4 + 329u^2 + 180)/(u^4 - 13u^2 + 60)$ and $\lambda = (u^4 - 13u^2 + 57)/3 - (2u^6 - 55u^4 + 329u^2 + 180)/(u^4 - 13u^2 + 60)$.

Lemma 15. *Let $d(u, v) = d(u, w) = d(v, w) = 2$. Then the triple intersection numbers are:*

$$\begin{aligned}
[111] &= r_2, [112] = -r_1 - r_2 + 3, [113] = r_1, [121] = -r_2 - r_3 + 3, \\
[122] &= r_1^2 + r_1 + r_2 + r_3 + r_4 - 15, [123] = -r_1 - r_4 + 3, [131] = r_3, \\
[132] &= -r_3 - r_4 + 3, [133] = r_4; \\
[211] &= -r_2 - r_4 + 3, [212] = [221] = r_1^2 + r_1 + r_2 + r_3 + r_4 - 15, [213] = \\
&= -r_1 - r_3 + 3, [222] = (r_1^4 - 19r_1^2)/3 - 2r_1 - 2r_2 - 2r_3 - 2r_4 + 48, [223] = \\
[232] &= r_1^2 + r_1 + r_2 + r_3 + r_4 - 15, [224] = [242] = 1, [233] = -r_2 - r_4 + 3; \\
[311] &= r_4, [312] = -r_3 - r_4 + 3, [313] = r_3, [321] = -r_1 - r_4 + 1, \\
[322] &= r_1^2 + r_1 + r_2 + r_3 + r_4 - 15, [323] = -r_2 - r_3 + 3, [331] = r_1, \\
[332] &= -r_1 - r_2 + 3, [333] = r_2; \\
[422] &= 1,
\end{aligned}$$

where $r_1 + r_2, r_1 + r_3, r_2 + r_4, r_3 + r_4, r_1 + r_4, r_2 + r_3 \leq 3$.

Proof. Simplification of formulas (+). □

By Lemma 15 we have $[222] = (r_1^4 - 19r_1^2)/3 - 2r_1 - 2r_2 - 2r_3 - 2r_4 + 48$

In case $r_1 = 3$ we have $[222] = 32$, and in case $r_1 \leq 2$ we have $[222] \leq 36$.

If $(u^4 - 13u^2 + 57)/3 - (2u^6 - 55u^4 + 329u^2 + 180)/(u^4 - 13u^2 + 60) = [222] \leq 36$, then $u \leq 4$.

A contradiction with the fact that intersection array $\{u^2-3, u^2-4, 3, 1; 1, 3, u^2-4, u^2-3\}$ is valid only for $u \geq 5$.

Theorem 3 is proved.

From Theorem 3 for $u = 5$ we obtain Corollary 2.

References

- [1] A.E. Brouwer, A.M. Cohen, A. Neumaier, *Distance-Regular Graphs*, Springer-Verlag, Berlin–Heidelberg–New-York, 1989.
- [2] K. Coolsaet, A. Jurishich, *Using equality in the Krein conditions to prove nonexistence of certain distance-regular graphs*, J. Comb. Theory, Series A., **115** (2008), 1086–1095.
- [3] A.A. Makhnev, D.V. Paduchikh, *Inverse problems in the class of distance-regular graphs with diameter 4*, Trudy Inst. Mat. Mekh. UrO RAN, **28**:1 (2022), 199–208.
- [4] A.A. Makhnev, M.P. Golubyatnikov, Wenbin Guo, *Inverse Problems in Graph Theory: Nets*, Communications in Mathematics and Statistics, **7**:1 (2019), 69–83.
- [5] A.A. Makhnev, Wenbin Guo, K.S. Efimov, *Distance-regular graphs of diameter 3 without triangles with $c_2 = 2$* , Communications in Mathematics and Statistics, **10**:1 (2022), 785–792.

ALEXANDR A. MAKHNEV

INSTITUTE OF MATHEMATICS AND MECHANICS, URAL BRANCH OF THE RUSSIAN ACADEMY OF SCIENCES, URAL FEDERAL UNIVERSITY NAMED AFTER THE FIRST PRESIDENT OF RUSSIA B. N. YELTSIN,

S. KOVALEVSKOY STR., 4,

625090, YEKATERINBURG, RUSSIA

Email address: makhnev@imm.uran.ru

MIKHAIL P. GOLUBYATNIKOV

INSTITUTE OF MATHEMATICS AND MECHANICS, URAL BRANCH OF THE RUSSIAN ACADEMY OF SCIENCES, URAL FEDERAL UNIVERSITY NAMED AFTER THE FIRST PRESIDENT OF RUSSIA B. N. YELTSIN,

S. KOVALEVSKOY STR., 4,

625090, YEKATERINBURG, RUSSIA

Email address: mike_ru1@mail.ru

KONSTANTIN S. EFIMOV

INSTITUTE OF MATHEMATICS AND MECHANICS, URAL BRANCH OF THE RUSSIAN ACADEMY OF SCIENCES, URAL FEDERAL UNIVERSITY NAMED AFTER THE FIRST PRESIDENT OF RUSSIA B. N. YELTSIN, URAL STATE MINING UNIVERSITY,

S. KOVALEVSKOY STR., 4,

625090, YEKATERINBURG, RUSSIA

Email address: konstantin.s.efimov@gmail.com