

## ON THE RECOGNITION OF FINITE SOLVABLE GROUPS WITH $\mathfrak{F}$ -SUBNORMAL PROJECTORS

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**Abstract:** For saturated formations  $\mathfrak{X} \subseteq \mathfrak{F}$  where  $\mathfrak{F}$  is hereditary it is proved that the class  $\mathfrak{X}_{sn_{\mathfrak{F}}}$  of groups all whose  $\mathfrak{X}$ -projectors are  $\mathfrak{F}$ -subnormal is a saturated formation and its local definitions are found. The computational methods of the  $\mathfrak{X}_{sn_{\mathfrak{F}}}$ -residual of a finite solvable group is discussed in the paper.

**Keywords:** Finite group; solvable group; local formation;  $\mathfrak{F}$ -residual;  $\mathfrak{F}$ -projector;  $\mathfrak{F}$ -subnormal subgroup.

### 1 Introduction and the Main Result

All the groups under consideration are finite. The classical method of the studying the structure of a finite group is to consider the properties of its chains of subgroups. For example, a group is nilpotent if every its cyclic primary (or Sylow) subgroup is subnormal. These results have been significantly developed within the framework of the theory of formations. The structure of groups with different systems of formational subnormal subgroups was studied, for example, in [1, 2, 3, 4, 5, 7, 6].

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**Definition 1** ([8, Definition 6.1.2], [9, 1.2.8]). *Let  $\mathfrak{F}$  be a class of groups. A subgroup  $H$  of  $G$  is called  $\mathfrak{F}$ -subnormal in  $G$ , if  $H = G$  or there exists a maximal chain of subgroups  $H = H_0 \subset H_1 \subset \dots \subset H_n = G$  such that  $H_i/\text{Core}_{H_i}(H_{i-1}) \in \mathfrak{F}$  for  $i = 1, \dots, n$ . Denoted by  $H\mathfrak{F}$ -sn  $G$ .*

In the work [10], the author proposed a construction that generalizes the constructions of the works [1, 2, 3, 4, 5] and allows to obtain their main results from a general position.

**Definition 2** ([10, Definition 1.2]). *Let  $\mathfrak{F}$  be a class of groups and  $\mathfrak{X}$  be a saturated homomorph. Denote by  $f_{\mathfrak{X}}(\mathfrak{F})$  the class of groups in which all  $\mathfrak{X}$ -subgroups are  $\mathfrak{F}$ -subnormal.*

Note that in an arbitrary case it is difficult to find all  $\mathfrak{X}$ -subgroups of a group from the computational point of view. This problem is solved in the theory of classes of groups by selecting a system of special  $\mathfrak{X}$ -subgroups in each group. In the case when  $\mathfrak{X}$  is a saturated formation, all  $\mathfrak{X}$ -projectors of the group  $G$  can be taken as such system. For example, the set of Sylow  $p$ -subgroups of a group coincides with the set of its  $\mathfrak{N}_p$ -projectors.

Let  $\mathfrak{X}$  be a class of groups. Recall [12, III, Definition 3.1] that a subgroup  $U$  of a group  $G$  is called  $\mathfrak{X}$ -maximal in  $G$  provided that (a)  $U \in \mathfrak{X}$ , and (b) if  $U \leq V \leq G$  and  $V \in \mathfrak{X}$ , then  $U = V$ ; a subgroup  $H$  of a group  $G$  is called  $\mathfrak{X}$ -projector [12, III, Definition 3.2] if  $HN/N$  is  $\mathfrak{X}$ -maximal in  $G/N$  for every  $N \trianglelefteq G$ . According to [12, III, Theorem 3.10]  $\mathfrak{X}$ -projectors exists in every group in case when  $\mathfrak{X}$  is a non-empty saturated formation.

In [11] the authors studied the question: how the structure of a group depends on the embedding of its  $\mathfrak{X}$ -projectors?

**Definition 3** ([11, Definition 2.5]). *Let  $\mathfrak{X}$  and  $\mathfrak{F}$  be non-empty classes of groups. Denote by  $\mathfrak{X}_{sn\mathfrak{F}}$  the class of groups all whose  $\mathfrak{X}$ -projectors are  $\mathfrak{F}$ -subnormal.*

In case when  $\mathfrak{X}$  is a saturated formation and  $\mathfrak{F}$  is a hereditary formation it was proved only that  $\mathfrak{X}_{sn\mathfrak{F}}$  is a formation [11, Theorem 2.14(2)]. At first we establish the connection between  $\mathfrak{X}_{sn\mathfrak{F}}$  and  $f_{\mathfrak{X}}(\mathfrak{F})$ . Recall that  $\mathfrak{X}^S$  denotes the class of groups all whose subgroups belong to  $\mathfrak{X}$  for a class of groups  $\mathfrak{X}$ .

**Theorem 1.** *Let  $\emptyset \neq \mathfrak{X}$  be a saturated formation and  $\mathfrak{F}$  be a hereditary formation. If  $\mathfrak{X} \subseteq \mathfrak{F}$  or  $\mathfrak{F}$  is saturated, then  $\mathfrak{X}_{sn\mathfrak{F}}^S = f_{\mathfrak{X}}(\mathfrak{F})$ .*

The aim of this paper is to describe the structure of  $\mathfrak{X}_{sn\mathfrak{F}}$  and suggest the algorithms to compute the  $\mathfrak{X}_{sn\mathfrak{F}}$ -residual of a (solvable) group, in particular to test a group for being an  $\mathfrak{X}_{sn\mathfrak{F}}$ -group. At first we are intersecting in conditions for  $\mathfrak{X}_{sn\mathfrak{F}}$  to be saturated (local) and in its canonical local definition.

**Theorem 2.** *Let  $\emptyset \neq \mathfrak{X} \subseteq \mathfrak{F}$  be saturated formations and  $F$  be the canonical local definition of  $\mathfrak{F}$ . If  $\mathfrak{F}$  is hereditary, then  $\mathfrak{X}_{sn\mathfrak{F}}$  is a local (saturated) formation locally defined by  $X(p)$  where  $X(p)$  is a class of all  $\mathfrak{X}_{sn\mathfrak{F}}$ -groups whose  $\mathfrak{X}$ -projectors belongs to  $F(p)$ .*

**Remark 1.** If  $\mathfrak{X} \supseteq \mathfrak{F}$  are saturated formations, then  $\mathfrak{X}_{sn_{\mathfrak{F}}} = \mathfrak{X}$  (see the proof of Remark 1).

Note that from this theorem the key results of [4] follow.

**Corollary 1** ([4, Theorems 3.4 and 3.6]). Let  $\pi$  be a set of primes and  $\mathfrak{F}$  be a hereditary saturated formation. Then the class  $W_{\pi}\mathfrak{F}$  of all groups  $G$ , all whose Sylow  $p$ -subgroups for every  $p \in \pi \cap \pi(G)$  and the unit subgroup are  $\mathfrak{F}$ -subnormal, is a hereditary saturated formation. If  $F$  and  $H$  are the canonical local definitions of  $\mathfrak{F}$  and  $W_{\pi}\mathfrak{F}$  respectively, then  $H(p)$  is the class of all  $W_{\pi}\mathfrak{F}$ -groups all whose Sylow  $\pi$ -subgroups belong to  $F(p)$  for all  $p \in \pi(\mathfrak{F})$  and  $H(p) = \emptyset$  otherwise.

The main idea of local formation is to reduce the computation of the group's properties associated with it to the computations for the values of its local definition (more simple formations). Therefore the presented in Theorem 2 local definition of  $\mathfrak{X}_{sn_{\mathfrak{F}}}$  is not good from the computational point of view because it requires to check that group belongs to  $\mathfrak{X}_{sn_{\mathfrak{F}}}$  for every its value. We obtained simpler local definition of  $\mathfrak{X}_{sn_{\mathfrak{F}}}$  in the solvable case.

**Theorem 3.** Let  $\emptyset \neq \mathfrak{X} \subseteq \mathfrak{F}$  be saturated formations of solvable groups,  $\mathfrak{F}$  be hereditary and  $F$  be the canonical local definition of  $\mathfrak{F}$ . Then  $\mathfrak{X}_{sn_{\mathfrak{F}}}$  is locally defined by  $X$ , where  $X(p)$  is a class of all solvable groups whose  $\mathfrak{X}$ -projectors belong to  $F(p)$ .

Recall that the  $\mathfrak{F}$ -residual of a group  $G$  is the smallest normal subgroup  $G^{\mathfrak{F}}$  of  $G$  with  $G/G^{\mathfrak{F}} \in \mathfrak{F}$ . If  $\mathfrak{F}$  is a local formation and  $f$  is its local definition, then from [12, IV, Theorem 3.2(b)] the formula for the  $\mathfrak{F}$ -residual follows:

$$G^{\mathfrak{F}} = O^{\pi(G) \cap \pi(\mathfrak{F})}(G) \prod_{p \in \pi(G) \cap \pi(\mathfrak{F})} O^{p'}(O^p(G^{f(p)}))$$

This formula was used to compute the  $\mathfrak{F}$ -residual of a solvable group in [13]. If the test for a group to belong  $\mathfrak{F}$  is known, then the method for computing the  $\mathfrak{F}$ -residual is suggested in [14]. For the formations of groups  $\mathfrak{F}$  defined by the action of a group on its chief series the polynomial time algorithms for the computation of the  $\mathfrak{F}$ -residual of a permutation group are suggested in [15]. Note that in case  $\mathfrak{F} \cap \mathfrak{X} \notin \{\mathfrak{F}, \mathfrak{X}\}$  we can not use any of these methods for the computation of the  $\mathfrak{X}_{sn_{\mathfrak{F}}}$ -residual of a group.

**Definition 4.** Let  $H$  be a subgroup of a group  $G$ ,  $\mathfrak{F}$  be a hereditary formation and a chain of subgroups  $G = G_0 \geq G_1 \geq G_2 \geq \dots$  be defined by  $G_i = HG_{i-1}^{\mathfrak{F}}$ . We will denote the final term of this chain by  $g(H, G, \mathfrak{F})$ .

**Remark 2.** Note that  $H$  is  $\mathfrak{F}$ -subnormal in  $G$  if and only if  $H = g(H, G, \mathfrak{F})$ .

Recall [12, III, Definition 3.5] that a subgroup  $\mathfrak{X}$  of a group  $G$  is called  $\mathfrak{X}$ -covering for a class of groups  $\mathfrak{X}$  if  $H$  is an  $\mathfrak{X}$ -projector of  $V$  for every  $H \leq V \leq G$ . For example a Sylow  $p$ -subgroup is an  $\mathfrak{N}_p$ -covering subgroup. It is known [12, III, Theorem 3.21] that for a saturated formation  $\mathfrak{X}$  and

a solvable group  $G$  the sets of its  $\mathfrak{X}$ -projectors and  $\mathfrak{X}$ -covering subgroups coincide. Our second approach in the computation of  $\mathfrak{X}_{sn\mathfrak{F}}$ -residual is based on the following results.

**Theorem 4.** *Let  $\emptyset \neq \mathfrak{X}$  be a saturated formation and  $\mathfrak{F}$  be a hereditary formation. Assume that in a group  $G$  every  $\mathfrak{X}$ -projector is an  $\mathfrak{X}$ -covering subgroup. Then the  $\mathfrak{X}_{sn\mathfrak{F}}$ -residual of  $G$  is  $\langle g(H, G, \mathfrak{F})^{\mathfrak{X}} \mid H \text{ is an } \mathfrak{X}\text{-projector of } G \text{ with } g(H, G, \mathfrak{F}) \neq H \rangle$ .*

In Section 4 we present the algorithms for the computation of the  $\mathfrak{X}_{sn\mathfrak{F}}$ -residual of a solvable group based on Theorems 3 and 4 and compare them to a known ones in case when they are applicable.

## 2 Preliminaries

We use standard definitions and notations (see [8, 12]). Recall some of them:  $\pi(G)$  is the set of all prime divisors of  $|G|$ ;  $\pi(\mathfrak{X}) = \bigcup_{G \in \mathfrak{X}} \pi(G)$ ;  $\Phi(G)$  is the Frattini subgroup of  $G$ ;  $O^\pi(G)$  is the smallest normal subgroup of  $G$  of  $\pi$ -index.

Recall that a *formation* is a class of groups  $\mathfrak{F}$  which is closed under taking epimorphic images (i.e. from  $G \in \mathfrak{F}$  and  $N \trianglelefteq G$  it follows that  $G/N \in \mathfrak{F}$ ) and subdirect products (i.e. from  $G/N_1 \in \mathfrak{F}$  and  $G/N_2 \in \mathfrak{F}$  it follows that  $G/(N_1 \cap N_2) \in \mathfrak{F}$ ). For a formation  $\mathfrak{F}$  let  $E_\Phi \mathfrak{F} = \{G \mid G/\Phi(G) \in \mathfrak{F}\}$ . A formation  $\mathfrak{F}$  is said to be: *saturated* if  $\mathfrak{F} = E_\Phi \mathfrak{F}$ ; *hereditary* if  $H \in \mathfrak{F}$  whenever  $H \leq G \in \mathfrak{F}$ .

A function of the form  $f : \mathbb{P} \rightarrow \{\text{formations}\}$  is called a *formation function*. Recall [12, IV, Definition 3.1] that a formation  $\mathfrak{F}$  is called *local* if

$$\mathfrak{F} = LF(f) = \{G \mid G/C_G(H/K) \in f(p)\}$$

for every  $p \in \pi(H/K)$  and chief factor  $H/K$  of  $G$

for some formation function  $f$ . In this case  $f$  is called a *local definition* of  $\mathfrak{F}$ . By the Gaschütz–Lubeseder–Schmid theorem [12, IV, Theorem 4.6], a non-empty formation is local if and only if it is saturated. Recall [12, IV, Proposition 3.8] that if  $\mathfrak{F}$  is a local formation, there exists a unique formation function  $F$ , defining  $\mathfrak{F}$ , such that  $F(p) = \mathfrak{N}_p F(p) \subseteq \mathfrak{F}$  for every  $p \in \mathbb{P}$ . In this case  $F$  is called the *canonical local definition* of  $\mathfrak{F}$ .

The basic properties of  $\mathfrak{F}$ -subnormality are contained in the following two lemmas.

**Lemma 1** ([8, Lemma 6.1.6]). *Let  $\mathfrak{F}$  be a formation,  $H, R \leq G$  and  $N \trianglelefteq G$ .*

- (1) *If  $H \mathfrak{F}$ -sn  $G$ , then  $HN/N \mathfrak{F}$ -sn  $G/N$ .*
- (2) *If  $H/N \mathfrak{F}$ -sn  $G/N$ , then  $H \mathfrak{F}$ -sn  $G$ .*
- (3) *If  $H \mathfrak{F}$ -sn  $R$  and  $R \mathfrak{F}$ -sn  $G$ , then  $H \mathfrak{F}$ -sn  $G$ .*

**Lemma 2** ([8, Lemma 6.1.7]). *Let  $\mathfrak{F}$  be a hereditary formation,  $G$  be a group and  $H, R \leq G$ .*

- (1) *If  $H \mathfrak{F}$ -sn  $G$ , then  $H \cap R \mathfrak{F}$ -sn  $R$ .*
- (2) *If  $H \mathfrak{F}$ -sn  $G$  and  $R \mathfrak{F}$ -sn  $G$ , then  $H \cap R \mathfrak{F}$ -sn  $G$ .*

Recall that the generalization  $\tilde{F}(G)$  of the Fitting subgroup is defined by  $\Phi(G) \subseteq \tilde{F}(G)$  and  $\tilde{F}(G)/\Phi(G) = \text{Soc}(G/\Phi(G))$ .

**Lemma 3** ([3, Lemma 2.7]). *Let  $\mathfrak{F}$  be a saturated formation and let a group  $G = H\tilde{F}(G)$ , where  $H$  is an  $\mathfrak{F}$ -subnormal  $\mathfrak{F}$ -subgroup of  $G$ . Then  $G \in \mathfrak{F}$ .*

### 3 Proves of the Main Results

**3.1. Proof of Theorem 1.** Since any  $\mathfrak{X}$ -projector is an  $\mathfrak{X}$ -subgroup, we see that  $f_{\mathfrak{X}}(\mathfrak{F}) \subseteq \mathfrak{X}_{sn_{\mathfrak{F}}}$ . Note that in the assumptions of the theorem every primitive section of an  $E_{\Phi}\mathfrak{F} \cap \mathfrak{X}$ -group belongs to  $\mathfrak{F}$ . Therefore  $f_{\mathfrak{X}}(\mathfrak{F})$  is a hereditary formation by [10, Theorem 3.1]. Note that  $\mathfrak{X}_{sn_{\mathfrak{F}}}^S$  is the greatest by inclusion hereditary subformation of  $\mathfrak{X}_{sn_{\mathfrak{F}}}$  by its definition and [16, Lemma 25.4]. Hence  $f_{\mathfrak{X}}(\mathfrak{F}) \subseteq \mathfrak{X}_{sn_{\mathfrak{F}}}^S$ .

Assume that  $f_{\mathfrak{X}}(\mathfrak{F}) \subset \mathfrak{X}_{sn_{\mathfrak{F}}}^S$ . Let  $G$  be a minimal order group from  $\mathfrak{X}_{sn_{\mathfrak{F}}}^S \setminus f_{\mathfrak{X}}(\mathfrak{F})$ . It means that  $G$  has a non- $\mathfrak{F}$ -subnormal  $\mathfrak{X}$ -subgroup  $H$ .

Let  $N$  be a minimal normal subgroup of  $G$ . Then  $G/N \in \mathfrak{X}_{sn_{\mathfrak{F}}}^S$ . Hence  $G/N \in f_{\mathfrak{X}}(\mathfrak{F})$  by our assumption. If  $G$  has two different minimal normal subgroups  $N_1$  and  $N_2$ , then from  $G/N_i \in f_{\mathfrak{X}}(\mathfrak{F})$  it follows that  $G \simeq G/(N_1 \cap N_2) \in f_{\mathfrak{X}}(\mathfrak{F})$ , a contradiction. Thus  $N$  is the unique minimal normal subgroup of  $G$ .

If  $G^{\mathfrak{F}} \simeq 1$ , then  $G \in \mathfrak{F}$ . Since  $\mathfrak{F}$  is a hereditary formation, every subgroup of  $G$  is  $\mathfrak{F}$ -subnormal in  $G$ . So  $G \in f_{\mathfrak{X}}(\mathfrak{F})$ , a contradiction. Hence  $G^{\mathfrak{F}} \neq 1$  and  $N \leq G^{\mathfrak{F}}$ . Let  $P$  be an  $\mathfrak{X}$ -projector of  $G$ . From  $P \mathfrak{F}$ -sn  $G$  it follows that  $PG^{\mathfrak{F}} < G$ . Hence  $PN < G$ . In particular,  $G/N \notin \mathfrak{X}$  by the definition of  $\mathfrak{X}$ -projector.

Assume that  $HN < G$ . Then  $HN \in f_{\mathfrak{X}}(\mathfrak{F})$ . It means that  $H \mathfrak{F}$ -sn  $HN$ . From  $G/N \in f_{\mathfrak{X}}(\mathfrak{F})$  it follows that  $HN/N \mathfrak{F}$ -sn  $G/N$ . Hence  $HN \mathfrak{F}$ -sn  $G$  and therefore  $H \mathfrak{F}$ -sn  $G$  by Lemma 1, a contradiction. Thus  $HN = G$ . Hence  $G/N \simeq H/(H \cap N) \in \mathfrak{X}$ , the final contradiction. It means that  $f_{\mathfrak{X}}(\mathfrak{F}) = \mathfrak{X}_{sn_{\mathfrak{F}}}^S$ .

**3.2. Proof of Theorem 2.** According to [11, Theorem 2.14]  $\mathfrak{X}_{sn_{\mathfrak{F}}}$  is a formation. Assume that  $\mathfrak{X}_{sn_{\mathfrak{F}}}$  is nonsaturated. Let  $G$  be a minimal order group from  $E_{\Phi}\mathfrak{X}_{sn_{\mathfrak{F}}} \setminus \mathfrak{X}_{sn_{\mathfrak{F}}}$ . Recall [12, A, Theorem 9.2(e)] that  $\Phi(G)N/N \leq \Phi(G/N)$  for any normal subgroup  $N$  of  $G$ . It means that  $(G/N)/\Phi(G/N)$  is a quotient group of  $G/\Phi(G) \in \mathfrak{X}_{sn_{\mathfrak{F}}}$ . Hence  $(G/N)/\Phi(G/N) \in \mathfrak{X}_{sn_{\mathfrak{F}}}$ . If  $N$  is non-trivial, then from  $|G/N| < |G|$  and our assumption it follows that  $G/N \in \mathfrak{X}_{sn_{\mathfrak{F}}}$ .

If  $N_1$  and  $N_2$  are different minimal normal subgroups of  $G$ , then from  $G/N_1, G/N_2 \in \mathfrak{X}_{sn_{\mathfrak{F}}}$  it follows that  $G \simeq G/(N_1 \cap N_2) \in \mathfrak{X}_{sn_{\mathfrak{F}}}$ , a contradiction. Thus  $G$  has the unique minimal normal subgroup  $N$ . It is clear that  $\Phi(G) \neq 1$ . Hence  $N \leq \Phi(G)$ . It means that  $N$  is a  $p$ -group for some prime  $p$ . Therefore  $\Phi(G)$  is a  $p$ -group and  $O_{p'}(G) = 1$ .

Let  $H$  be an  $\mathfrak{X}$ -projector of  $G$ . Note that  $H\Phi(G)/\Phi(G)$  is an  $\mathfrak{X}$ -projector of  $G/\Phi(G)$ . From  $G/\Phi(G) \in \mathfrak{X}_{sn_{\mathfrak{F}}}$  it follows that  $H\Phi(G)/\Phi(G) \mathfrak{F}$ -sn  $G/\Phi(G)$ . From (2) of Lemma 1 it follows that  $H\Phi(G) \mathfrak{F}$ -sn  $G$ .

Let  $K = H\tilde{F}(G)$ . Since  $\tilde{F}(G)/\Phi(G)$  is a direct product of simple groups by definition, it is a normal quasinilpotent subgroup of  $K/\Phi(G)$ . Therefore  $\tilde{F}(G)/\Phi(G) \leq F^*(K/\Phi(G))$ . Hence  $\tilde{F}(G)/\Phi(G) \leq \tilde{F}(K/\Phi(G))$  by [3, Lemma 1.1(4)]. Now  $K/\Phi(G) = (H\Phi(G)/\Phi(G))\tilde{F}(K/\Phi(G))$ . Since  $\mathfrak{F}$  is hereditary and  $H\Phi(G)/\Phi(G) \mathfrak{F}$ -sn  $G/\Phi(G)$ , we see that  $H\Phi(G)/\Phi(G) \mathfrak{F}$ -sn  $K/\Phi(G)$  by Lemma 2. From  $\mathfrak{X} \subseteq \mathfrak{F}$  it follows that  $H\Phi(G)/\Phi(G) \in \mathfrak{F}$ . Therefore  $K/\Phi(G) \in \mathfrak{F}$  by Lemma 3. It means that  $K$  acts  $F$ -hypercentrally on  $\tilde{F}(G)/\Phi(G)$ . From  $O_{p'}(G) = 1$  and [16, Theorem 9.18] it follows that  $K$  acts  $F$ -hypercentrally on  $\Phi(G)$ . Thus  $K \in \mathfrak{F}$ . Since  $\mathfrak{F}$  is hereditary,  $H\Phi(G) \in \mathfrak{F}$  and therefore  $H \mathfrak{F}$ -sn  $H\Phi(G)$ . Thus  $H \mathfrak{F}$ -sn  $G$  by Lemma 1. It means that  $G \in \mathfrak{X}_{sn_{\mathfrak{F}}}$ , the contradiction. Thus  $\mathfrak{X}_{sn_{\mathfrak{F}}}$  is a saturated formation.

According to [12, IV, Proposition 1.2] the class of groups all whose  $\mathfrak{X}$ -projectors are  $F(p)$ -subgroups is a formation. Therefore  $X(p)$  is the intersection of formations and hence a formation.

Let  $\mathfrak{H} = LF(X)$ . Assume that  $\mathfrak{H} \setminus \mathfrak{X}_{sn_{\mathfrak{F}}} \neq \emptyset$  and  $G$  is a minimal order group from  $\mathfrak{H} \setminus \mathfrak{X}_{sn_{\mathfrak{F}}}$ . Since  $\mathfrak{H}$  and  $\mathfrak{X}_{sn_{\mathfrak{F}}}$  are saturated formations, we see that  $\Phi(G) = 1$  and  $G$  has the unique minimal normal subgroup  $N$ . If  $N$  is non-abelian, then  $C_G(N) = 1$  and for every  $p \in \pi(N)$  holds  $G \simeq G/C_G(N) \in X(p) \subseteq \mathfrak{X}_{sn_{\mathfrak{F}}}$ , a contradiction. Thus  $N$  is abelian. Hence  $C_G(N) = N$  and  $N$  is a  $p$ -group for some prime  $p$ . Let  $H$  be an  $\mathfrak{X}$ -projector of  $G$ . Note that  $HN/N$  is an  $\mathfrak{X}$ -projector of  $G/N$ . It means that  $HN/N \mathfrak{F}$ -sn  $G/N$ . Hence  $HN \mathfrak{F}$ -sn  $G$ . From  $G/N = G/C_G(N) \in X(p)$  it follows that  $HN/N \in F(p)$ . Now  $HN \in \mathfrak{N}_p F(p) = F(p) \subseteq \mathfrak{F}$ . Since  $\mathfrak{F}$  is hereditary, we see that  $H \mathfrak{F}$ -sn  $HN$ . Therefore  $H \mathfrak{F}$ -sn  $G$ . It means that  $G \in \mathfrak{X}_{sn_{\mathfrak{F}}}$ , a contradiction. Thus  $\mathfrak{H} \subseteq \mathfrak{X}_{sn_{\mathfrak{F}}}$ .

Assume that  $\mathfrak{X}_{sn_{\mathfrak{F}}} \setminus \mathfrak{H} \neq \emptyset$ . Let  $G$  be a minimal order group from  $\mathfrak{X}_{sn_{\mathfrak{F}}} \setminus \mathfrak{H}$ . Since  $\mathfrak{H}$  and  $\mathfrak{X}_{sn_{\mathfrak{F}}}$  are saturated formations, we see that  $\Phi(G) = 1$  and  $G$  has the unique minimal normal subgroup  $N$ . Let  $H$  be an  $\mathfrak{X}$ -projector of  $G$ . From  $G \in \mathfrak{X}_{sn_{\mathfrak{F}}}$  it follows that  $H \mathfrak{F}$ -sn  $G$ . Since  $\mathfrak{F}$  is hereditary, we see that  $H \mathfrak{F}$ -sn  $HN$ . Now from  $\mathfrak{X} \subseteq \mathfrak{F}$  and  $N \leq \tilde{F}(HN)$  it follows that  $HN \in \mathfrak{F}$  by Lemma 3. Let  $p \in \pi(N)$ . Since  $N$  is the unique minimal normal subgroup of  $G$ , we see that  $C_G(N) \subseteq N$ . Hence  $O_{p'}(HN) = 1$ . Now  $HN \simeq HN/O_{p'}(HN) \in F(p)$ . Since  $\mathfrak{F}$  is hereditary,  $F(p)$  is hereditary for every  $p$  too by [12, IV, Proposition 3.16]. Thus  $H \in F(p)$ . It means that  $G$  belongs to a formation  $X(p)$  for every  $p \in \pi(N)$ . Therefore  $G/C_G(N) \in X(p)$  for every  $p \in \pi(N)$ . From  $G/N \in LF(X)$  it follows that  $G \in LF(X) = \mathfrak{H}$ , a contradiction. Hence  $\mathfrak{H} \subseteq \mathfrak{X}_{sn_{\mathfrak{F}}}$ . Thus  $\mathfrak{H} = \mathfrak{X}_{sn_{\mathfrak{F}}}$ .

Note that  $X(p) \subseteq \mathfrak{X}_{sn_{\mathfrak{F}}}$  by the definition of  $X$ . Hence  $X$  is an inner definition of  $\mathfrak{X}_{sn_{\mathfrak{F}}}$ . Let prove that  $X$  is a full definition of  $\mathfrak{X}_{sn_{\mathfrak{F}}}$ . Note that if  $F(p) = \emptyset$ , then  $X(p) = \emptyset$ . That is why we may assume that  $F(p) \neq \emptyset$ . Assume that  $G \in \mathfrak{N}_p X(p)$ . It means that  $G$  has a normal  $p$ -subgroup  $N$  with  $G/N \in X(p)$ . Note that  $G \in \mathfrak{X}_{sn_{\mathfrak{F}}}$  by [12, IV, Proposition 3.8(a)]. Let  $H$  be an  $\mathfrak{X}$ -projector of  $G$ . Then  $HN/N \simeq H/(H \cap N)$  is an  $\mathfrak{X}$ -projector of  $G/N$  and  $H \cap N$  is a  $p$ -group. From  $G/N \in X(p)$  it follows that  $H/(H \cap N) \in F(p)$ .

Now  $H \in \mathfrak{N}_p F(p) = F(p)$ . Thus  $G \in X(p)$ . So  $X$  is a full definition of  $\mathfrak{X}_{sn_{\mathfrak{F}}}$ . Thus  $X$  is the canonical local definition of  $\mathfrak{X}_{sn_{\mathfrak{F}}}$ .

**3.2.1. Proof of Remark 1.** Assume that  $\mathfrak{X} \supseteq \mathfrak{F}$  are saturated formations and  $G \in \mathfrak{X}_{sn_{\mathfrak{F}}}$ . Let  $H$  be an  $\mathfrak{X}$ -projector of  $G$ . Then  $H \mathfrak{F}$ -sn  $G$ . If  $H \neq G$ , then  $HG^{\mathfrak{F}} < G$  by the definition of  $\mathfrak{F}$ -subnormality. Hence  $HG^{\mathfrak{F}}/G^{\mathfrak{F}} < G/G^{\mathfrak{F}} \in \mathfrak{F} \subseteq \mathfrak{X}$ , a contradiction with the definition of  $\mathfrak{X}$ -projector. Thus  $H = G \in \mathfrak{X}$ . So  $\mathfrak{X}_{sn_{\mathfrak{F}}} \subseteq \mathfrak{X}$ . From the other hand an  $\mathfrak{X}$ -group is the unique  $\mathfrak{X}$ -projector in itself and hence  $\mathfrak{F}$ -subnormal. It means that  $\mathfrak{X} \subseteq \mathfrak{X}_{sn_{\mathfrak{F}}}$ . Thus  $\mathfrak{X} = \mathfrak{X}_{sn_{\mathfrak{F}}}$ .

**3.2.2. Proof of Corollary 1.** Note that the unit group is the unique (1)-projector and all Sylow  $p$ -subgroups are all  $\mathfrak{N}_p$ -projectors in every group. It means that  $\mathfrak{F}_0 = (1)_{sn_{\mathfrak{F}}}$  and  $\mathfrak{F}_p = (\mathfrak{N}_p)_{sn_{\mathfrak{F}}}$  are saturated formations by Theorem 2. Then  $W_{\pi}\mathfrak{F} = \bigcap_{i \in \pi \cup \{0\}} \mathfrak{F}_i$  is a saturated formation. The direct check or [4, Theorem 3.1(5)] can be used to prove that  $W_{\pi}\mathfrak{F}$  is hereditary. Let  $F_0(p) = \mathfrak{F}_0$  and  $F_q(p)$  be the class of all  $\mathfrak{F}_q$ -groups all whose Sylow  $q$ -subgroups belong to  $F(p)$  for all  $p \in \pi(\mathfrak{F})$  and  $F_0(p) = F_q(p) = \emptyset$  otherwise. Then  $F_0$  and  $F_q$  are the canonical local definitions of  $\mathfrak{F}_0$  and  $\mathfrak{F}_q$  respectively by Theorem 2. Let  $H(p) = \bigcap_{i \in \pi \cup \{0\}} F_i(p) \subseteq W_{\pi}\mathfrak{F}$  be an inner local definition of  $W_{\pi}\mathfrak{F}$ . From  $\mathfrak{N}_p F_i(p) = F_i(p)$  for every  $i \in \pi \cup \{0\}$  it follows that  $\mathfrak{N}_p H(p) = H(p)$ . Thus  $H$  is the canonical local definition of  $W_{\pi}\mathfrak{F}$ .

**3.3. Proof of Theorem 3.** From  $\emptyset \neq \mathfrak{X} \subseteq \mathfrak{F} \subseteq \mathfrak{S}$  it follows that 1  $\mathfrak{S}$ -sn  $G$  for every  $\mathfrak{X}_{sn_{\mathfrak{F}}}$ -group  $G$ . Thus  $\mathfrak{X}_{sn_{\mathfrak{F}}}$  consists of solvable groups by [3, Lemma 2.6]. Let  $X(p)$  be the class of all solvable groups all whose  $\mathfrak{X}$ -projectors are  $F(p)$ -groups. From [12, IV, Proposition 1.2] it follows that  $X(p)$  is a formation. Let  $\mathfrak{H} = LF(X)$ . Then  $\mathfrak{H}$  is a saturated formation of solvable groups.

Assume that  $\mathfrak{H} \setminus \mathfrak{X}_{sn_{\mathfrak{F}}} \neq \emptyset$ . Let  $G$  be a minimal order group from  $\mathfrak{H} \setminus \mathfrak{X}_{sn_{\mathfrak{F}}}$ . Since  $\mathfrak{H}$  and  $\mathfrak{X}_{sn_{\mathfrak{F}}}$  are saturated formations of solvable groups, we see that  $\Phi(G) = 1$ ,  $G$  is solvable and has the unique minimal normal subgroup  $N$ . So  $N$  is abelian. Hence  $C_G(N) = N$  and  $N$  is a  $p$ -group for some prime  $p$ . Let  $H$  be an  $\mathfrak{X}$ -projector of  $G$ . Note that  $HN/N$  is an  $\mathfrak{X}$ -projector of  $G/N$ . It means that  $HN/N \mathfrak{F}$ -sn  $G/N$ . Hence  $HN \mathfrak{F}$ -sn  $G$  by Lemma 1. From  $G/N = G/C_G(N) \in X(p)$  it follows that  $HN/N \in F(p)$ . Now  $HN \in \mathfrak{N}_p F(p) = F(p) \subseteq \mathfrak{F}$ . Since  $\mathfrak{F}$  is hereditary, we see that  $H \mathfrak{F}$ -sn  $HN$  by Lemma 2. Therefore  $H \mathfrak{F}$ -sn  $G$ . It means that  $G \in \mathfrak{X}_{sn_{\mathfrak{F}}}$ , a contradiction. Thus  $\mathfrak{H} \subseteq \mathfrak{X}_{sn_{\mathfrak{F}}}$ .

Assume that  $\mathfrak{X}_{sn_{\mathfrak{F}}} \setminus \mathfrak{H} \neq \emptyset$ . Let  $G$  be a minimal order group from  $\mathfrak{X}_{sn_{\mathfrak{F}}} \setminus \mathfrak{H}$ . Since  $\mathfrak{H}$  and  $\mathfrak{X}_{sn_{\mathfrak{F}}}$  are saturated formations of solvable groups, we see that  $\Phi(G) = 1$ ,  $G$  is solvable and has the unique minimal normal subgroup  $N$ . Now  $N$  is abelian  $p$ -group for some  $p \in \pi(G)$  and  $C_G(N) = N$ . Let  $H$  be an  $\mathfrak{X}$ -projector of  $G$ . From  $G \in \mathfrak{X}_{sn_{\mathfrak{F}}}$  it follows that  $H \mathfrak{F}$ -sn  $G$ . Since  $\mathfrak{F}$  is hereditary, we see that  $H \mathfrak{F}$ -sn  $HN$  by Lemma 2. Now from  $\mathfrak{X} \subseteq \mathfrak{F}$  and  $N \leq F(HN) = \tilde{F}(HN)$  it follows that  $HN \in \mathfrak{F}$  by Lemma 3. Note that  $O_{p'}(HN) = 1$ . Now  $HN \simeq HN/O_{p'}(HN) \in F(p)$ . Since  $\mathfrak{F}$  is hereditary,

$F(p)$  is hereditary for every  $p$  too by [12, IV, Proposition 3.16]. Thus  $H \in F(p)$ . It means that  $G$  belongs to a formation  $X(p)$ . Therefore  $G/C_G(N) \in X(p)$ . From  $G/N \in LF(X)$  it follows that  $G \in LF(X) = \mathfrak{H}$ , a contradiction. Hence  $\mathfrak{H} \subseteq \mathfrak{X}_{sn_{\mathfrak{F}}}$ . Thus  $\mathfrak{H} = \mathfrak{X}_{sn_{\mathfrak{F}}}$ .

**3.4. Proof of Theorem 4.** Assume that

$$D = \langle g(H, G, \mathfrak{F})^{\mathfrak{X}} \mid H \text{ is an } \mathfrak{X}\text{-projector of } G \text{ with } g(H, G, \mathfrak{F}) \neq H \rangle.$$

Note that  $(S^{\mathfrak{F}})^x = (S^x)^{\mathfrak{F}}$  for any  $x \in G$  and  $S \leq G$ . Therefore  $g(H^x, G, \mathfrak{F}) = g(H, G, \mathfrak{F})^x$ . Thus  $D$  is a normal subgroup of  $G$ .

Let  $H$  be an  $\mathfrak{X}$ -projector of  $G$  with  $K = g(H, G, \mathfrak{F}) \neq H$ . Note that  $HK^{\mathfrak{F}} = K$  and  $K \mathfrak{F}$ -sn  $G$ . Assume  $N$  is a normal subgroup of  $G$  such that  $HN/N \mathfrak{F}$ -sn  $G/N$ . Since  $\mathfrak{F}$  is hereditary,  $HN/N \mathfrak{F}$ -sn  $KN/N$ . Note that  $K^{\mathfrak{F}} \subseteq (KN)^{\mathfrak{F}}$  and  $(KN/N)^{\mathfrak{F}} = (KN)^{\mathfrak{F}}N/N$ . If  $HN/N < KN/N$ , then by the definition of  $\mathfrak{F}$ -subnormality

$$KN/N = HK^{\mathfrak{F}}N/N \subseteq (HN/N)(KN/N)^{\mathfrak{F}} < KN/N,$$

a contradiction. Thus  $HN/N = KN/N \simeq K/(K \cap N) \in \mathfrak{X}$ . Hence  $K^{\mathfrak{X}} \leq K \cap N \leq N$ . It means that  $D \leq G^{\mathfrak{X}_{sn_{\mathfrak{F}}}}$ .

Assume that  $D < G^{\mathfrak{X}_{sn_{\mathfrak{F}}}}$ . It means that there is an  $\mathfrak{X}$ -projector  $S/D$  which is not  $\mathfrak{F}$ -subnormal in  $G/D$ . Note that [12, III, Proposition 3.7] there is an  $\mathfrak{X}$ -projector  $H$  of  $G$  with  $HD/D = S/D$ . Let  $K = g(H, G, \mathfrak{F})$ . If  $K = H$ , then  $H \mathfrak{F}$ -sn  $G$  and hence  $S/D = HD/D \mathfrak{F}$ -sn  $G/D$ , a contradiction. Hence  $H < K$ . From the definition of  $D$  it follows that  $K^{\mathfrak{X}} \leq D$ . By the assumption of the theorem  $H$  is an  $\mathfrak{X}$ -covering subgroup of  $G$ . Hence  $H$  is an  $\mathfrak{X}$ -projector of  $K$ . It means that  $HK^{\mathfrak{X}} = K$ . Hence  $HD/D = KD/D \mathfrak{F}$ -sn  $G/D$ , the final contradiction. Thus  $D = G^{\mathfrak{X}_{sn_{\mathfrak{F}}}}$ .

## 4 The computation of the $\mathfrak{X}_{sn_{\mathfrak{F}}}$ -residual of a solvable group

Recall [17] that a subgroup  $H$  of a group  $G$  is called  $\mathbb{P}$ -subnormal if  $H = G$  or there is a chain of subgroups  $H = H_0 < H_1 < \dots < H_n = G$  such that  $|H_i : H_{i-1}|$  is a prime for  $1 \leq i \leq n$ . Groups with various systems of  $\mathbb{P}$ -subnormal subgroups were studied in [10, 17, 18]. In the universe of all solvable groups the concepts of  $\mathbb{P}$ -subnormal and  $\mathfrak{U}$ -subnormal subgroups coincide where  $\mathfrak{U}$  denotes the class of all solvable groups. Recall that  $\mathfrak{N}$ -covering subgroup (of a solvable group) is called a Carter subgroup. According to Theorem 2 the class  $\mathfrak{N}_{sn_{\mathfrak{U}}}$  of all solvable groups all whose Carter subgroups are  $\mathbb{P}$ -subnormal is a saturated formation. Note that the symmetric group of degree 4 belongs to this class but the alternating group of degree 4 does not belong to it. Therefore  $\mathfrak{N}_{sn_{\mathfrak{U}}}$  is not hereditary.

Let  $\mathcal{A}$  denotes the class of all solvable groups all whose Sylow subgroups are abelian. The dihedral group of order 8 shows that  $\mathcal{A}$  is a non-saturated formation. So  $\mathfrak{U}_{sn_{\mathcal{A}}}$  is a formation but we can not tell either it is saturated or not.

If not stated the opposite for the rest of the section all *groups considered in this section are solvable*. It is known [12, III, Theorem 3.21] that in the universe of all solvable groups for a saturated formation  $\mathfrak{F}$  in every group there is a single conjugacy class of  $\mathfrak{F}$ -projectors which coincides with the set of all  $\mathfrak{F}$ -covering subgroups.

Let  $\mathfrak{X} \subseteq \mathfrak{F}$  be saturated formations of solvable groups,  $\mathfrak{F}$  be hereditary and  $F$  be the canonical local definition of  $\mathfrak{F}$ . If we are able to compute  $\mathfrak{X}$ -projectors and  $F(p)$ -residuals, then we are able to compute  $X(p)$ -residuals of a group where  $X$  is the local definition of  $\mathfrak{X}_{sn\mathfrak{F}}$  that is described in Theorem 3:

$$G^{X(p)} = (H^{F(p)})^G, H \text{ is an } \mathfrak{X}\text{-projector of } G \quad (1)$$

From Theorem 4 the algorithm for the computation of the  $\mathfrak{X}_{sn\mathfrak{F}}$ -residual of a group for a saturated formation  $\mathfrak{X}$  and a hereditary formation  $\mathfrak{F}$  follows.

---

**Algorithm 1:** ResidualXsnF( $G, \mathfrak{X}, \mathfrak{F}$ )

---

**Result:**  $G^{\mathfrak{X}_{sn\mathfrak{F}}}$ .

**Data:**  $G$  is a solvable group, functions to compute the  $\mathfrak{X}$ -residual, the  $\mathfrak{F}$ -residual and an  $\mathfrak{X}$ -projector.

$G_0 \leftarrow$  generators of an  $\mathfrak{X}$ -projector of  $G$ ;

$G_1 \leftarrow$  generators of  $G$ ;

$G_2 \leftarrow$  generators of  $G^{\mathfrak{F}}$ ;

$L \leftarrow G_0 \cup G_2$ ;

**while**  $|\langle L \rangle| \neq |\langle G_1 \rangle|$  **do**

$G_1 \leftarrow L$ ;

$G_2 \leftarrow$  generators of  $\langle L \rangle^{\mathfrak{F}}$ ;

$L \leftarrow G_0 \cup G_2$ ;

**end**

**if**  $|\langle L \rangle| \neq |\langle G_0 \rangle|$  **then**

**return**  $(\langle L \rangle^{\mathfrak{X}})^G$ ;

**else**

**return** 1;

**end**

---

In the computer system algebra GAP [19] there is two packages which deal with formations, their residuals and projectors: FORMAT [20] and CRISP [21]. We implemented the described above methods to compute the  $\mathfrak{X}_{sn\mathfrak{F}}$ -residual of a group in GAP:

- Method 1 uses FORMAT and Formula (1).
- Method 2 uses FORMAT and Algorithm 1.
- Method 3 uses CRISP and Formula (1).
- Method 4 uses CRISP and Algorithm 1.

We use these 4 methods to compute the  $\mathfrak{N}_{sn\mathfrak{U}}$ -residual of a group. In the next table all groups of orders 1296, 1200 and 1120 are taken from the package SmallGroups while The Dark's group, the maximal subgroup  $MF_{22}$  of Fischer's group and the wreath product  $U$  of the subgroup of upper

triangular matrices in  $GL(4, 7)$  with the cyclic group of order 3 are taken from FORMAT. The timings (in seconds) were obtained using GAP 4.11.0 started with 4 GB of RAM on Intel(R) Core(TM) i5-8250U CPU 1.60GHz 1.80GHz.

$\mathfrak{N}_{sn_{\mathcal{U}}}$ -residual				
Group(s)	Method 1	Method 2	Method 3	Method 4
All 3609 groups of order $2^4 \cdot 3^4$	17.61	20.83	84.89	53.52
All 1201 solvable groups of order $2^4 \cdot 3 \cdot 5^2$	5.42	5.85	21.11	13.56
All 1092 groups of order $2^5 \cdot 5 \cdot 7$	5.10	6.32	22.20	12.88
The Dark's group of order $2^3 \cdot 3^9 \cdot 5^{24} \cdot 7 \cdot 31^8$	0.19	0.23	0.94	0.63
$MF_{22}$ of order $2^8 \cdot 3^9$	0.03	0.02	0.06	0.06
$U$ of order $2^{12} \cdot 3^{13} \cdot 7^{18}$	0.05	0.03	0.28	0.13

The methods based on Formula (1) (i.e. Methods 1 and 3) are not applicable for the computation of the  $\mathfrak{U}_{sn_{\mathcal{A}}}$ -residual. So we can use only Algorithm 1 and based on it methods in this case.

$\mathfrak{U}_{sn_{\mathcal{A}}}$ -residual		
Group(s)	Method 2	Method 4
All 3609 groups of order $2^4 \cdot 3^4$	18.89	33.22
All 1201 solvable groups of order $2^4 \cdot 3 \cdot 5^2$	5.11	7.75
All 1092 groups of order $2^5 \cdot 5 \cdot 7$	5.03	7.52
The Dark's group of order $2^3 \cdot 3^9 \cdot 5^{24} \cdot 7 \cdot 31^8$	0.31	0.30
$MF_{22}$ of order $2^8 \cdot 3^9$	0.02	0.05
$U$ of order $2^{12} \cdot 3^{13} \cdot 7^{18}$	0.06	0.27

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