

ON THE SOLUTION OF THE DIRICHLET PROBLEM
FOR THE WAVE EQUATIONYU.I. SUKHAREV, V.P. TANANA, B.A. MARKOV *Представлено П.П. ПЕТРОВЫМ*

Abstract: In this paper we consider the Dirichlet problem for the wave equation on a finite time interval. The difficulty of solving the problem lies in the fact that under natural conditions the solution of this problem does not always exist, which complicates the problem formulation.

This problem is solved by comparing the Dirichlet problem with the Cauchy problem in this paper. Under certain boundary conditions, it is possible to construct a solution to the Dirichlet problem by reducing it to the Cauchy problem for the wave equation, thus proving the existence of a solution. Due to the uniqueness and stability of the solution of the Cauchy problem, it is possible to prove the uniqueness of the solution of the Dirichlet problem under rather strong constraints on the functions in the boundary conditions.

Keywords: Dirichlet problem, finite time interval, wave equation, conditionally ill-posed problem, Cauchy problem, non-equivalence of the Cauchy problem and the Dirichlet problem.

Introduction

The concept of a colloid as a set of layers is common in colloid chemistry [1], [2]. The layers will have one or another electrical moment in the case of rare metal oxyhydrates, since they are composed of electrically polar long molecules [3].

It has been experimentally discovered that between electrodes placed in a colloidal substance and connected via a measuring device, a certain potential difference arises, and the device records a spontaneous electric current [4]. The diagram of the current's dependence on time appears chaotic, but the Fourier transform over time for a finite time interval (over the measurement interval) shows with high reliability the presence of harmonic oscillations of various frequencies. In this case, the frequencies are discrete and increase linearly with the frequency number.

The conclusion seems possible that the detected oscillations are a consequence of oscillations of long molecules having electric moments due to the layered structure of the colloidal substance, and the measuring device records these oscillations. Note that such an assumption is confirmed by the presence of long rod-shaped fragments in the rare metal colloid, visible at x1000 magnification.

The experimenter has access, at best, to the initial and final location of the fragment, since all measurements occur over a finite time interval. At the same time, the change in the shape of a long molecule is of considerable importance, since it can be used to draw conclusions about the chemical composition of the substance, its changes over time, the homogeneities and heterogeneities of the colloid, the flows of substances filtered by the fragments, the sorption properties of the gel, and much more.

Oscillations of structural units on a finite time interval with conditions at the initial and final moments are determined by the solution of the Dirichlet boundary value problem, which is ill-posed [5], [6]. Therefore, a complexity arises from the fact that the solution of such a problem may not exist, be non-unique or unstable with respect to the initial data.

1 Mathematical statement of the problem

Let the Dirichlet problem be given for the colloidal substance linear fragment oscillation time on a unit time interval:

$$\begin{cases} \frac{\partial^2 u(x, t)}{\partial t^2} = \frac{\partial^2 u(x, t)}{\partial x^2}, & x \in (0; \pi), \quad t \in (0; 1), \\ u(0, t) = 0, \quad t \in [0; 1], \quad u(1, t) = 0, \quad t \in [0; 1]; \\ u(x, 0) = f(x), \quad x \in [0; \pi], \quad u(x, 1) = h(x), \quad x \in [0; \pi], \end{cases} \quad (1)$$

where $u(x, t)$ is the linear fragment rod deviation from the equilibrium position. We are looking for a solution to the problem (1):

Definition 1. *A solution to the boundary value problem (1) [7], p. 254 is a function $u(x, t)$ such that:*

1. $u(x, t)$ is continuous with first derivatives on the set $t \in [0; 1], x \in [0; \pi]$,
2. $u(x, t)$ has continuous second-order derivatives on the set $t \in (0; 1), x \in (0; \pi)$,
3. $u(x, t)$ satisfies the differential equation (1) on $t \in (0; 1), x \in (0; \pi)$,
4. $u(x, t)$ satisfies all the boundary conditions of the problem (1).

The complexity of the problem (1) is that its solution for certain lengths of the time interval is not unique [5] or may not exist at all (we choose the case corresponding to the value $\alpha = 1$ in the notation of [5]). In addition, the problem is unstable with respect to the initial data [5]. Therefore, we will construct and consider an auxiliary Cauchy problem, the classical solution of which is well known (for example, [7]). It will be necessary to answer the question of whether the solution of the auxiliary problem coincides with the solution of problem (1) as well.

Let us consider the question of eigenfunctions, since we will construct a solution by expanding it in terms of the eigenfunctions of the corresponding Sturm-Liouville problem.

We also need to consider the auxiliary Cauchy problem (the direct problem), since we will use it later.

$$\begin{cases} \frac{\partial^2 u(x, t)}{\partial t^2} = \frac{\partial^2 u(x, t)}{\partial x^2}, & x \in (0; \pi), \quad t \in (0; 1), \\ u(0, t) = 0, \quad t \in [0; 1], \quad u(1, t) = 0, \quad t \in [0; 1]; \\ u(x, 0) = f(x), \quad x \in [0; \pi], \quad \frac{\partial u(x, 0)}{\partial t} = g(x), \quad x \in [0; \pi], \end{cases} \quad (2)$$

We do not indicate the spaces in which we construct the solution, because their clarification is preceded by the properties of the solution constructed.

1.1. Sturm-Liouville problem. Consider an operator $A : H \rightarrow H$, where $H \equiv L_2[0; \pi]$, such that $D(A) = C_0^2[0; \pi]$, and defined by the formula

$$\begin{cases} A(U(x)) \equiv \frac{d^2 U(x)}{dx^2}, \\ U(0) = 0, \quad U(\pi) = 0. \end{cases} \quad (3)$$

Let us construct the closure of the operator A in the space $L_2[0; \pi]$. We introduce the scalar product to do this:

Definition 2. *The scalar product in the space $L_2[0; \pi]$ is the integral*

$$(W(x), U(x)) = \int_0^\pi U(x)W(x)dx,$$

where $W(x), U(x) \in L_2[0; \pi]$.

The operator A is symmetric on its domain. The symmetry of the operator implies its closure in $L_2[0; \pi]$. Since the operator is symmetric on $D(A)$, defined by Definition 2, then, according to [8], p. 354:

Definition 3. The closure of the operator A is the operator \bar{A} , such that $D(A) \subset D(\bar{A})$, and the graph of $\Gamma(\bar{A})$ is closed in the Cartesian product $L_2[0; \pi] \times L_2[0; \pi]$.

The solution (1) has the form of a series in the Sturm-Liouville problem eigenfunctions:

$$\begin{cases} \frac{d^2U(x)}{dx^2} = \Lambda U(x), x \in (0; \pi), \\ U(0) = 0, U(\pi) = 0. \end{cases} \quad (4)$$

Lemma 1. The eigenvalues of the Sturm-Liouville problem (4) are $\Lambda = -n^2$. The system of eigenfunctions corresponding to these numbers, $\{\sin(nx)\}_{n=1}^{\infty}$, is complete and orthogonal in the space $L_2[0; \pi]$.

Proof: The proof is given in, for example, [9]. \square

1.2. Construction of the solution to the problem (2) and some of its properties. Let us consider an auxiliary Cauchy problem to construct a solution to problem (1). The auxiliary problem is comparable with the Dirichlet problem (1):

$$\begin{cases} \frac{\partial^2 U(x, t)}{\partial t^2} = \frac{\partial^2 U(x, t)}{\partial x^2}, & x \in (0; \pi), \quad t \in (0; 1), \\ U(0, t) = 0, \quad t \in [0; 1], \quad U(1, t) = 0, \quad t \in [0; 1], \\ U(x, 0) = f(x), \quad x \in [0; \pi], \quad \frac{\partial U(x, 0)}{\partial t} = g(x), \quad x \in [0; \pi], \\ f(x) \in H_0^4[0; \pi], \quad g(x) \in H_0^3[0; \pi]. \end{cases} \quad (5)$$

Note that such a problem is considered in [7], p. 259. The following statement is true:

Lemma 2. A solution to problem (5) exists $\forall x \in [0; \pi], \forall t \in [0; +\infty)$, is unique, is continuous in the initial data and can be represented as a series:

$$\begin{aligned} U(x, t) &= \sum_{n=1}^{\infty} \left\{ f_n \cos(nt) + \frac{g_n}{n} \sin(nt) \right\} \sin(nx), \\ f_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx, \quad g_n = \frac{2}{\pi} \int_0^{\pi} g(x) \sin(nx) dx, \end{aligned} \quad (6)$$

if $f(x) \in H_0^4[0; \pi], g(x) \in H_0^3[0; \pi]$.

The proof is given in [7], pp. 259-260.

The following lemma is true:

Lemma 3. Let a solution to problem (5) exist, i.e. $f(x) \in H_0^4[0; \pi], g(x) \in H_0^3[0; \pi]$. Then $\frac{\partial^4 U(x, t)}{\partial x^3 \partial t} \in L_2[0; \pi] \forall t \in [0; +\infty)$, $\frac{\partial U(x, t)}{\partial t} \in H_0^3[0; \pi] \forall t \in [0; +\infty)$.

Proof: We differentiate the series (6), since the solution (5) is defined by formula (6):

$$\frac{\partial^4 U(x, t)}{\partial x^3 \partial t} = \sum_{n=1}^{\infty} n^4 \left\{ -f_n \sin(nt) + \frac{g_n}{n} \cos(nt) \right\} \cos(nx). \quad (7)$$

Since $f(x) \in H_0^4[0; \pi]$, $g(x) \in H_0^3[0; \pi]$, then

$$f_n = \frac{F_n}{n^4}, g_n = \frac{G_n}{n^3}, \exists C_1 > 0, \sum_{n=1}^{\infty} F_n^2 \leq C_1, \sum_{n=1}^{\infty} G_n^2 \leq C_1, \quad (8)$$

where

$$F_n = \frac{2}{\pi} \int_0^{\pi} f^{(4)}(x) \cos(nx) dx, G_n = -\frac{2}{\pi} \int_0^{\pi} g'''(x) \sin(nx) dx.$$

The equality is true:

$$\frac{\partial^4 U(x, t)}{\partial x^3 \partial t} = \sum_{n=1}^{\infty} \{F_n \sin(nt) - G_n \cos(nt)\} \sin(nx).$$

We obtain, composing the scalar product in the space $L_2[0; \pi]$, and taking into account the norm in this space $\|s(x)\|_{L_2[0; \pi]}^2 \equiv (s(x); s(x))$:

$$\begin{aligned} & \left| \left(\frac{\partial^4 U(x, t)}{\partial x^3 \partial t}, \frac{\partial^4 U(x, t)}{\partial x^3 \partial t} \right) \right| = \\ & = \sum_{n=1}^{\infty} \{F_n^2 \sin^2(nt) + F_n G_n \sin(2nt) + G_n^2 \cos^2(nt)\} \leq 2 \sum_{n=1}^{\infty} \{F_n^2 + G_n^2\}. \end{aligned}$$

This series converges absolutely by (8), therefore, $\frac{\partial^4 U(x, t)}{\partial x^3 \partial t} \in L_2[0; \pi] \forall t \in [0; +\infty)$.

The assertion of the lemma follows from this. \square

1.3. Solution to the problem (1). Let us now consider the problem (1) and the series:

$$u(x, t) = \sum_{n=1}^{\infty} \left\{ f_n \cos(nt) + \frac{h_n - f_n \cos n}{\sin n} \sin(nt) \right\} \sin(nx), \quad (9)$$

where f_n are given by the formula (6), $h_n = \frac{2}{\pi} \int_0^{\pi} h(x) \sin(nx) dx$.

Note that the series (9) upon formal substitution into all relations (1) turns them into true equality.

We have not defined the classes of functions $f(x), h(x)$ for solving the problem (1) yet. We turn to the solution of the problem (5): $f(x) \in H_0^4[0; \pi]$, $g(x) \in H_0^3[0; \pi]$ to determine them. However, the series (9) does not contain components of the expansion of the function $g(x)$ into a Fourier series.

We compare the series (9) and the series (6) in order to determine the components g_n :

$$\sum_{n=1}^{\infty} \left\{ f_n \cos(nt) + \frac{h_n - f_n \cos n}{\sin n} \sin(nt) \right\} \sin(nx) = \quad (10)$$

$$= \sum_{n=1}^{\infty} \left\{ f_n \cos(nt) + \frac{g_n}{n} \sin(nt) \right\} \sin(nx). \quad (11)$$

If we choose

$$g_n = \frac{h_n - f_n \cos(n)}{\sin n} \cdot n, \quad (12)$$

then the solutions of the Dirichlet (1) and Cauchy (5) problems will coincide. Let us find out under what conditions this is possible.

We will use a result from number theory for what follows:

Theorem 1. *There are only finitely many non-multiplicative numbers $n, m \in \mathbb{N}$, such that $\left| \pi - \frac{m}{n} \right| \leq \frac{C_\pi}{n^\mu}$, $2 \leq \mu \leq 7.7$.*

Proof: According to a result of [11], there is a number $C_\pi > 0$, such that there are only finitely many non-multiplicative numbers $n, m \in \mathbb{N}$, such that $\left| \pi - \frac{m}{n} \right| \leq \frac{C_\pi}{n^{7.7}}$. Therefore, $\exists N$, such that $\forall n > N \left| \pi - \frac{m}{n} \right| \geq \frac{C_\pi}{n^\mu}$.

Note also that the number π is transcendental, and the degree of its irrationality μ cannot be less than 2. Thus, $\mu \geq 2$.

From these two statements we obtain that $\mu \in [2; 7.7]$. \square

Lemma 4. *Let the function $\eta(x) \equiv \sum_{n=1}^{\infty} \frac{h_n - f_n \cos n}{\sin n} \sin(nx)$, $h(x), f(x) \in$*

$H_0^k[0; \pi]$, where $k = \mu + 3$, if μ is an integer, and $k = [\mu] + 4$, if it is not an integer. Then $\eta(x) \in H_0^3[0; \pi]$.

Proof: Consider the series

$$\eta(x) = \sum_{n=1}^{\infty} \frac{h_n - f_n \cos n}{\sin n} \sin(nx). \quad (13)$$

Since $\exists N$, that $\forall n, m > N \left| \pi - \frac{m}{n} \right| \geq \frac{C_\pi}{n^\mu}$, then $|\pi n - m| \geq \frac{C_\pi}{n^{\mu-1}}$. Let's choose the numbers n such that the difference $0 < |\pi n - m| < \frac{\pi}{2}$. The function $\sin(a)$ is increasing provided that $0 < a < \frac{\pi}{2}$, so the following is true:

$$\sin(|\pi n - m|) \geq \sin\left(\frac{C_\pi}{n^{\mu-1}}\right),$$

whence for sufficiently large n :

$$|\sin(m)| \geq \sin\left(\frac{C_\pi}{n^{\mu-1}}\right).$$

Therefore,

$$\frac{1}{|\sin(m)|} \leq 2 \frac{n^{\mu-1}}{C_\pi}. \quad (14)$$

Note that the sequence of numbers n is determined by the value m , but in order for the condition $|\pi n - m| \leq \pi/2$ to be satisfied, n will not exceed m . Thus, from this reasoning and (14) for numbers n starting from some value N the following relation is true:

$$\frac{1}{|\sin(n)|} \leq 2 \frac{n^{\mu-1}}{C_\pi}. \quad (15)$$

The estimate $\forall n > N$ follows from (15) and (12) :

$$\left| \frac{h_n - f_n \cos n}{\sin n} \right| \leq \frac{2n^{\mu-1}}{C_\pi} (h_n - f_n \cos n). \quad (16)$$

Since only the estimate for μ is known, we choose $f(x), h(x) \in H_0^{12}[0; \pi]$. This estimate may be greatly overestimated, but we have no other data.

Then the relation is true:

$$\Phi_n = \frac{2}{\pi} \int_0^\pi f^{(12)}(x) \sin(nx) dx, \Psi_n = \frac{2}{\pi} \int_0^\pi h^{(12)}(x) \sin(nx) dx,$$

$$\sum_{n=1}^{infy} \Phi_n^2 < C_1, \sum_{n=1}^{\infty} \Psi_n^2 < C_1, f_n = \frac{\Phi_n}{n^{12}}, h_n = \frac{\Psi_n}{n^{12}}.$$

And the chain of estimates is correct:

$$\begin{aligned} \eta'''(x) &= \sum_{n=1}^{\infty} \frac{h_n - f_n \cos(n)}{\sin n} (-n^3) \cos(nx) \cdot n = \\ &= \sum_{n=1}^{\infty} \frac{\Psi_n - \Phi_n \cos(n)}{n^{12} \sin n} (-n^3) \cos(nx), \\ &= \left| \sum_{n=1}^{\infty} \frac{\Psi_n - \Phi_n \cos(n)}{n^{12} \sin n} (-n^3) \cos(nx) \right| \leq \\ &\leq \sum_{n=1}^{\infty} \frac{|\Psi_n - \Phi_n \cos(n)| n^8 (n^3)}{n^{12} C_\pi} \leq \sum_{n=1}^{\infty} \frac{|\Psi_n - \Phi_n \cos(n)|}{n \cdot C_\pi}. \end{aligned}$$

This series is convergent, therefore, $\eta(x) \in H_0^3[0; \pi]$. \square

Lemma 5. *Let $f(x), h(x) \in H_0^{12}[0; \pi]$. Then the solution (1) exists.*

Proof: By Lemma 4, if $f(x), h(x) \in H_0^{12}[0; \pi]$, then $\exists g(x) \in H_0^3[0; \pi]$. By Lemma 2, the solution (5) exists and is unique in this case. Moreover, the constructed solution satisfies all the conditions of the problem (1). Therefore, it is also a solution to the problem (1). Thus, a solution to (1) exists. \square

Conclusion

The theorem is true thus

Theorem 2. *Let $f(x) \in H_0^{1,2}[0; \pi]$, $h(x) \in H_0^{(1,2)}$. Then a solution to the Dirichlet problem (1) exists and is unique.*

Proof: The existence is proved in Lemma 5.

Let us prove that the solution is unique. Indeed, since the solution (1) $u(x, t)$ exists under the conditions of the theorem, it can be found in the form of series (11). According to Lemma 3 and Lemma 4 this solution has a derivative $\frac{\partial u(x, t)}{\partial t} \in H_0^3[0; \pi] \forall t \in [0; 1]$. Having declared that $\frac{\partial u(x, 1)}{\partial t} = g(x)$, $g(x) \in H_0^3[0; \pi]$, problem (1) can be viewed as the Cauchy problem (5).

Thus, problem (1) is reduced to the Cauchy problem (2), whose solution is unique. □

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