# THE IMPULSIVE KELVIN-VOIGT EQUATIONS FOR TWO-COMPONENT MIXTURES

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**Abstract:** We study the multidimensional initial boundary value problem for the system of Kelvin-Voigt equations for a two-component mixture of viscoelastic fluids with a nonlinear convective term and a linear impulse term — a regular minor term describing impulsive phenomena. The impulsive summand depends on the positive integer parameter $n$ and at $n \to \infty$ weakly* converges to an expression including the Dirac delta-function modelling the impulsive phenomena at the initial moment of time. The main results of the paper are related to the limit transition at $n \to \infty$ in the family of regular weak solutions of the studied initial boundary value problem.

**Keywords:** Kelvin-Voigt equation, two-component mixture, impulsive partial differential equation, initial layer.

# 1    Introduction

In the present paper, we proceed the research initiated in [11, 12]. Here, we study the Dirichlet problem for the Kelvin-Voigt equations for a two-component mixture of viscoelastic fluids in the presence of impulsive phenomena:

$$
\begin{cases}
\partial_t \boldsymbol{v}_i^n + \operatorname{div}(\boldsymbol{v}_i^n \otimes \boldsymbol{v}_i^n) = -\nabla \pi_i^n + \sum_{j=1}^{2} \mu_{ij} \Delta \boldsymbol{v}_j^n \\
+\varkappa_i \Delta(\partial_t \boldsymbol{v}_i^n) + \gamma \varphi^n(t) \boldsymbol{v}_i^n \text{ in } Q_T, \quad i = 1, 2, \\
\operatorname{div} \boldsymbol{v}_i^n = 0 \text{ in } Q_T, \quad i = 1, 2, \\
\boldsymbol{v}_i^n(\boldsymbol{x}, 0) = \boldsymbol{v}_{0i}(\boldsymbol{x}) \text{ in } \Omega, \quad i = 1, 2, \\
\boldsymbol{v}_i^n(\boldsymbol{x}, t) = \boldsymbol{0} \text{ on } \Gamma_T, \quad i = 1, 2,
\end{cases}
\tag{1}
$$

where $Q_T = \Omega \times (0, T)$, $\Omega$ is a bounded domain of $\mathbb{R}^d$, $d \in \{2, 3\}$, $\gamma = \pm 1$, and T is a given positive constant, and $\Gamma_T = \partial\Omega \times (0, T)$, with $\partial\Omega$ denoting the boundary of $\Omega$. The vector–valued functions $\boldsymbol{v}_i^n(\boldsymbol{x}, t)$, $i = 1, 2$, with $\boldsymbol{x} \in \mathbb{R}^d$ and $t \in \mathbb{R}_0^+$, and the scalar-valued functions $\pi_i^n$, $i = 1, 2$ are the sought solutions of the problem (1), whereas the vector–valued functions $\boldsymbol{v}_{0i}(\boldsymbol{x})$, $i = 1, 2$ are given data, and $\mu_{ij}$, $i, j = 1, 2$ and $\varkappa_i$, $i = 1, 2$ are assumed to be constants,

$$
\{\mu_{i,j}\}_{i,j=1}^{2} > 0, \quad \varkappa_i > 0, \quad i = 1, 2.
\tag{2}
$$

In (2), the expression $\{\mu_{i,j}\}_{i,j=1}^{2} > 0$ means the positive definiteness of the matrix $\mathbf{M} = \{\mu_{i,j}\}_{i,j=1}^{2}$, i. e., for all $\boldsymbol{\zeta} \in \mathbb{R}^d \setminus \{\boldsymbol{0}\}$, the inequality $(\mathbf{M}\boldsymbol{\zeta}, \boldsymbol{\zeta}) > 0$ holds. The sequence $\varphi^n(t)$ approximates the Dirac delta–function. For every natural $n$, $n \geq n_0 = \left[\dfrac{1}{T}\right] + 2$, the function $\varphi^n = \varphi^n(t)$ is defined by the formula $\varphi^n(t) = n\Phi(nt)$, $t \in [0, T]$, where $\Phi$ is a nonnegative smooth function with $\operatorname{supp} \Phi = [0, 1]$ and the unit mean: $\displaystyle\int_0^1 \Phi(\vartheta)\, d\vartheta = 1$.

The system under study is a generalization of the well-known Kelvin-Voigt system of equations for the dynamics of a one-component viscoelastic fluid to the two-component case. One of its characteristic features, in addition to nonlinearity and the presence of an impulse term, is the presence of second-order derivatives of the velocities of both components in the equations $(1)_1$. Unlike the one-component case where the viscosity is a scalar, in the two-component case the viscosity coefficients $\mu_{ij}$, $i, j = 1, 2$ form a matrix $\mathbf{M}$ whose elements are responsible for viscous friction. The diagonal elements of the matrix $\mathbf{M}$ are responsible for the viscous friction within each component, and the non-diagonal elements are responsible for the viscous friction between the components. If the matrix $\mathbf{M}$ is diagonal, then the equations $(1)_1$ will not be related and then the results known for the Kelvin-Voigt equations for the dynamics of a one-component viscoelastic fluid [1–10, 27–29, 33, 36–38] are

automatically transferred to the two-component case. We consider the more complicated case of a non-diagonal and non-triangular matrix $\mathbf{M}$ and do not impose any simplifying assumptions on it except the standard requirement of positive definiteness. Solvability issues for hydrodynamic equations with non-diagonal viscosity matrices were studied in [14–26, 30, 31].

We introduce the following functional spaces widely used in the Mathematical Fluid Mechanics:

$$\mathcal{V}(\Omega) = \{\boldsymbol{v} \in C_0^\infty(\Omega) : \operatorname{div} \boldsymbol{v} = 0\},$$

$$\mathbf{H}(\Omega) = \text{ closure of } \mathcal{V}(\Omega) \text{ in the norm of } L^2(\Omega),$$

$$\mathbf{V}^l(\Omega) = \text{ closure of } \mathcal{V}(\Omega) \text{ in the norm of } W^{l,2}(\Omega), \quad l = 1, 2.$$

In case $l = 1$, we denote $\mathbf{V}^1(\Omega)$ simply by $\mathbf{V}(\Omega)$.

The weak solution to problem (1) is defined as follows.

**Definiton 1.** *A functions $\boldsymbol{v}_i^n \in L^\infty(0, T; \mathbf{V}^2(\Omega)) \cap C([0, T]; \mathbf{H}(\Omega))$, $\partial_t \boldsymbol{v}_i^n \in L^2(0, T; \mathbf{V}(\Omega))$, $i = 1, 2$ is called a weak solution to problem (1) if $\boldsymbol{v}_i^n(\boldsymbol{x}, 0) = \boldsymbol{v}_{0i}(\boldsymbol{x}) \in \mathbf{V}^2(\Omega)$ a. e. in $\Omega$, $i = 1, 2$, and for any $\boldsymbol{\phi}_i \in L^\infty(0, T; \mathbf{V}^2(\Omega))$, $i = 1, 2$ such that $\partial_t \boldsymbol{\phi}_i \in L^2(0, T; \mathbf{V}(\Omega))$, $i = 1, 2$ and any $s \in [0, T]$*

$$\int\limits_0^s \int\limits_\Omega \Big( \partial_t \boldsymbol{v}_i^n(\boldsymbol{x}, t) \cdot \boldsymbol{\phi}_i(\boldsymbol{x}, t) + \operatorname{div}(\boldsymbol{v}_i^n(\boldsymbol{x}, t) \otimes \boldsymbol{v}_i^n(\boldsymbol{x}, t)) \cdot \boldsymbol{\phi}_i(\boldsymbol{x}, t)$$

$$+ \varkappa_i \nabla(\partial_t \boldsymbol{v}_i^n)(\boldsymbol{x}, t) : \nabla \boldsymbol{\phi}_i(\boldsymbol{x}, t) + \sum_{j=1}^2 \mu_{ij} \nabla \boldsymbol{v}_j^n(\boldsymbol{x}, t) : \nabla \boldsymbol{\phi}_i(\boldsymbol{x}, t) \Big) \, d\boldsymbol{x} dt$$

$$= \int\limits_0^s \gamma \varphi^n(t) \int\limits_\Omega \boldsymbol{v}_i^n(\boldsymbol{x}, t) \cdot \boldsymbol{\phi}_i(\boldsymbol{x}, t) \, d\boldsymbol{x} dt, \quad i = 1, 2,$$

$$\|\boldsymbol{v}_i^n(\cdot, t) - \boldsymbol{v}_{0i}(\cdot)\|_{\mathbf{H}(\Omega)} \to 0 \ as \ t \to 0+, \quad i = 1, 2.$$

## 2    The existence and uniqueness results for a fixed $n \in \mathbb{N}$

**Theorem 1 (Global in time existence and uniqueness).** *Let $\partial \Omega \in C^2$. Then the problem (1) has at most one global in time solution in the sense of Definition 1. The solution satisfies the estimates*

$$\sum_{i=1}^2 \operatorname*{ess\,sup}_{(0,T)} \|\boldsymbol{v}_i^n(\cdot, t)\|_{\mathbf{V}^2(\Omega)} \leqslant C, \tag{3}$$

$$\sum_{i=1}^2 \Big( \|\partial_t \boldsymbol{v}_i^n\|_{L^1(0,T;\mathbf{H}(\Omega))} + \|\nabla(\partial_t \boldsymbol{v}_i^n)\|_{L^1(0,T;L^2(\Omega)^{d\times d})} \Big) \leqslant C, \tag{4}$$

$$\sum_{i=1}^{2}\left(\|\partial_t \boldsymbol{v}_i^n\|_{L^2(0,T;\mathbf{H}(\Omega))}+\|\nabla(\partial_t \boldsymbol{v}_i^n)\|_{L^2(0,T;L^2(\Omega)^{d\times d})}\right)\leqslant C\left(T+\int_0^T(\varphi^n(t))^2\,dt\right),$$

$$(5)$$

*where constants $C$ do not depend on $n$.*

The proof of Theorem 1 is split into several subsections.

**2.1. Galerkin's approximations.** From the spectral theorem for self-adjoint compact operators (see, e.g., [32, Ch. 10, Th. 10.12]), it follows that there exist sequences $\{\boldsymbol{\psi}_k\}_{k=1}^{\infty}\subset \mathbf{V}^2(\Omega)$ and $\{\lambda_k\}_{k=1}^{\infty}\subset(0,+\infty)$ such that

$$-\Delta\boldsymbol{\psi}_k=\lambda_k\boldsymbol{\psi}_k,\quad k\in\{1,2,\ldots\},$$

and $\{\boldsymbol{\psi}_k\}_{k=1}^{\infty}$ is an orthonormal basis of the space $\mathbf{H}(\Omega)$. We construct the solution of problem (1) as the limit of the sequence of Galerkin's approximations $(m=1,2,\ldots,\ i=1,2)$

$$\boldsymbol{v}_i^{m,n}(\boldsymbol{x},t)=\sum_{k=1}^{m}v_{k,i}^{m,n}(t)\boldsymbol{\psi}_k(\boldsymbol{x}),\quad \boldsymbol{v}_i^m(\boldsymbol{x},0)=\boldsymbol{v}_{0i}^m(\boldsymbol{x})=\sum_{k=1}^{m}v_{k,i}^m(0)\boldsymbol{\psi}_k(\boldsymbol{x}).$$

According to the Galerkin method, the coefficients $v_{k,i}^{m,n}(t)$ are defined by the following ordinary differential equations

$$(1+\varkappa_i\lambda_k)\frac{dv_{k,i}^{m,n}(t)}{dt}+\int_\Omega \mathrm{div}(\boldsymbol{v}_i^{m,n}(\boldsymbol{x},t)\otimes\boldsymbol{v}_i^{m,n}(\boldsymbol{x},t))\cdot\boldsymbol{\psi}_k(\boldsymbol{x})\,d\boldsymbol{x}$$

$$+\int_\Omega\left(\sum_{j=1}^{2}\mu_{ij}\nabla\boldsymbol{v}_j^{m,n}(\boldsymbol{x},t):\nabla\boldsymbol{\psi}_k(\boldsymbol{x})\right. \tag{6}$$

$$\left.-\gamma\varphi^n(t)\boldsymbol{v}_i^{m,n}(\boldsymbol{x},t)\cdot\boldsymbol{\psi}_k(\boldsymbol{x})\right)d\boldsymbol{x}=0,\quad k=1,\ldots,m,\quad i=1,2,$$

along with the following initial conditions

$$v_{k,i}^m(0)=\int_\Omega \boldsymbol{v}_{0i}(\boldsymbol{x})\cdot\boldsymbol{\psi}_k(\boldsymbol{x})\,d\boldsymbol{x},\quad k=1,\ldots,m,\quad i=1,2. \tag{7}$$

Since $1+\varkappa_i\lambda_k>1$, $k=1,\ldots,m$, $i=1,2$, by Peano theorem, system (6), (7) has a solution $v_{k,i}^{m,n}(t)$, $k=1,\ldots,m$, $i=1,2$ for each $m\in\mathbb{N}$ on some interval $(0,T^{m,n})$. Accordingly, an approximate solution $\boldsymbol{v}_i^{m,n}$, $i=1,2$ exists in $\Omega\times(0,T^{m,n})$.

In this section we aim to study some regularity properties of a solution $\boldsymbol{v}_i^n$, $i=1,2$ to the problem (1). We start by establishing a result that improves the regularity of $\boldsymbol{v}_i^n$, $i=1,2$, provided the boundary $\partial\Omega$ and the initial velocity $\boldsymbol{v}_{0i}$, $i=1,2$ are sufficiently regular, and if the space dimensions are restricted to $d=2$ or $d=3$.

## 2.2. Uniform estimates for $\boldsymbol{v}_i^{m,n}$ and $\nabla \boldsymbol{v}_i^{m,n}$.

**Lemma 1.** *Let $\boldsymbol{v}_i^{m,n}$, $i = 1, 2$ be a Galerkin approximation considered in section 2.1. Then*

$$
\sum_{i=1}^{2} \left( \operatorname*{ess\,sup}_{(0,T)} \|\boldsymbol{v}_i^{m,n}(\cdot, t)\|_{\mathbf{H}(\Omega)} + \operatorname*{ess\,sup}_{(0,T)} \|\nabla \boldsymbol{v}_i^{m,n}(\cdot, t)\|_{L^2(\Omega)^{d \times d}} \right)
$$

$$
+ \sum_{i=1}^{2} \int_0^T \|\nabla \boldsymbol{v}_i^{m,n}(\cdot, t)\|_{L^2(\Omega)}^2 \, dt \leqslant C \left( 1 + \sum_{i=1}^{2} \|\boldsymbol{v}_{0i}\|_{L^2(\Omega)} \right. \tag{8}
$$

$$
\left. + \sum_{i=1}^{2} \|\nabla \boldsymbol{v}_{0i}\|_{L^2(\Omega)} \right).
$$

*Proof.* We multiply the equation of (6) by $2v_{k,i}^{m,n}(t)$ and add up the resulting equation, from $k = 1, \ldots, m$, $i = 1, 2$, to obtain

$$
\sum_{i=1}^{2} \int_{\Omega} \partial_t \left( |\boldsymbol{v}_i^{m,n}(\boldsymbol{x}, t)|^2 \right) \, d\boldsymbol{x} + \sum_{i=1}^{2} \varkappa_i \int_{\Omega} \partial_t \left( |\nabla \boldsymbol{v}_i^{m,n}(\boldsymbol{x}, t)|^2 \right) \, d\boldsymbol{x}
$$

$$
+ 2 \sum_{i,j=1}^{2} \mu_{ij} \int_{\Omega} \nabla \boldsymbol{v}_i^{m,n}(\boldsymbol{x}, t) : \nabla \boldsymbol{v}_j^{m,n}(\boldsymbol{x}, t) \, d\boldsymbol{x}
$$

$$
+ 2 \sum_{i=1}^{2} \int_{\Omega} \operatorname{div} \left( \boldsymbol{v}_i^{m,n}(\boldsymbol{x}, t) \otimes \boldsymbol{v}_i^{m,n}(\boldsymbol{x}, t) \right) \cdot \boldsymbol{v}_i^{m,n}(\boldsymbol{x}, t) \, d\boldsymbol{x} \tag{9}
$$

$$
= 2\gamma\varphi^n(t) \sum_{i=1}^{2} \int_{\Omega} |\boldsymbol{v}_i^{m,n}(\boldsymbol{x}, t)|^2 \, d\boldsymbol{x}
$$

for a. a. $t \in [0, T]$. Taking into account

$$
\int_{\Omega} \operatorname{div} \left( \boldsymbol{v}_i^{m,n}(\boldsymbol{x}, t) \otimes \boldsymbol{v}_i^{m,n}(\boldsymbol{x}, t) \right) \cdot \boldsymbol{v}_i^{m,n}(\boldsymbol{x}, t) \, d\boldsymbol{x} = 0, \quad i = 1, 2,
$$

$$
2 \sum_{i,j=1}^{2} \mu_{ij} \int_{\Omega} \nabla \boldsymbol{v}_i^{m,n}(\boldsymbol{x}, t) : \nabla \boldsymbol{v}_j^{m,n}(\boldsymbol{x}, t) \, d\boldsymbol{x} \geq C \sum_{i=1}^{2} \int_{\Omega} |\nabla \boldsymbol{v}_i^{m,n}(\boldsymbol{x}, t)|^2 \, d\boldsymbol{x},
$$

and integrating inequality (9) with respect to $t$, we derive (8). $\qquad \square$

**Remark 1.** *We derive from* (9) *with $\gamma = 1$*

$$
\frac{d}{dt} Y^{m,n}(t) \leqslant \frac{d}{dt} Y^{m,n}(t) + C \sum_{i=1}^{2} \int_{\Omega} |\nabla \boldsymbol{v}_i^{m,n}(\boldsymbol{x}, t)|^2 \, d\boldsymbol{x}
$$

$$
\leqslant 2\gamma\varphi^n(t) \sum_{i=1}^{2} \int_{\Omega} |\boldsymbol{v}_i^{m,n}(\boldsymbol{x}, t)|^2 \, d\boldsymbol{x} \leqslant 2\varphi^n(t) Y^{m,n}(t), \tag{10}
$$

*where*

$$Y^{m,n}(t) = \sum_{i=1}^{2} \left( \|\boldsymbol{v}_i^{m,n}(\cdot,t)\|_{L^2(\Omega)}^2 + \varkappa_i \|\nabla \boldsymbol{v}_i^{m,n}(\cdot,t)\|_{L^2(\Omega)}^2 \right).$$

*Last inequality implies*

$$\ln \left( \frac{Y^{m,n}(t)}{Y^m(0)} \right) \leqslant 2 \int_0^T \varphi^n(t) \, dt \leqslant 2,$$

*and respectively*

$$Y^{m,n}(t) = \sum_{i=1}^{2} \left( \|\boldsymbol{v}_i^{m,n}(\cdot,t)\|_{L^2(\Omega)}^2 + \varkappa_i \|\nabla \boldsymbol{v}_i^{m,n}(\cdot,t)\|_{L^2(\Omega)}^2 \right) \leqslant e^2 Y^m(0)$$

$$= e^2 \sum_{i=1}^{2} \left( \|\boldsymbol{v}_{0i}^m\|_{L^2(\Omega)}^2 + \varkappa_i \|\nabla \boldsymbol{v}_{0i}^m\|_{L^2(\Omega)}^2 \right). \tag{11}$$

**Remark 2.** *Asymptotic behavior solution in time when $\gamma = -1$. In this case the inequality* (10) *takes the form*

$$\frac{d}{dt} Y^{m,n}(t) + C \sum_{i=1}^{2} \int_\Omega |\nabla \boldsymbol{v}_i^{m,n}(\boldsymbol{x},t)|^2 \, d\boldsymbol{x} \leq 0. \tag{12}$$

*Taking into account*

$$\sum_{i=1}^{2} \|\boldsymbol{v}_i^{m,n}(\cdot,t)\|_{L^2(\Omega)}^2 \leq \sum_{i=1}^{2} C \|\nabla \boldsymbol{v}_i^{m,n}(\cdot,t)\|_{L^2(\Omega)}^2,$$

$$Y^{m,n}(t) \leq C \sum_{i=1}^{2} \left( \|\nabla \boldsymbol{v}_i^{m,n}(\cdot,t)\|_{L^2(\Omega)}^2 \right),$$

*we arrive at inequality*

$$\frac{d}{dt} Y^{m,n}(t) + C Y^{m,n}(t) \leq 0, \tag{13}$$

*which implies*

$$Y^{m,n}(t) \leq Y^{m,n}(0) e^{-Ct} \to 0 \text{ as } t \to +\infty. \tag{14}$$

## 2.3. Uniform estimates for $\nabla \boldsymbol{v}_i^{m,n}$ and $\Delta \boldsymbol{v}_i^{m,n}$.

**Lemma 2.** *Let* $\boldsymbol{v}_i^{m,n}$, $i = 1,2$ *be a Galerkin approximation considered in section 2.1. Then*

$$
\sum_{i=1}^{2} \left( \operatorname*{ess\,sup}_{(0,T)} \|\nabla \boldsymbol{v}_i^{m,n}(\cdot,t)\|_{L^2(\Omega)} + \operatorname*{ess\,sup}_{(0,T)} \|\Delta \boldsymbol{v}_i^{m,n}(\cdot,t)\|_{L^2(\Omega)} \right)
$$
$$
+ \sum_{i=1}^{2} \|\Delta \boldsymbol{v}_i^{m,n}\|_{L^2(Q_T)} \leqslant C \left( 1 + \sum_{i=1}^{2} \|\boldsymbol{v}_{0i}\|_{L^2(\Omega)} + \sum_{i=1}^{2} \|\nabla \boldsymbol{v}_{0i}\|_{L^2(\Omega)} \right. \tag{15}
$$
$$
\left. + \sum_{i=1}^{2} \|\Delta \boldsymbol{v}_{0i}\|_{L^2(\Omega)} \right).
$$

*Proof.* We multiply (6) by $\lambda_k v_{k,i}^{m,n}(t)$ and sum up the resulting equation by $k = 1, \ldots, m$, $i = 1,2$:

$$
\frac{1}{2} \frac{d}{dt} \sum_{i=1}^{2} \left( \|\nabla \boldsymbol{v}_i^{m,n}(\cdot,t)\|_{L^2(\Omega)}^2 + \varkappa_i \|\Delta \boldsymbol{v}_i^{m,n}(\cdot,t)\|_{L^2(\Omega)}^2 \right) \tag{16}
$$
$$
+ \sum_{i,j=1}^{2} \mu_{ij} \int_\Omega \Delta \boldsymbol{v}_i^{m,n}(\boldsymbol{x},t) \cdot \Delta \boldsymbol{v}_j^{m,n}(\boldsymbol{x},t) \, d\boldsymbol{x}
$$
$$
= \sum_{i=1}^{2} \int_\Omega \operatorname{div}(\boldsymbol{v}_i^{m,n}(\boldsymbol{x},t) \otimes \boldsymbol{v}_i^{m,n}(\boldsymbol{x},t)) \cdot \Delta \boldsymbol{v}_i^{m,n}(\boldsymbol{x},t) \, d\boldsymbol{x}
$$
$$
- \gamma \varphi^n(t) \sum_{i=1}^{2} \int_\Omega \boldsymbol{v}_i^{m,n}(\boldsymbol{x},t) \cdot \Delta \boldsymbol{v}_i^{m,n}(\boldsymbol{x},t) \, d\boldsymbol{x}.
$$

We integrate by parts the second term in the right hand side

$$
-\gamma \varphi^n(t) \sum_{i=1}^{2} \int_\Omega \boldsymbol{v}_i^{m,n}(\boldsymbol{x},t) \cdot \Delta \boldsymbol{v}_i^{m,n}(\boldsymbol{x},t) \, d\boldsymbol{x} = \gamma \varphi^n(t) \sum_{i=1}^{2} \int_\Omega |\nabla \boldsymbol{v}_i^{m,n}(\boldsymbol{x},t)|^2 \, d\boldsymbol{x}.
$$

Using the Sobolev inequalities

$$
\begin{aligned}
\|\boldsymbol{v}_i^{m,n}(\cdot,t)\|_{L^4(\Omega)} &\leqslant C \|\nabla \boldsymbol{v}_i^{m,n}(\cdot,t)\|_{L^2(\Omega)}, \quad i = 1,2, \\
\|\nabla \boldsymbol{v}_i^{m,n}(\cdot,t)\|_{L^4(\Omega)} &\leqslant C \|\Delta \boldsymbol{v}_i^{m,n}(\cdot,t)\|_{L^2(\Omega)}, \quad i = 1,2,
\end{aligned} \tag{17}
$$

and the Hölder, Cauchy and Young inequalities, we estimate

$$\left| \sum_{i=1}^{2} \int_{\Omega} \operatorname{div}(\boldsymbol{v}_i^{m,n}(\boldsymbol{x},t) \otimes \boldsymbol{v}_i^{m,n}(\boldsymbol{x},t)) \cdot \Delta \boldsymbol{v}_i^{m,n}(\boldsymbol{x},t)\, d\boldsymbol{x} \right| \tag{18}$$

$$\leqslant \frac{1}{2} \sum_{i=1}^{2} \|(\operatorname{div}(\boldsymbol{v}_i^{m,n} \otimes \boldsymbol{v}_i^{m,n}))(\cdot,t)\|_{L^2(\Omega)}^2 + \frac{1}{2} \sum_{i=1}^{2} \|\Delta \boldsymbol{v}_i^{m,n}(\cdot,t)\|_{L^2(\Omega)}^2,$$

$$\|(\operatorname{div}(\boldsymbol{v}_i^{m,n} \otimes \boldsymbol{v}_i^{m,n}))(\cdot,t)\|_{L^2(\Omega)}^2$$

$$\leqslant C \left( \int_{\Omega} |\boldsymbol{v}_i^{m,n}(\boldsymbol{x},t)|^4 \, d\boldsymbol{x} \right)^{\frac{1}{2}} \left( \int_{\Omega} |\nabla \boldsymbol{v}_i^{m,n}(\boldsymbol{x},t)|^4 \, d\boldsymbol{x} \right)^{\frac{1}{2}} \tag{19}$$

$$\leqslant C \|\Delta \boldsymbol{v}_i^{m,n}(\cdot,t)\|_{L^2(\Omega)}^2, \quad i = 1,2.$$

Therefore

$$\sum_{i=1}^{2} \int_{\Omega} \operatorname{div}(\boldsymbol{v}_i^{m,n}(\boldsymbol{x},t) \otimes \boldsymbol{v}_i^{m,n}(\boldsymbol{x},t)) \cdot \Delta \boldsymbol{v}_i^{m,n}(\boldsymbol{x},t)\, d\boldsymbol{x}$$

$$- \gamma \varphi^n(t) \sum_{i=1}^{2} \int_{\Omega} \boldsymbol{v}_i^{m,n}(\boldsymbol{x},t) \cdot \Delta \boldsymbol{v}_i^{m,n}(\boldsymbol{x},t)\, d\boldsymbol{x}$$

$$\leqslant \left| \sum_{i=1}^{2} \int_{\Omega} \operatorname{div}(\boldsymbol{v}_i^{m,n}(\boldsymbol{x},t) \otimes \boldsymbol{v}_i^{m,n}(\boldsymbol{x},t)) \cdot \Delta \boldsymbol{v}_i^{m,n}(\boldsymbol{x},t)\, d\boldsymbol{x} \right|$$

$$+ \varphi^n(t) \sum_{i=1}^{2} \int_{\Omega} |\nabla \boldsymbol{v}_i^{m,n}(\boldsymbol{x},t)|^2 \, d\boldsymbol{x} \leqslant C \sum_{i=1}^{2} \|\Delta \boldsymbol{v}_i^{m,n}(\cdot,t)\|_{L^2(\Omega)}^2$$

$$+ \varphi^n(t) \sum_{i=1}^{2} \|\nabla \boldsymbol{v}_i^{m,n}(\cdot,t)\|_{L^2(\Omega)}^2.$$

Introducing

$$Z^{m,n}(t) = \sum_{i=1}^{2} \left( \|\nabla \boldsymbol{v}_i^{m,n}(\cdot,t)\|_{L^2(\Omega)}^2 + \varkappa_i \|\Delta \boldsymbol{v}_i^{m,n}(\cdot,t)\|_{L^2(\Omega)}^2 \right)$$

and using (16), we arrive at differential inequality

$$\frac{dZ^{m,n}(t)}{dt} + C \sum_{i=1}^{2} \|\Delta \boldsymbol{v}_i^{m,n}(\cdot,t)\|_{L^2(\Omega)}^2 \leqslant C \left( 1 + \varphi^n(t) \right) Z^{m,n}(t).$$

Integrating last one we obtain

$$\ln \frac{Z^{m,n}(t)}{Z^m(0)} \leqslant \int_{0}^{T} C \left( 1 + \varphi^n(t) \right) dt \leqslant C \left( T + 1 \right) \tag{20}$$

and

$$Z^{m,n}(t) \leqslant Z^m(0)e^{C(T+1)}. \tag{21}$$

Lemma is proven. □

## 2.4. Uniform estimates for $\partial_t v_i^{m,n}$ and $\nabla(\partial_t v_i^{m,n})$.

**Lemma 3.** *Let* $v_i^{m,n}$, $i = 1,2$ *be a Galerkin approximation considered in section 2.1. Then*

$$\operatorname*{ess\,sup}_{(0,T)} \sum_{i=1}^2 \left( \left( \int_\Omega |\partial_t v_i^{m,n}(\boldsymbol{x},t)|^2 \, d\boldsymbol{x} \right)^{\frac{1}{2}} \right. \tag{22}$$
$$\left. + \left( \int_\Omega |\nabla(\partial_t v_i^{m,n})(\boldsymbol{x},t)|^2 \, d\boldsymbol{x} \right)^{\frac{1}{2}} \right) \leqslant C(1 + \varphi^n(t)),$$

*and*

$$\sum_{i=1}^2 \int_0^T \left( \left( \int_\Omega |\partial_t v_i^{m,n}(\boldsymbol{x},t)|^2 \, d\boldsymbol{x} \right)^{\frac{1}{2}} + \left( \int_\Omega |\nabla(\partial_t v_i^{m,n})(\boldsymbol{x},t)|^2 \, d\boldsymbol{x} \right)^{\frac{1}{2}} \right) dt$$
$$\leqslant C \left( T + \int_0^T \varphi^n(t) \, dt \right) = C(T+1). \tag{23}$$

*Proof.* We multiply the equation of (6) by $2\dfrac{dv_{k,i}^{m,n}(t)}{dt}$, and sum up the resulting equations by $k = 1, \ldots, m$, $i = 1,2$, to obtain

$$2\sum_{i=1}^2 \int_\Omega |\partial_t v_i^{m,n}(\boldsymbol{x},t)|^2 \, d\boldsymbol{x} + 2\sum_{i=1}^2 \varkappa_i \int_\Omega |\nabla(\partial_t v_i^{m,n})(\boldsymbol{x},t)|^2 \, d\boldsymbol{x}$$
$$= -2\sum_{i=1}^2 \int_\Omega \operatorname{div}(v_i^{m,n}(\boldsymbol{x},t) \otimes v_i^{m,n}(\boldsymbol{x},t)) \cdot \partial_t v_i^{m,n}(\boldsymbol{x},t) \, d\boldsymbol{x}$$
$$- 2\sum_{i,j=1}^2 \mu_{ij} \int_\Omega \nabla v_j^{m,n}(\boldsymbol{x},t) : \nabla(\partial_t v_i^{m,n})(\boldsymbol{x},t) \, d\boldsymbol{x}$$
$$+ 2\gamma\varphi^n(t) \sum_{i=1}^2 \int_\Omega v_i^{m,n}(\boldsymbol{x},t) \cdot \partial_t v_i^{m,n}(\boldsymbol{x},t) \, d\boldsymbol{x}.$$

Here we apply the estimates

$$- 2 \sum_{i=1}^{2} \int_{\Omega} \mathrm{div} \left( \boldsymbol{v}_i^{m,n}(\boldsymbol{x}, t) \otimes \boldsymbol{v}_i^{m,n}(\boldsymbol{x}, t) \right) \cdot \partial_t \boldsymbol{v}_i^{m,n}(\boldsymbol{x}, t) \, d\boldsymbol{x}$$

$$\leqslant \epsilon \sum_{i=1}^{2} \| \partial_t \boldsymbol{v}_i^{m,n}(\cdot, t) \|_{L^2(\Omega)}^2 + \frac{1}{\epsilon} \sum_{i=1}^{2} \| \mathrm{div} (\boldsymbol{v}_i^{m,n} \otimes \boldsymbol{v}_i^{m,n})(\cdot, t)) \|_{L^2(\Omega)}^2 ,$$

$$- 2 \sum_{i,j=1}^{2} \mu_{ij} \int_{\Omega} \nabla \boldsymbol{v}_j^{m,n}(\boldsymbol{x}, t) : \nabla(\partial_t \boldsymbol{v}_i^{m,n})(\boldsymbol{x}, t) \, d\boldsymbol{x}$$

$$\leqslant \epsilon C \sum_{i=1}^{2} \| \nabla(\partial_t \boldsymbol{v}_i^{m,n})(\cdot, t) \|_{L^2(\Omega)}^2 + \frac{C}{\epsilon} \sum_{i=1}^{2} \| \nabla \boldsymbol{v}_i^{m,n}(\cdot, t) \|_{L^2(\Omega)}^2 ,$$

$$2 \gamma \varphi^n(t) \sum_{i=1}^{2} \int_{\Omega} \boldsymbol{v}_i^{m,n}(\boldsymbol{x}, t) \cdot \partial_t \boldsymbol{v}_i^{m,n}(\boldsymbol{x}, t) \, d\boldsymbol{x} \leqslant \epsilon \sum_{i=1}^{2} \| \partial_t \boldsymbol{v}_i^{m,n}(\cdot, t) \|_{L^2(\Omega)}^2$$

$$+ \frac{(\varphi^n(t))^2}{\epsilon} \sum_{i=1}^{2} \| \boldsymbol{v}_i^{m,n}(\cdot, t) \|_{L^2(\Omega)}^2 ,$$

and (19). Combining last formulas and estimates of the Lemma 2.2 we derive

$$\sum_{i=1}^{2} \left( \int_{\Omega} |\partial_t \boldsymbol{v}_i^{m,n}(\boldsymbol{x}, t)|^2 \, d\boldsymbol{x} + \int_{\Omega} |\nabla(\partial_t \boldsymbol{v}_i^{m,n})(\boldsymbol{x}, t)|^2 \, d\boldsymbol{x} \right) \qquad (24)$$

$$\leqslant C \left( 1 + (\varphi^n(t))^2 \right).$$

The last one entails (22). Lemma is proven.    $\square$

**Remark 3.** *Note that estimate* (24) *implies boundedness of norms* $\| \partial_t \boldsymbol{v}_i^{m,n} \|_{L^2(Q_T)}$, $\| \nabla(\partial_t \boldsymbol{v}_i^{m,n}) \|_{L^2(Q_T)}$, $i = 1, 2$ *for any finite n because the*

$$\sum_{i=1}^{2} \int_0^T \int_{\Omega} \left( |\partial_t \boldsymbol{v}_i^{m,n}(\boldsymbol{x}, t)|^2 + |\nabla(\partial_t \boldsymbol{v}_i^{m,n})(\boldsymbol{x}, t)|^2 \right) \, d\boldsymbol{x} dt$$

$$\leqslant C \left( T + \int_0^T (\varphi^n(t))^2 \, dt \right). \qquad (25)$$

*Using the estimates of lemmas* (8), (15), (23) *and* (24), *we continue the* $\boldsymbol{v}_i^{m,n}$, $i = 1, 2$ *from* $(0, T^{m,n})$ *to* $(0, T)$ *and realize the passage to the limit with respect to m for a fixed finite n.*

**2.5. The limit passage as $m \to \infty$.** A similar result is obtained in [4]. From Lemma 2 and Remark 3, we have

- $\boldsymbol{v}_i^{m,n}$, $i = 1, 2$ are uniformly bounded in $L^2(0, T; \mathbf{V}^2(\Omega))$,
- $\partial_t \boldsymbol{v}_i^{m,n}$, i=1,2 are uniformly bounded in $L^2(0, T; \mathbf{V}^1(\Omega))$.

Then, due to the compact embedding $\mathbf{V}^2(\Omega) \hookrightarrow\hookrightarrow \mathbf{V}(\Omega)$, we can use the Aubin-Lions compactness lemma so that $\boldsymbol{v}_i^{m,n}$ strongly converges to $\boldsymbol{v}_i^n$ in $L^2(0,T;\mathbf{V}(\Omega))$, as $i = 1,2$ and $n \in \mathbb{N}$. In fact, writing the corresponding integral as $(i = 1,2)$

$$
\int\limits_{Q_T} ((\boldsymbol{v}_i^{m,n}(\boldsymbol{x},t) \cdot \nabla)\boldsymbol{v}_i^{m,n}(\boldsymbol{x},t) - (\boldsymbol{v}_i^n(\boldsymbol{x},t) \cdot \nabla)\boldsymbol{v}_i^n(\boldsymbol{x},t))\, d\boldsymbol{x}dt
$$
$$
= \int\limits_{Q_T} (((\boldsymbol{v}_i^{m,n}(\boldsymbol{x},t) - \boldsymbol{v}_i^n(\boldsymbol{x},t)) \cdot \nabla)\boldsymbol{v}_i^{m,n}(\boldsymbol{x},t)
$$
$$
+ (\boldsymbol{v}_i^n(\boldsymbol{x},t) \cdot \nabla)(\boldsymbol{v}_i^{m,n}(\boldsymbol{x},t) - \boldsymbol{v}_i^n(\boldsymbol{x},t)))\, d\boldsymbol{x}dt \to 0 \text{ as } m \to \infty.
$$
(26)

Here we repeat formulas (3.53)–(3.57) from [4]. From $\boldsymbol{v}_i^n \in L^2(0,T;W_0^{1,2}(\Omega))$, $\partial_t \boldsymbol{v}_i^n \in L^2(0,T;W_0^{1,2}(\Omega))$, $i = 1,2$ it follows that $\boldsymbol{v}_i^n \in C([0,T];W_0^{1,2}(\Omega))$, $i = 1,2$. For any functions $(k = 1,\ldots,M,\ i = 1,2,\ M \leqslant m)$

$$
\boldsymbol{\eta}_i^M(\boldsymbol{x},t) = \sum_{k=1}^{M} \theta_{k,i}(t)\boldsymbol{\psi}_k(\boldsymbol{x}), \quad \theta_{k,i} \in C^1[0,T],
$$
(27)

and any $s \in [0,T]$ it follows from (6) that

$$
\int\limits_0^s \int\limits_\Omega \Big( \partial_t \boldsymbol{v}_i^{m,n}(\boldsymbol{x},t) \cdot \boldsymbol{\eta}_i^M(\boldsymbol{x},t) + \varkappa_i \nabla(\partial_t \boldsymbol{v}_i^{m,n})(\boldsymbol{x},t) : \nabla\boldsymbol{\eta}_i^M(\boldsymbol{x},t)
$$
$$
+ \operatorname{div}(\boldsymbol{v}_i^{m,n}(\boldsymbol{x},t) \otimes \boldsymbol{v}_i^{m,n}(\boldsymbol{x},t)) \cdot \boldsymbol{\eta}_i^M(\boldsymbol{x},t)
$$
$$
+ \sum_{j=1}^{2} \mu_{ij} \nabla\boldsymbol{v}_j^{m,n}(\boldsymbol{x},t) : \nabla\boldsymbol{\eta}_i^M(\boldsymbol{x},t) \Big)\, d\boldsymbol{x}dt
$$
$$
= \int\limits_0^s \gamma\varphi^n(t) \int\limits_\Omega \boldsymbol{v}_i^{m,n}(\boldsymbol{x},t) \cdot \boldsymbol{\eta}_i^M(\boldsymbol{x},t)\, d\boldsymbol{x}dt, \quad i = 1,2.
$$
(28)

According to what was said above and (26), it is possible to realize the limit for $m \to \infty$ and fixed $n$. This will lead us to

$$
\int\limits_0^s \int\limits_\Omega \Big( \partial_t \boldsymbol{v}_i^n(\boldsymbol{x},t) \cdot \boldsymbol{\eta}_i^M(\boldsymbol{x},t) + \operatorname{div}(\boldsymbol{v}_i^n(\boldsymbol{x},t) \otimes \boldsymbol{v}_i^n(\boldsymbol{x},t)) \cdot \boldsymbol{\eta}_i^M(\boldsymbol{x},t)
$$
$$
+ \varkappa_i \nabla(\partial_t \boldsymbol{v}_i^n)(\boldsymbol{x},t) : \nabla\boldsymbol{\eta}_i^M(\boldsymbol{x},t)
$$
$$
+ \sum_{j=1}^{2} \mu_{ij} \nabla\boldsymbol{v}_j^n(\boldsymbol{x},t) : \nabla\boldsymbol{\eta}_i^M(\boldsymbol{x},t) \Big)\, d\boldsymbol{x}dt
$$
$$
= \int\limits_0^s \gamma\varphi^n(t) \int_\Omega \boldsymbol{v}_i^n(\boldsymbol{x},t) \cdot \boldsymbol{\eta}_i^M(\boldsymbol{x},t)\, d\boldsymbol{x}dt, \quad i = 1,2.
$$
(29)

Since the set of functions of the form (27) is dense in $\left\{\boldsymbol{\phi}:\ \boldsymbol{\phi}\in L^{\infty}(0,T;\mathbf{V}^{2}(\Omega)),\partial_{t}\boldsymbol{\phi}\in L^{2}(0,T;\mathbf{V}(\Omega))\right\}$, the existence of a solution has been proven.

Let us consider the uniqueness of the solution. Let $\boldsymbol{v}_{1,i}^{n}$, $i=1,2$ and $\boldsymbol{v}_{2,i}^{n}$, $i=1,2$ be two solutions to the problem (1). We set $\boldsymbol{W}_{i}^{n}=\boldsymbol{v}_{1,i}^{n}-\boldsymbol{v}_{2,i}^{n}$, $i=1,2$. Then we arrive at the following energy relation

$$
\frac{1}{2}\sum_{i=1}^{2}\frac{d}{dt}\left(\|\boldsymbol{W}_{i}^{n}(\cdot,t)\|_{L^{2}(\Omega)}^{2}+\varkappa_{i}\|\nabla\boldsymbol{W}_{i}^{n}(\cdot,t)\|_{L^{2}(\Omega)}^{2}\right)
$$

$$
+\sum_{i,j=1}^{2}\mu_{ij}\int_{\Omega}\nabla\boldsymbol{W}_{i}^{n}(\boldsymbol{x},t):\nabla\boldsymbol{W}_{j}^{n}(\boldsymbol{x},t)\,d\boldsymbol{x} \tag{30}
$$

$$
=\gamma\varphi^{n}(t)\sum_{i=1}^{2}\|\boldsymbol{W}_{i}^{n}(\cdot,t)\|_{L^{2}(\Omega)}^{2}+\boldsymbol{I}^{n}(t),
$$

where

$$
|\boldsymbol{I}^{n}(t)|\leqslant\sum_{i=1}^{2}\int_{\Omega}|\boldsymbol{W}_{i}^{n}(\boldsymbol{x},t)|^{2}\left|\nabla\boldsymbol{v}_{1,i}^{n}(\boldsymbol{x},t)\right|\,d\boldsymbol{x}
$$

$$
+\sum_{i=1}^{2}\int_{\Omega}|\nabla\boldsymbol{W}_{i}^{n}(\boldsymbol{x},t)|\,|\boldsymbol{W}_{i}^{n}(\boldsymbol{x},t)|\left|\boldsymbol{v}_{2,i}^{n}(\boldsymbol{x},t)\right|\,d\boldsymbol{x} \tag{31}
$$

$$
=\boldsymbol{I}_{1}^{n}(t)+\boldsymbol{I}_{2}^{n}(t).
$$

Applying estimates of Lemma (2) we derive

$$
|\boldsymbol{I}_{1}^{n}(t)|\leqslant\sum_{i=1}^{2}\|\boldsymbol{W}_{i}^{n}(\cdot,t)\|_{L^{\frac{2d}{d-2}}(\Omega)}^{2}\|\nabla\boldsymbol{v}_{1,i}^{n}(\cdot,t)\|_{L^{\frac{d}{2}}(\Omega)}
$$

$$
\leqslant C\sum_{i=1}^{2}\|\nabla\boldsymbol{W}_{i}^{n}(\cdot,t)\|_{L^{2}(\Omega)}^{2}\|\Delta\boldsymbol{v}_{1,i}^{n}(\cdot,t)\|_{L^{2}(\Omega)}
$$

$$
\leqslant C\sum_{i=1}^{2}\|\nabla\boldsymbol{W}_{i}^{n}(\cdot,t)\|_{L^{2}(\Omega)}^{2},
$$

$$
|\boldsymbol{I}_{2}^{n}(t)|\leqslant\sum_{i=1}^{2}\|\nabla\boldsymbol{W}_{i}^{n}(\cdot,t)\|_{L^{2}(\Omega)}\|\boldsymbol{W}_{i}^{n}(\cdot,t)\|_{L^{\frac{2d}{d-2}}(\Omega)}\|\boldsymbol{v}_{2,i}^{n}(\cdot,t)\|_{L^{d}(\Omega)}
$$

$$
\leqslant C\sum_{i=1}^{2}\|\nabla\boldsymbol{W}_{i}^{n}(\cdot,t)\|_{L^{2}(\Omega)}^{2}\|\Delta\boldsymbol{v}_{2,i}^{n}(\cdot,t)\|_{L^{2}(\Omega)}
$$

$$
\leqslant C\sum_{i=1}^{2}\|\nabla\boldsymbol{W}_{i}^{n}(\cdot,t)\|_{L^{2}(\Omega)}^{2}.
$$

Introducing function $Y(t) = \sum_{i=1}^{2} \left( \|\boldsymbol{W}_i^n(\cdot, t)\|_{L^2(\Omega)}^2 + \varkappa_i \|\nabla \boldsymbol{W}_i^n(\cdot, t)\|_{L^2(\Omega)}^2 \right)$ and combining (30) and last formulas we arrive at the problem

$$\frac{dY(t)}{dt} \leqslant C(1 + \varphi^n(t))Y(t), \ Y(0) = 0.$$

Last one has only trivial solution $Y(t) = 0$. Uniqueness of solution is proved.

## 3   The main results

**Theorem 2.** *Assume $\{\boldsymbol{v}_i^n\}_{n \geq n_0}$, $i = 1, 2$, is the family of regular weak solutions to problem* (1) *in the sense of Definition* 1. *Then the following assertions hold true.*

    **1.** *The family $\{\boldsymbol{v}_i^n\}_{n \geq n_0}$, $i = 1, 2$, is relatively compact in $L^2(0, T; \mathbf{V}(\Omega))$ and relatively weakly$^\star$ compact in $L^\infty(0, T; \mathbf{V}^2(\Omega))$, as $n \to \infty$. More precisely, there exist a subsequence from $\{\boldsymbol{v}_i^n\}_{n \geq n_0}$, $i = 1, 2$, still labeled by $n$, and limit functions $\boldsymbol{v}_i \in L^\infty(0, T; \mathbf{V}^2(\Omega))$, $i = 1, 2$ such that*

$$\boldsymbol{v}_i^n \underset{n \to +\infty}{\longrightarrow} \boldsymbol{v}_i \ \text{strongly in } L^2(0, T; \mathbf{V}(\Omega)),$$
$$\text{weakly}^\star \ \text{in } L^\infty(0, T; \mathbf{V}^2(\Omega)), \quad i = 1, 2. \tag{32}$$

    **2.** *The family of rescalled solutions $\overline{\boldsymbol{v}}_i^n$ in the sense of distribution*

$$\begin{cases} \partial_\vartheta \overline{\boldsymbol{v}}_i^n + \dfrac{1}{n} \operatorname{div}\left(\overline{\boldsymbol{v}}_i^n \otimes \overline{\boldsymbol{v}}_i^n\right) \\[2mm] \quad = \operatorname{div}\left(\dfrac{1}{n} \sum_{j=1}^{2} \mu_{ij} \nabla \overline{\boldsymbol{v}}_j^n + \varkappa_i \nabla(\partial_\vartheta \overline{\boldsymbol{v}}_i^n)\right) \\[2mm] \quad + \gamma \Phi(\vartheta) \overline{\boldsymbol{v}}_i^n(x, \vartheta), \quad i = 1, 2, \\[1mm] \operatorname{div} \overline{\boldsymbol{v}}_i^n = 0 \ \text{in } Q_T, \quad i = 1, 2, \\[1mm] \overline{\boldsymbol{v}}_i^n(x, 0) = \boldsymbol{v}_{i0}(x) \ \text{in } \Omega, \quad i = 1, 2, \\[1mm] \overline{\boldsymbol{v}}_i^n(x, t) = \boldsymbol{0} \ \text{on } \Gamma_T, \quad i = 1, 2, \end{cases} \tag{33}$$

*defined by the formula*

$$\overline{\boldsymbol{v}}_i^n(x, \vartheta) \overset{\text{def}}{=} \boldsymbol{v}_i^n\left(x, \frac{\vartheta}{n}\right), \quad i = 1, 2, \quad \vartheta \in [0, 1], \tag{34}$$

*is relatively compact in $L^1(0, 1; \mathbf{V}(\Omega))$ and relatively weakly$^\star$ compact in $L^\infty(0, 1; \mathbf{V}^2(\Omega))$. In other terms, there exist a subsequence from $\{\overline{\boldsymbol{v}}_i^n\}_{n \geq n_0}$, $i = 1, 2$, still labeled by $n$, and a limit function $\overline{\boldsymbol{v}}_i \in L^\infty(0, 1; \mathbf{V}^2(\Omega))$, $i = 1, 2$ such that*

$$\overline{\boldsymbol{v}}_i^n \underset{n \to +\infty}{\longrightarrow} \overline{\boldsymbol{v}} \ \text{strongly in } L^2(0, 1; \mathbf{H}(\Omega))$$
$$\text{and weakly}^\star \ \text{in } L^\infty(0, 1; \mathbf{V}^2(\Omega)), \quad i = 1, 2. \tag{35}$$

**3.** *Define* $\widetilde{\boldsymbol{v}}_i^n$, $i = 1, 2$ *by the formula*

$$\widetilde{\boldsymbol{v}}_i^n(x, \widetilde{t}) \stackrel{\text{def}}{=} \boldsymbol{v}_i^n\left(x, \frac{1}{n} + \widetilde{t}\right), \quad \widetilde{t} \in [0, T - \epsilon), \quad i = 1, 2.$$

*There exist a subsequence from* $\{\widetilde{\boldsymbol{v}}_i^n\}_{n \geq n_0}$, $i = 1, 2$, *still labeled by* $n$, *and a limit function* $\widetilde{\boldsymbol{v}}_i \in L^\infty(0, T - \epsilon; \mathbf{V}(\Omega))$, $i = 1, 2$ *such that*

$$\begin{aligned} \widetilde{\boldsymbol{v}}_i^n \underset{n \to +\infty}{\longrightarrow} \widetilde{\boldsymbol{v}}_i \text{ strongly in } L^2(0, T - \epsilon; \mathbf{H}(\Omega)) \\ \text{and weakly}^\star \text{ in } L^\infty(0, T - \epsilon; \mathbf{V}(\Omega)), \quad i = 1, 2. \end{aligned} \tag{36}$$

**4.** *In the limit in* $\boldsymbol{v}_i^n$, $i = 1, 2$ *and* $\widetilde{\boldsymbol{v}}_i^n$, $i = 1, 2$ *coincides in the space* $L^1(0, T - \epsilon; \mathbf{V}(\Omega))$.

**5.** *The functions* $(\overline{\boldsymbol{v}}_1, \overline{\boldsymbol{v}}_2, \widetilde{\boldsymbol{v}}_1, \widetilde{\boldsymbol{v}}_2)$ *is* ***a strong solution*** *of the two Cauchy–Dirichlet problems that should be solved successively:*
**5a.** *Firstly, find* $\overline{\boldsymbol{v}}_i = \overline{\boldsymbol{v}}_i(x, \vartheta)$, $i = 1, 2$ *satisfying*

$$\begin{cases} \partial_\vartheta \overline{\boldsymbol{v}}_i = \varkappa_i \Delta \overline{\boldsymbol{v}}_{i\vartheta} + \gamma \Phi(\vartheta) \overline{\boldsymbol{v}}_i, \quad i = 1, 2, \\ \operatorname{div} \overline{\boldsymbol{v}}_i = 0 \text{ in } Q_T, \quad i = 1, 2, \\ \overline{\boldsymbol{v}}_i(x, 0) = \boldsymbol{v}_{i0}(x) \text{ in } \Omega, \quad i = 1, 2, \\ \overline{\boldsymbol{v}}_i(x, t) = \boldsymbol{0} \text{ on } \Gamma_T, \quad i = 1, 2, \end{cases} \tag{37}$$

**5b.** *Secondly, find* $\widetilde{\boldsymbol{v}}_i = \widetilde{\boldsymbol{v}}_i(x, t)$, $i = 1, 2$ *satisfying*

$$\begin{cases} \partial_t \widetilde{\boldsymbol{v}}_i + \operatorname{div}(\widetilde{\boldsymbol{v}}_i \otimes \widetilde{\boldsymbol{v}}_i) \\ \quad = \operatorname{div}\left(\sum_{j=1}^{2} \mu_{ij} \nabla \widetilde{\boldsymbol{v}}_j + \varkappa_i \nabla(\partial_t \widetilde{\boldsymbol{v}}_i)\right), \quad i = 1, 2, \\ \operatorname{div} \widetilde{\boldsymbol{v}}_i = 0 \text{ in } Q_T, \quad i = 1, 2, \\ \widetilde{\boldsymbol{v}}_i(x, 0) = \overline{\boldsymbol{v}}_i(x, 1) \text{ in } \Omega, \quad i = 1, 2, \\ \widetilde{\boldsymbol{v}}_i(x, t) = \boldsymbol{0} \text{ on } \Gamma_T, \quad i = 1, 2, \end{cases} \tag{38}$$

*where the initial function* $\widetilde{\boldsymbol{v}}_i(\cdot, 0)$, $i = 1, 2$ *is the solution of* (37) *at the moment* $\vartheta = 1$.

Following the idea of the previous section, we split the proof of Theorem 2 into several subsections.

## 3.1. Relative compactness of $\{\boldsymbol{v}_i^n\}$ and limiting passage in $Q_T$.

This subsection is similar to subsection 2.5. From the estimates (3),(4), we have

- $\boldsymbol{v}_i^n$ is uniformly bounded in $L^2(0, T; \mathbf{V}^2(\Omega))$, $i = 1, 2$,
- $\boldsymbol{v}_{it}^n$ is uniformly bounded in $L^1(0, T; \mathbf{V}(\Omega))$, $i = 1, 2$.

Then, due to the compact embedding $\mathbf{V}^2(\Omega) \hookrightarrow\hookrightarrow \mathbf{V}(\Omega)$, we can use the Aubin-Lions compactness lemma so that $\boldsymbol{v}_i^n$, $i = 1, 2$ converges strongly in $L^2(0, T; \mathbf{V}(\Omega))$, as $n \in \mathbb{N}$.

**3.2. Rescaling and shift.** Let us fulfill some more preliminary considerations before we turn to the limiting passage as $n \to \infty$. We assume that the test function in equation (6) belongs to $H^1(0, T; \mathbf{V}(\Omega))$ and vanishes in a neighborhood of section $\{t = T\}$ and we integrate (6) in $t$ by parts. We write out the resulting equality in an expanded form, separating the integrals over segments $(0, 1/n)$ and $(1/n, T)$ from each other:

$$
\begin{aligned}
&\int\limits_0^{\frac{1}{n}} \int\limits_\Omega \bigg( -\boldsymbol{v}_i^n \cdot \partial_t \boldsymbol{\phi}_i + \operatorname{div}(\boldsymbol{v}_i^n \otimes \boldsymbol{v}_i^n) \cdot \boldsymbol{\phi}_i - \varkappa_i \nabla \boldsymbol{v}_i^n : \nabla \partial_t \boldsymbol{\phi}_i \\
&\qquad + \sum_{j=1}^2 \mu_{ij} \nabla \boldsymbol{v}_j^n : \nabla \boldsymbol{\phi}_i - \gamma \varphi^n(t) \boldsymbol{v}_i^n \cdot \boldsymbol{\phi}_i \bigg) \, dx dt \\
&- \int\limits_\Omega \boldsymbol{v}_{i0}(x) \cdot \boldsymbol{\phi}_i(x, 0) \, dx - \varkappa_i \int\limits_\Omega \nabla \boldsymbol{v}_{i0}(x) \cdot \nabla \boldsymbol{\phi}_i(x, 0) \, dx \\
&+ \int\limits_{\frac{1}{n}}^T \int\limits_\Omega \bigg( \partial_t \boldsymbol{v}_i^n \cdot \boldsymbol{\phi}_i + \operatorname{div}(\boldsymbol{v}_i^n \otimes \boldsymbol{v}_i^n) \cdot \boldsymbol{\phi}_i + \varkappa_i \nabla \partial_t \boldsymbol{v}_i^n : \nabla \boldsymbol{\phi}_i \\
&\qquad + \sum_{j=1}^2 \mu_{ij} \nabla \boldsymbol{v}_j^n : \nabla \boldsymbol{\phi}_i \bigg) \, dx dt = 0, \quad i = 1, 2.
\end{aligned}
\tag{39}
$$

In (39), we change the independent variable $t$ and the sought variable $\boldsymbol{v}_i^n$, $i = 1, 2$ on segments $\{0 < t < 1/n\}$ and $\{1/n < t \leqslant T\}$ as follows.

On $(1/n, T]$ we shift the timescale backwards and take $(i = 1, 2)$

$$
\widetilde{t} := t - 1/n, \quad \widetilde{\boldsymbol{v}}_i^n(x, \widetilde{t}) := \boldsymbol{v}_i^n(x, t) \equiv \boldsymbol{v}_i^n(x, \widetilde{t} + 1/n) \quad \text{for } t \in (1/n, T]. \tag{40}
$$

Note that $\widetilde{t} \in (0, T - 1/n]$, $dt = d\widetilde{t}$, $\partial_t = \partial_{\widetilde{t}}$, and $t = \widetilde{t} + 1/n$. Further, following the idea of rescaling from [34], we take (i=1,2)

$$
\vartheta := nt, \quad \overline{\boldsymbol{v}}_i^n(x, \vartheta) := \boldsymbol{v}_i^n(x, t) \equiv \boldsymbol{v}_i^n(x, n^{-1}\vartheta) \quad \text{for } t \in [0, 1/n]. \tag{41}
$$

Note that $\vartheta \in [0, 1]$, $dt = n^{-1} d\vartheta$, $\partial_t = n\partial_\vartheta$, and $t = n^{-1}\vartheta$. Thus, (39) takes the form

$$
\begin{aligned}
\int\limits_0^1 \int\limits_\Omega \bigg( &-\overline{\boldsymbol{v}}_i^n(x, \vartheta) \cdot \partial_\vartheta \boldsymbol{\phi}_i(x, n^{-1}\vartheta) \\
&- \varkappa_i \nabla \overline{\boldsymbol{v}}_i^n(x, \vartheta) : \partial_\vartheta \nabla \boldsymbol{\phi}_i(x, n^{-1}\vartheta) \\
&+ \frac{1}{n} \sum_{j=1}^2 \mu_{ij} \nabla \overline{\boldsymbol{v}}_j^n(x, \vartheta) : \nabla \boldsymbol{\phi}_i(x, n^{-1}\vartheta) \\
&+ \frac{1}{n} (\overline{\boldsymbol{v}}_i^n(x, \vartheta) \otimes \overline{\boldsymbol{v}}_i^n(x, \vartheta)) : \nabla \boldsymbol{\phi}_i(x, n^{-1}\vartheta)
\end{aligned}
\tag{42}
$$

$$- \gamma \Phi(\vartheta) \overline{\boldsymbol{v}}_i^n(x, \vartheta) \cdot \boldsymbol{\phi}_i(x, n^{-1}\vartheta) \bigg) dx d\vartheta$$

$$- \int_\Omega \boldsymbol{v}_{i0}(x) \cdot \boldsymbol{\phi}_i(x, 0) \, dx - \varkappa_i \int_\Omega \nabla \boldsymbol{v}_{i0}(x) : \nabla \boldsymbol{\phi}_i(x, 0) \, dx$$

$$+ \int_0^{T-1/n} \int_\Omega \bigg( - \widetilde{\boldsymbol{v}}_i^n(x, \widetilde{t}) \partial_{\widetilde{t}} \boldsymbol{\phi}_i(x, \widetilde{t} + 1/n)$$

$$- \varkappa_i \nabla \widetilde{\boldsymbol{v}}_i^n(x, \widetilde{t}) \cdot \partial_{\widetilde{t}} \nabla \boldsymbol{\phi}_i(x, \widetilde{t} + 1/n)$$

$$- \big( \widetilde{\boldsymbol{v}}_i^n(x, \widetilde{t}) \otimes \widetilde{\boldsymbol{v}}_i^n(x, \widetilde{t}) \big) : \nabla_x \boldsymbol{\phi}_i(x, \widetilde{t} + 1/n)$$

$$+ \sum_{j=1}^2 \mu_{ij} \nabla \widetilde{\boldsymbol{v}}_j^n(x, \widetilde{t}) : \nabla \boldsymbol{\phi}_i(x, \widetilde{t} + 1/n) \bigg) dx d\widetilde{t} = 0, \quad i = 1, 2.$$

Keeping in mind the further limiting passage as $n \to +\infty$, in (42) we take the test function $\boldsymbol{\phi}_i = \boldsymbol{\phi}_i^n(x, t)$, $i = 1, 2$ depending on $n$ in the following form ($i = 1, 2$):

$$\boldsymbol{\phi}_i^n(x, t) = \begin{cases} \overline{\boldsymbol{\phi}}_i(x, \vartheta) \equiv \overline{\boldsymbol{\phi}}_i(x, nt), t \in [0, 1/n], \vartheta \in [0, 1], \\ \widetilde{\boldsymbol{\phi}}_i(x, \widetilde{t}) \equiv \widetilde{\boldsymbol{\phi}}_i(x, t - 1/n), t \in (1/n, T], \widetilde{t} \in (0, T - 1/n], \end{cases} \quad (43)$$

where $\overline{\boldsymbol{\phi}}_i = \overline{\boldsymbol{\phi}}_i(x, \vartheta)$, $i = 1, 2$ and $\widetilde{\boldsymbol{\phi}}_i = \widetilde{\boldsymbol{\phi}}_i(x, \widetilde{t})$, $i = 1, 2$ are arbitrary smooth test functions defined on $\overline{\Omega} \times [0, 1]$ and $\overline{\Omega} \times (0, T]$, respectively, such that $\overline{\boldsymbol{\phi}}_i = \widetilde{\boldsymbol{\phi}}_i \equiv 0$, $i = 1, 2$ in a neighborhood of $\partial\Omega$, $\widetilde{\boldsymbol{\phi}}_i \equiv 0$, $i = 1, 2$ in a neighborhood of the plane $\{\widetilde{t} = T\}$ and the matching condition

$$\overline{\boldsymbol{\phi}}_i(x, 1 - 0) = \widetilde{\boldsymbol{\phi}}_i(x, 0+), \quad i = 1, 2 \quad (44)$$

holds. We notice that condition (44) yields that the weak derivatives $\partial_t \boldsymbol{\phi}_i^n$, $i = 1, 2$ and $\partial_t \nabla_x \boldsymbol{\phi}_i^n$, $i = 1, 2$ are essentially bounded in $Q_T$, which implies that $\boldsymbol{\phi}_i^n \in L^2(0, T; \mathbf{V}(\Omega))$, $i = 1, 2$, $\partial_t \boldsymbol{\phi}_i^n \in L^2(0, T; \mathbf{V}(\Omega))$, $i = 1, 2$ and therefore $\boldsymbol{\phi}_i^n$, $i = 1, 2$ is an admissible test function for (39) and, equivalently, for (42). Inserting (43) into (42), we get

$$\int_0^1 \int_\Omega \bigg( - \overline{\boldsymbol{v}}_i^n(x, \vartheta) \cdot \partial_\vartheta \overline{\boldsymbol{\phi}}_i(x, \vartheta) - \varkappa_i \nabla \overline{\boldsymbol{v}}_i^n(x, \vartheta) \cdot \partial_\vartheta \nabla \overline{\boldsymbol{\phi}}_i(x, \vartheta)$$

$$+ \frac{1}{n} \sum_{j=1}^2 \mu_{ij} \nabla \overline{\boldsymbol{v}}_j^n(x, \vartheta) \cdot \nabla \overline{\boldsymbol{\phi}}_i(x, \vartheta)$$

$$- \frac{1}{n} \big( \overline{\boldsymbol{v}}_i^n(x, \vartheta) \otimes \overline{\boldsymbol{v}}_i^n(x, \vartheta) \big) : \nabla \overline{\boldsymbol{\phi}}(x, \vartheta)$$

$$- \gamma \Phi(\vartheta) \overline{\boldsymbol{v}}_i^n(x, \vartheta) \cdot \overline{\boldsymbol{\phi}}_i(x, \vartheta) \bigg) dx d\vartheta \quad (45)$$

$$
- \int\limits_{\Omega} \boldsymbol{v}_{i0}(x) \cdot \overline{\boldsymbol{\phi}}_i(x,0) \, dx - \varkappa_i \int\limits_{\Omega} \nabla \boldsymbol{v}_{i0}(x) : \nabla \overline{\boldsymbol{\phi}}_i(x,0) \, dx
$$

$$
+ \int\limits_{0}^{T-1/n} \int\limits_{\Omega} \bigg( -\widetilde{\boldsymbol{v}}_i^n(x,\widetilde{t}) \cdot \partial_{\widetilde{t}} \widetilde{\boldsymbol{\phi}}_i(x,\widetilde{t}) - \varkappa_i \nabla \widetilde{\boldsymbol{v}}_i^n(x,\widetilde{t}) : \partial_{\widetilde{t}} \nabla \widetilde{\boldsymbol{\phi}}_i(x,\widetilde{t})
$$

$$
+ (\widetilde{\boldsymbol{v}}_i^n(x,\vartheta) \otimes \widetilde{\boldsymbol{v}}_i^n(x,\vartheta)) : \nabla \widetilde{\boldsymbol{\phi}}_i(x,\widetilde{t})
$$

$$
+ \sum_{j=1}^{2} \mu_{ij} \nabla \widetilde{\boldsymbol{v}}_j^n(x,\widetilde{t}) : \nabla \widetilde{\boldsymbol{\phi}}_i(x,\widetilde{t}) \bigg) \, dx d\widetilde{t} = 0, \quad i = 1, 2.
$$

Furthermore, we notice that

$$
\overline{\boldsymbol{v}}_i^n(x, 1-0) = \widetilde{\boldsymbol{v}}_i^n(x, 0+) \quad \text{in } \overline{\Omega}, \quad i = 1, 2 \tag{46}
$$

due to (40), (41) and the regularity properties of $\boldsymbol{v}_i^n$, $i = 1, 2$, see Definition 1.

The rest of the proof of Theorem 1 is based on the systematical study of (45) with account of (46).

### 3.3. Limiting passage in $\Omega \times \{0 < \vartheta < 1\}$. The initial layer equation.

In (45) take $\widetilde{\boldsymbol{\phi}}_i \equiv 0$, $i = 1, 2$ and get

$$
\int\limits_{0}^{1} \int\limits_{\Omega} \bigg( -\overline{\boldsymbol{v}}_i^n(x,\vartheta) \cdot \partial_{\vartheta} \overline{\boldsymbol{\phi}}_i(x,\vartheta) - \varkappa_i \nabla \overline{\boldsymbol{v}}_i^n(x,\vartheta) \cdot \partial_{\vartheta} \nabla \overline{\boldsymbol{\phi}}_i(x,\vartheta)
$$

$$
+ \frac{1}{n} \sum_{j=1}^{2} \mu_{ij} \nabla \overline{\boldsymbol{v}}_j^n(x,\vartheta) \cdot \nabla \overline{\boldsymbol{\phi}}_i(x,\vartheta) - \frac{1}{n} (\overline{\boldsymbol{v}}_i^n(x,\vartheta) \otimes \overline{\boldsymbol{v}}_i^n(x,\vartheta)) : \nabla \overline{\boldsymbol{\phi}}_i(x,\vartheta)
$$

$$
- \gamma \Phi(\vartheta) \overline{\boldsymbol{v}}_i^n(x,\vartheta) \cdot \overline{\boldsymbol{\phi}}_i(x,\vartheta) \bigg) \, dx d\vartheta - \int\limits_{\Omega} \boldsymbol{v}_{i0}(x) \cdot \overline{\boldsymbol{\phi}}_i(x,0) \, dx
$$

$$
- \varkappa_i \int\limits_{\Omega} \nabla \boldsymbol{v}_{i0}(x) : \nabla \overline{\boldsymbol{\phi}}_i(x,0) \, dx = 0, \quad i = 1, 2. \tag{47}
$$

From Theorem 1 it follows that

$$
\sum_{i=1}^{2} \bigg( \operatorname*{ess\,sup}_{(0,1)} \|\nabla \overline{\boldsymbol{v}}_i^n\|_{L^2(\Omega)} + \operatorname*{ess\,sup}_{(0,1)} \|\Delta \overline{\boldsymbol{v}}_i^n\|_{L^2(\Omega)}
$$

$$
+ \|\partial_{\vartheta} \overline{\boldsymbol{v}}_i^n\|_{L^1(0,1;L^2(\Omega))} + \|\nabla(\partial_{\vartheta} \overline{\boldsymbol{v}}_i^n)\|_{L^1(0,1;L^2(\Omega))} \bigg) \leqslant C, \tag{48}
$$

where $C$ does not depend on $n$. Scaling Galerkin's approximation, we get and passing to the limits as $m \to \infty$

$$
\sum_{i=1}^{2} \bigg( \operatorname*{ess\,sup}_{(0,1)} \|\partial_{\vartheta} \overline{\boldsymbol{v}}_i^n\|_{L^2(\Omega)} + \operatorname*{ess\,sup}_{(0,1)} \|\nabla(\partial_{\vartheta} \overline{\boldsymbol{v}}_i^n)\|_{L^2(\Omega)} \bigg) \leqslant C. \tag{49}
$$

Moreover, due to (48) and (49) we can choose this sequence such that

$$\overline{\boldsymbol{v}}_i^n \underset{n \to +\infty}{\longrightarrow} \overline{\boldsymbol{v}}_i \quad \text{weakly}^\star \text{ in } L^\infty(0,1;\mathbf{V}^2(\Omega)), \quad i = 1,2, \quad (50a)$$

$$\partial_\vartheta \overline{\boldsymbol{v}}_i^n \underset{n \to +\infty}{\longrightarrow} \partial_\vartheta \overline{\boldsymbol{v}}_i \text{ weakly}^\star \text{ in } L^\infty(0,1;\mathbf{H}(\Omega)), \quad i = 1,2, \quad (50b)$$

$$\nabla(\partial_\vartheta \overline{\boldsymbol{v}}_i^n) \underset{n \to +\infty}{\longrightarrow} \nabla(\partial_\vartheta \overline{\boldsymbol{v}}_i) \quad \text{weakly}^\star \text{ in } L^\infty(0,1;L^2(\Omega)), \quad i = 1,2. \quad (50c)$$

In particular, (35) holds and $\overline{\boldsymbol{v}}_i$, $i = 1,2$ is a solution to the problem (37) in the sense of distributions.

**3.4. Limiting passage in $\Omega \times \{0 < \widetilde{t} \leqslant T\}$.** This paragraph is similar to subsection 3.3.

**3.5. Matching condition at $\vartheta = 1 - 0$. Completion of the proof of assertion 3 of Theorem 1.** The bound

$$\sum_{i=1}^{2} \|\partial_\vartheta \overline{\boldsymbol{v}}_i^n\|_{L^\infty(0,1;\mathbf{H}(\Omega))} \leq C$$

implies that the family of mappings $\overline{\boldsymbol{v}}_i^n \colon [0,1] \mapsto \mathbf{H}(\Omega)$, $i = 1,2$ is equicontinuous. On the other hand, by estimate

$$\sum_{i=1}^{2} \operatorname*{ess\,sup}_{(0,1)} \|\nabla \overline{\boldsymbol{v}}_i^n\|_{\mathbf{H}(\Omega)} \leq C,$$

the values of functions $\vartheta \mapsto \overline{\boldsymbol{v}}_i^n(\cdot, \vartheta)$, $i = 1,2$ are uniformly bounded in $\mathbf{V}(\Omega)$, which is a compact subset of $\mathbf{H}(\Omega)$, due to the Rellich theorem. By the Arcel theorem, this implies that the set $\{\overline{\boldsymbol{v}}_i^n\}_{n \geq n_0}$, $i = 1,2$ is relatively compact in $C([0,1];\mathbf{H}(\Omega))$. Hence, there exists a subsequence, still labeled by $n$, such that $\overline{\boldsymbol{v}}_i^n(\cdot, \vartheta) \underset{n \to +\infty}{\longrightarrow} \overline{\boldsymbol{v}}_i(\cdot, \vartheta)$ in $\mathbf{H}(\Omega)$ uniformly on $\{0 \leqslant \vartheta \leqslant 1\}$, $i = 1,2$. Quite analogously, we deduce that $\widetilde{\boldsymbol{v}}_i^n(\cdot, \widetilde{t}) \underset{n \to +\infty}{\longrightarrow} \widetilde{\boldsymbol{v}}_i(\cdot, \widetilde{t})$ strongly in $\mathbf{H}(\Omega)$ uniformly on $\{0 \leqslant \widetilde{t} \leqslant T - 1/n_0\}$, $i = 1,2$. These two limiting relations, along with (46) and inclusions $\overline{\boldsymbol{v}}_i(\cdot, 1 - 0) \in \mathbf{H}(\Omega)$, $i = 1,2$ and $\widetilde{\boldsymbol{v}}_i(\cdot, 0+) = \boldsymbol{v}_i(\cdot, 0+) \in \mathbf{H}(\Omega)$, $i = 1,2$, yield the matching relation $(38)_2$ in the strong sense in $\mathbf{H}(\Omega)$.

Theorem 2 is fully proved.

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