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СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ
Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru
ISSN 1813-3304

Том 20, № 2, стр. 144–145 (2023) УДК 517.958
https://doi.org/10.33048/semi.2023.20.??? MSC 35L05

DIRICHLET PROBLEM FOR THE WAVE EQUATION

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Abstract: The paper considers the Dirichlet problem for the wave equation on a finite time interval. The difficulty in solving the problem lies in the fact that under natural conditions the solution to this problem may not exist, which complicates the problem formulation.

The paper solves this problem by comparing the Dirichlet problem with the Cauchy problem. Under certain boundary conditions, it is possible to construct a solution to the Dirichlet problem by reducing it to the Cauchy problem for the wave equation, thus proving the existence of a solution. Due to the uniqueness and stability of the solution to the Cauchy problem, it is possible to prove the uniqueness of the solution to the Dirichlet problem under rather strong constraints on the functions in the boundary conditions.

Keywords: Dirichlet problem, finite time interval, wave equation, conditionally ill-posed problem, Cauchy problem, non-equivalence of the Cauchy problem and the Dirichlet problem.

Introduction

The concept of a colloid as a set of layers is common in colloid chemistry [1], [2]. The layers will have one or another electrical moment in the case of

Sukharev, Yu.I., Tanana V.P., Markov B.A. On the solution of the Dirichlet problem for the wave equation.

Поступила 22 октября 2024 г., опубликована 31 декабря 2024 г.

rare metal oxyhydrates, since they are composed of electrically polar long molecules [3].

It has been experimentally discovered that between electrodes placed in a colloidal substance and connected by a measuring device, a certain potential difference arises, and the device records a spontaneous electric current [4]. The diagram of the current's dependence on time appears chaotic, but the Fourier transform over time for a finite time interval (over the measurement interval) shows with high reliability the presence of harmonic oscillations of different frequencies. In this case, the frequencies are discrete and increase linearly with the number of frequences.

It seems likely that the detected oscillations are a result of oscillations of long molecules with electric moments due to the layered structure of the colloidal substance, and the measuring device records these oscillations. This idea is supported by the presence of long rod-shaped fragments in the rare metal colloid, which can be seen at x1000 magnification.

At best, the experimenter has access to the initial and final location of the fragment, since all measurements are made over a finite time interval. At the same time, the change in the shape of a long molecule is of considerable importance, since it can be used to infer the chemical composition of the substance, its changes over time, the homogeneities and heterogeneities of the colloid, the fluxes of substances filtered by the fragments, the sorption properties of the gel, and much more.

Oscillations of structural units on a finite time interval with conditions at the initial and final moments are determined by the solution of the Dirichlet boundary value problem, which is ill-posed [5], [6]. Therefore, a complexity arises from the fact that the solution to such a problem may not exist, may be non-unique or may unstable with respect to the initial data.

1 Mathematical statement of the problem

Let the Dirichlet problem for the colloidal substance be given by linear fragment oscillation time on a unit interval of time.

$$\begin{cases} \dfrac{\partial^2 u(x,t)}{\partial t^2} = \dfrac{\partial^2 u(x,t)}{\partial x^2}, \quad x \in (0;\pi), \quad t \in (0;1), \\ u(0,t) = 0, \quad t \in [0;1], \quad u(1,t) = 0, \quad t \in [0;1]; \\ u(x,0) = f(x), \quad x \in [0;\pi], \quad u(x,1) = h(x), \quad x \in [0;\pi], \end{cases} \tag{1}$$

where $u(x,t)$ is the deviation of the linear fragment rod from the equilibrium position.

We are looking for a solution to the problem (1).

Definition 1. *A solution to the boundary value problem (1)* [7]*, p. 254 is function $u(x,t)$ such that:*
1. $u(x,t)$ is continuous with first derivatives on set $t \in [0;1], x \in [0;\pi]$,
2. $u(x,t)$ has continuous second order derivatives on set $t \in (0;1), x \in (0;\pi)$,

3. $u(x,t)$ satisfies the differential equation (1) on set $t \in (0;1), x \in (0;\pi)$,
4. $u(x,t)$ satisfies all the boundary conditions of the problem (1).

The complexity of the problem (1) is that its solution for certain lengths of the time interval is not unique [5] or may not exist at all (we choose the case corresponding to value $\alpha = 1$ in the notation of [5]). Moreover, the problem is unstable with respect to the initial data [5]. Therefore, we construct and consider an auxiliary Cauchy problem, whose classical solution is well known (for example, [7]). It is necessary to answer the question of whether the solution to the auxiliary problem also coincides with the solution to the problem (1).

Let us consider the question of eigenfunctions, since we construct a solution by expanding it in terms of the eigenfunctions of the corresponding Sturm-Liouville problem.

We also need to consider the auxiliary Cauchy problem (the direct problem), since we use it later.

$$\begin{cases} \dfrac{\partial^2 u(x,t)}{\partial t^2} = \dfrac{\partial^2 u(x,t)}{\partial x^2}, \quad x \in (0;\pi), \quad t \in (0;+\infty), \\ u(0,t) = 0, \quad t \in [0;+\infty), \quad u(1,t) = 0, \quad t \in [0;+\infty); \\ u(x,0) = f(x), \quad x \in [0;\pi], \quad \dfrac{\partial u(x,0)}{\partial t} = g(x), \quad x \in [0;\pi], \end{cases} \tag{2}$$

We do not specify the spaces in which we construct the solution, because their clarification is preceded by the properties of the constructed solution.

1.1. Sturm-Liouville problem. Consider an operator $A : H \to H$, where $H \equiv L_2[0;\pi]$, such that $D(A) = C_0^2[0;\pi]$, and is defined by

$$\begin{cases} A(U(x)) \equiv \dfrac{d^2 U(x)}{dx^2}, \\ U(0) = 0, \quad U(\pi) = 0. \end{cases} \tag{3}$$

Let us construct the closure of operator A in the space $L_2[0;\pi]$ by introducing the scalar product.

Definition 2. *The scalar product in space $L_2[0;\pi]$ is integral*

$$(W(x), U(x)) = \int_0^\pi U(x)W(x)dx,$$

where $W(x), U(x) \in L_2[0;\pi]$.

Operator A is symmetric on its domain. The symmetry of the operator implies its closure in $L_2[0;\pi]$. Since the operator is symmetric on $D(A)$, then, according to [8], p. 354

Definition 3. *The closure of Operator A is Operator $\bar{A}$ such that $D(A) \subset D(\bar{A})$, and the graph of $\Gamma(\bar{A})$ is closed in the Cartesian product $L_2[0;\pi] \times L_2[0;\pi]$.*

The solution to (1) has the form of a series in eigenfunctions of the Sturm-Liouville problem

$$\begin{cases} \dfrac{d^2U(x)}{dx^2} = \Lambda U(x), x \in (0;\pi), \\ U(0) = 0, U(\pi) = 0. \end{cases} \tag{4}$$

Lemma 1. *The eigenvalues of the Sturm-Liouville problem (4) are* $\Lambda = -n^2$. *The system of eigenfunctions corresponding to these numbers* $\{\sin(nx)\}_{n=1}^{\infty}$ *is complete and orthogonal in the space* $L_2[0;\pi]$.

Proof: The proof is given in, for example, [9]. □

1.2. Construction of the solution to the problem (2) and some of its properties. Let us consider an auxiliary Cauchy problem to construct a solution to problem (1). The auxiliary problem is comparable with the Dirichlet problem (1)

$$\begin{cases} \dfrac{\partial^2 U(x,t)}{\partial t^2} = \dfrac{\partial^2 U(x,t)}{\partial x^2}, \quad x \in (0;\pi), \quad t \in (0;+\infty), \\ U(0,t) = 0, \quad t \in [0;+\infty), \quad U(1,t) = 0, \quad t \in [0;+\infty), \\ U(x,0) = f(x), \quad x \in [0;\pi], \quad \dfrac{\partial U(x,0)}{\partial t} = g(x), \quad x \in [0;\pi], \\ f(x) \in H_0^4[0;\pi], \quad g(x) \in H_0^3[0;\pi]. \end{cases} \tag{5}$$

Note that such a problem is considered in [7], p. 259.

The following statement is true.

Lemma 2. *A solution to the problem (5) exists* $\forall x \in [0;\pi], \forall t \in [0;+\infty)$, *it is unique, continuous in the initial data and can be represented as a series*

$$\begin{aligned} U(x,t) &= \sum_{n=1}^{\infty} \left\{ f_n \cos(nt) + \frac{g_n}{n} \sin(nt) \right\} \sin(nx), \\ f_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx, \quad g_n = \frac{2}{\pi} \int_0^{\pi} g(x) \sin(nx) dx, \end{aligned} \tag{6}$$

if $f(x) \in H_0^4[0;\pi]$, $g(x) \in H_0^3[0;\pi]$.

The proof is given in [7], pp. 259-260.

The following lemma is true.

Lemma 3. *Let a solution to problem (5) exist, i.e.* $f(x) \in H_0^4[0;\pi]$, $g(x) \in H_0^3[0;\pi]$. *Then* $\dfrac{\partial^4 U(x,t)}{\partial x^3 \partial t} \in L_2[0;\pi]$ $\forall t \in [0;+\infty)$, $\dfrac{\partial U(x,t)}{\partial t} \in H_0^3[0;\pi]$ $\forall t \in [0;+\infty)$.

Proof: We differentiate series (6), since solution to (5) is defined by formula (6)

$$\frac{\partial^4 U(x,t)}{\partial x^3 \partial t} = \sum_{n=1}^{\infty} n^4 \left\{ -f_n \sin(nt) + \frac{g_n}{n} \cos(nt) \right\} \cos(nx). \tag{7}$$

Since $f(x) \in H_0^4[0;\pi]$, $g(x) \in H_0^3[0;\pi]$, then

$$f_n = \frac{F_n}{n^4}, g_n = \frac{G_n}{n^3}, \exists C_1 > 0, \sum_{n=1}^{\infty} F_n^2 \le C_1, \sum_{n=1}^{\infty} G_n^2 \le C_1, \tag{8}$$

where

$$F_n = \frac{2}{\pi} \int_0^{\pi} f^{(4)}(x)\cos(nx)dx, G_n = -\frac{2}{\pi}\int_0^{\pi} g'''(x)\sin(nx)dx.$$

The following equality is true

$$\frac{\partial^4 U(x,t)}{\partial x^3 \partial t} = \sum_{n=1}^{\infty} \{F_n \sin(nt) - G_n \cos(nt)\}\sin(nx).$$

We obtain, by composing the scalar product in space $L_2[0;\pi]$ and considering the norm in this space $||s(x)||^2_{L_2[0;\pi]} \equiv (s(x); s(x))$

$$\left| \left(\frac{\partial^4 U(x,t)}{\partial x^3 \partial t}, \frac{\partial^4 U(x,t)}{\partial x^3 \partial t} \right) \right| =$$

$$= \sum_{n=1}^{\infty} \left\{F_n^2 \sin^2(nt) + F_n G_n \sin(2nt) + G_n^2 \cos^2(nt)\right\} \le 2\sum_{n=1}^{\infty}\left\{F_n^2 + G_n^2\right\}.$$

This series converges absolutely by (8). Therefore, $\dfrac{\partial^4 U(x,t)}{\partial x^3 \partial t} \in L_2[0;\pi] \forall t \in [0;+\infty)$.

The assertion of the lemma follows from this. □

1.3. Solution to the problem (1). Let us now consider problem (1) and the series

$$u(x,t) = \sum_{n=1}^{\infty} \left\{ f_n \cos(nt) + \frac{h_n - f_n \cos n}{\sin n} \sin(nt) \right\} \sin(nx), \tag{9}$$

where f_n are given by formula (6), $h_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} h(x)\sin(nx)dx$.

Note that series (9) upon formal substitution into all relations (1) turns them into a true equality.

We have not yet defined the classes of functions $f(x), h(x)$ to solve problem (1). We turn to the solution to problem (5): $f(x) \in H_0^4[0;\pi]$, $g(x) \in H_0^3[0;\pi]$ to determine them. However, series (9) does not contain the components of the expansion of function $g(x)$ into a Fourier series. We compare series (9) and series (6) in order to determine components g_n

$$\sum_{n=1}^{\infty}\left\{f_n\cos(nt)+\frac{h_n-f_n\cos n}{\sin n}\sin(nt)\right\}\sin(nx)= \tag{10}$$

$$=\sum_{n=1}^{\infty}\left\{f_n\cos(nt)+\frac{g_n}{n}\sin(nt)\right\}\sin(nx). \tag{11}$$

If we choose

$$g_n=\frac{h_n-f_n\cos(n)}{\sin n}\cdot n, \tag{12}$$

then the solutions to the Dirichlet (1) and Cauchy (5) problems coincide. Let us determine the conditions under which this is possible.

We can use a result from number theory for what follows.

Theorem 1. *There are only finitely many relatively prime numbers $n, m \in \mathbb{N}$ such that*

$$\left|\pi-\frac{m}{n}\right|\le\frac{C_\pi}{n^\mu},\quad \mu\le 7.7. \tag{13}$$

Proof: According to the result of [11], there is a number $C_\pi > 0$ such that there are only finitely many relatively prime numbers $n, m \in \mathbb{N}$ such that (13) is true.

Let us prove that exists N such that $\forall n, m > N$ $\left|\pi-\frac{m}{n}\right|\ge\frac{C_\pi}{n^\mu}$. Suppose that this is not so, that is, there are infinitely many numbers $n, m \in \mathbb{N}$ such that (13) is true. Since there are only finitely many such relatively prime numbers, there are infinitely many natural numbers n, m, which have a common factor and inequality (13) is true for these numbers.

Suppose that pair of numbers $n_0, m_0 \in \mathbb{N}$ is such that (13) is true. Therefore, there are infinitely many numbers k, $n_k = n_0k, m_k = m_0k, k \in \mathbb{N}$, and $\left|\pi-\frac{m_k}{n_k}\right|\le\frac{C_\pi}{n_k^{7.7}}$ is true for n_k, m_k. Since m_0/n_0 is rational and π is transcendental, then $\exists\alpha > 0$ such that $\alpha < \left|\pi-\frac{m_0}{n_0}\right|$.

Then

$$0<\alpha<\left|\pi-\frac{m_0}{n_0}\right|=\left|\pi-\frac{m_k}{n_k}\right|\le\frac{C_\pi}{n_0^{7.7}k^{7.7}}. \tag{14}$$

Since $k \in \mathbb{N}$, it is always possible to choose k such that $\alpha > \frac{C_\pi}{n_0^{7.7}k^{7.7}}$. This contradicts (14), and the set of admissable factors k for numbers $m_0, n_0 \in \mathbb{N}$ is finite. The theorem is true since there are finitely many relatively prime numbers $n, m \in \mathbb{N}$ such that (13) is true and there are finitely many of their admissible factors. □

Lemma 4. *Let* $\eta(x) \equiv \sum_{n=1}^{\infty} \frac{h_n - f_n \cos n}{\sin n} n \sin(nx)$, $h(x), f(x) \in H_0^k[0;\pi]$, *where* $k = \mu + 4$, *if* μ *is an integer, and* $k = [\mu] + 5$, *if it is not an integer. Then* $\eta(x) \in H_0^3[0;\pi]$.

Proof: Consider series

$$\eta(x) = \sum_{n=1}^{\infty} \frac{h_n - f_n \cos n}{\sin n} n \sin(nx). \tag{15}$$

Since theorem 1 is true, $\exists N$, that $\forall n, m > N$ $\left|\pi - \frac{m}{n}\right| \geq \frac{C_\pi}{n^\mu}$, then $|\pi n - m| \geq \frac{C_\pi}{n^{\mu-1}}$. Let us use natural number m for summation in series (15). We have to find how numbers n and m are related to make this summation possible.

Let us choose numbers n so that $0 < |\pi - \frac{m}{n}| < \frac{\pi}{2n}$ or $\frac{\pi}{2}(2n-1) < m < \frac{\pi}{2}(2n+1)$. This inequality is true for any natural $m > 1$. We obtain that the whole set of real numbers is divided by intervals $\left(\frac{\pi}{2}(2n-1); \frac{\pi}{2}(2n+1)\right)$ by enumerating all the possible natural values of number n. In this case, only irrational numbers $\frac{\pi}{2}(2k-1), k \in \mathbb{N}$ are excluded from the set of real numbers, while all the natural numbers greater than one remain, and each number $m(n)$ falls into one of the partition intervals. Thus, each natural $m > 1$ corresponds to number n, and for sufficiently large n a following estimate is true: $n < m(n) < 4n$. This choice of numbers does not contradict the 1 theorem discussed above, since condition $0 < |\pi - \frac{m(n)}{n}| < \frac{\pi}{2n}$ does not contradict inequality (13).

The function $\sin(a)$ is increasing provided that $0 < a < \frac{\pi}{2}$, so the following inequality

$$|\sin(|\pi n - m(n)|)| \geq \sin\left(\frac{C_\pi}{n^{\mu-1}}\right),$$

is true, whence for sufficiently large m, n

$$|\sin(m(n))| \geq \sin\left(\frac{C_\pi}{n^{\mu-1}}\right).$$

Therefore,

$$\frac{1}{|\sin(m(n))|} \leq 2\frac{n^{\mu-1}}{C_\pi},$$

and

$$\frac{1}{|\sin(m)|} \leq 2\frac{m^{\mu-1}}{C_\pi}. \tag{16}$$

Thus, from (16) for numbers n starting from some value N the following relation is true

$$\frac{1}{|\sin(n)|} \leq 2\frac{n^{\mu-1}}{C_\pi}. \tag{17}$$

The estimate $\forall n > N$ follows from (17) and (12) is

$$\Big|\frac{h_n - f_n \cos n}{\sin n}\Big| \leq \frac{2n^{\mu-1}}{C_\pi}|h_n - f_n \cos n|. \tag{18}$$

Since only the estimate for μ is known, we choose $f(x), h(x) \in H_0^{12}[0;\pi]$. This estimate may be greatly overestimated, but we have no other data.

Then the following relation is true.

$$\Phi_n = \frac{2}{\pi}\int_0^\pi f^{(12)}(x)\sin(nx)dx, \Psi_n = \frac{2}{\pi}\int_0^\pi h^{(12)}(x)\sin(nx)dx,$$

$$\sum_{n=1}^{\infty}\Phi_n^2 < C_1, \sum_{n=1}^{\infty}\Psi_n^2 < C_1, f_n = \frac{\Phi_n}{n^{12}}, h_n = \frac{\Psi_n}{n^{12}}.$$

The following chain of estimates is correct.

$$|\eta'''(x)| = \Big|\sum_{n=1}^{\infty}\frac{h_n - f_n\cos(n)}{\sin n}(-n^3)\cos(nx)\cdot n\Big| \leq$$

$$\leq 2\sum_{n=1}^{\infty}\frac{|\Psi_n - \Phi_n\cos(n)|}{n^{12}C_\pi\sin(n)}n^4|\cos(nx)|,$$

$$\sum_{n=1}^{\infty}\frac{2|\Psi_n - \Phi_n\ cos(n)|}{C_\pi n^{12}\sin(n)}n^4|\cos(nx)| \leq$$

$$\leq \sum_{n=1}^{\infty}\frac{2|\Psi_n - \Phi_n\cos(n)|n^7}{n^{12}}\frac{(n^4)}{C_\pi} \leq \sum_{n=1}^{\infty}\frac{2|\Psi_n - \Phi_n\cos(n)|}{n\cdot C_\pi} \leq$$

$$\leq \frac{4}{C_\pi}\sum_{n=1}^{\infty}\left\{(\Psi_n - \Phi_n\cos(n))^2 + \frac{1}{n^2}\right\} \leq \frac{8}{C_\pi}\sum_{n=1}^{\infty}\left\{\Psi_n^2 + \Phi_n^2 + \frac{1}{n^2}\right\}.$$

This series is convergent because series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent, $\sum_{n=1}^{\infty}\Phi_n^2$ is convergent for $\Phi(x) \in H[0;1]$, $\sum_{n=1}^{\infty}\Psi_n^2$ is convergent for $\Psi(x) \in H[0;1]$. Therefore, $\eta(x) \in H_0^3[0;\pi]$. □

Lemma 5. *Let $f(x), h(x) \in H_0^{12}[0;\pi]$. Then the solution to (1) exists.*

Proof: If $f(x), h(x) \in H_0^{12}[0;\pi]$, then $g(x) \equiv \sum_{n=1}^{\infty} n\frac{h_n - f_n\cos(n)}{\sin n}\sin(nx) \in H_0^3[0;\pi]$ according to lemma 4. Solution to (5) exists and is unique in this case by lemma 2. Moreover, the constructed solution satisfies all the conditions of problem (1). Therefore, it is also a solution to problem (1). Thus, a solution to (1) exists. □

Conclusion

Theorem 2. *Let* $f(x) \in H_0^{12}[0;\pi]$, $h(x) \in H_0^{(12)}[0;\pi]$. *Then the solution to the Dirichlet problem (1) exists and is unique.*

Proof: The existence is proved in Lemma 5.

Let us prove that the solution is unique. Indeed, since solution (1) $u(x,t)$ exists under the conditions of the theorem, it can be found in the form of series (11). According to Lemma 3 and Lemma 4 this solution has a derivative $\dfrac{\partial u(x,t)}{\partial t} \in H_0^3[0;\pi] \forall t \in [0;1]$. Having found that $\dfrac{\partial u(x,1)}{\partial t} = g(x)$, $g(x) \in H_0^3[0;\pi]$, we can view the problem (1) as the Cauchy problem (5).

Thus, problem (1) is reduced to the Cauchy problem (2), whose solution is unique. □

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