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TOTAL COALITION GRAPHS OF CYCLES AND PATHS

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Abstract: A set of vertices of a graph G is a total dominating set of G if every vertex of G is adjacent to at least one vertex in the set. Two non-total dominating sets forms a total coalition in a graph if their union is a total dominating set. A vertex partition $\pi = \{V_1, V_2, ..., V_k\}$ of $V(G)$ into non-total dominating sets is the total coalition partition if every set of π forms a total coalition set with another member of π. Vertices of the total coalition graph $TCG(G, \pi)$ correspond one-to-one with the sets of π and two vertices are adjacent in $TCG(G, \pi)$ if and only if the corresponding sets form a total coalition. In this paper, we show that the cycle C_{4k} is a universal total coalition cycle for $k \geq 2$, that is a cycle whose total coalition partitions define all possible total coalition graphs of cycles. We also demonstrate that the path P_n is a universal path for $n > 4$.

Keywords: total coalition graph, total dominating set.

Dobrynin, A.A., Golmohammadi H., On cubic graphs having the maximum coalition number.

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1 Introduction

We consider finite, undirected and simple graphs with no isolated vertices having vertex set $V(G)$. We generally follow [22] for notation and graph theory terminology. A dominating set in a graph G is a set S of vertices of G, such that every vertex outside S is adjacent to at least one vertex in S. Domination in graphs appears as a model for facility location problems, and has many applications in design and analysis of transportation networks, wireless sensor networks and so on [18]. The detailed information on graph domination and related topics can be found in books [16, 17, 18, 19]. Cockayne, Dawes, and Hedetniemi, in [8], introduced a variation of domination called the total domination. A set $D \subseteq V$ is a total dominating set of a graph G if every vertex in G is adjacent to at least one vertex in D. For a thorough monograph on total dominating sets, we refer the reader to [21]. In 2020, Haynes et al. [11] introduced a novel graph invariant, called the coalition, based on dominating sets in graphs. Coalitions can be used for changing the policies in industrial business to solve the problems or achieve a common goal [7].

A coalition in a graph G is made up of two disjoint sets $V_1, V_2 \subset V(G)$, such that neither V_1 nor V_2 is a dominating set, but the union $V_1 \cup V_2$ is a dominating set in G. A coalition partition of $V(G)$ is a vertex partition $\pi(G) = \{V_1, V_2, \ldots, V_k\}$, such that for every $i \in \{1, 2, \ldots, k\}$ the set V_i is either a dominating set and $|V_i| = 1$, or forms a coalition with another set V_j. Haynes et al. initiated the study of the coalitions graphs in [14]. To describe coalition formation in $\pi(G)$, they associate with the partition its coalition graph $CG(G, \pi)$. Vertices of this graphs correspond to sets of the partition and two vertices are adjacent if and only if the corresponding sets form a coalition. A path is called coalition universal if its coalition partitions define all possible coalition graphs of paths. In [14], the authors demonstrated that there are only 18 coalition graphs of paths. Henning et al. [5] proved that there are no universal coalition paths and P_{10} is the shortest path that defines the maximal number of coalition graphs. Haynes et al. [14] showed that there are precisely 27 graphs of order at most 6 that can be coalition graphs of cycles and asked about the shortest cycle having the maximum number of coalition graphs. Dobrynin and Golmohammadi showed that C_{15} is the shortest cycle satisfying this property [10].

One of the important modifications of the coalition concept is the total coalition [1]. A total coalition in a graph G consists of two disjoint sets of vertices V_1 and V_2 of G, neither of which is a total dominating set but whose union is a total dominating set in G. Such sets V_1 and V_2 are said to form a total coalition. A total coalition partition is a vertex partition $\pi(G) = \{V_1, V_2, \ldots, V_k\}$ of $V(G)$ into non-total dominating sets, such that every set V_i forms a total coalition with another set V_j. The maximum cardinality of a total coalition partition is called the total coalition number of the graph, and denoted by $TC(G)$. The total coalition graph $TCG(G, \pi)$, defined by a total coalition partition π of a graph G, is constructed by the same way as the coalition graph $CG(G, \pi)$. This concept was recently introduced by Barát and Blázsik in [6]. To get some insights into results on the coalition and its variations in graphs, we refer to the survey articles [2, 3, 4, 9, 12, 13, 15, 20].

In this work, we show that the cycle C_{4k} is a universal total coalition cycle for $k \geq 2$ and the path P_n is a total coalition universal path for $n > 4$.

2 Main results

In this section we describe the total coalition graphs of paths and cycles. We begin with the following definitions.

Definition 1. *A path is called a universal total coalition path if its total coalition partitions define all possible total coalition graphs of cycles.*

Definition 2. *A cycle is called a universal total coalition cycle if its total coalition partitions define all possible total coalition graphs of paths.*

According to the following known result that gives the total coalition numbers of cycles, we realize that the number of total coalition graphs of cycles is finite.

Theorem 1. [1] *For any cycle C_n,*

$$TC(C_n) = \begin{cases} 4, & n \equiv 0 \ (mod\ 4) \\ 3, & \text{otherwise.} \end{cases}$$

The next result gives the maximum degree of a total coalition graph.

Lemma 1. [6] *The maximum degree of $TCG(G, \pi)$ cannot be greater than the maximum degree of G, i.e., $\Delta(TCG(G, \pi)) \leq \Delta(G)$.*

By Lemma 1, for each vertex v of a total coalition graph of cycles $1 \leq \deg(v) \leq 2$. Theorem 1 shows that the order of a total coalition graphs of the cycle P_n is at most 4 for $n \geq 4$. This implies that the possible total coalition graphs of the cycle C_n are K_2, P_3, K_3, $2K_2$, P_3, P_4, and C_4. It is easy to see that C_3 has the unique total coalition partition $\pi = \{\{v_1\}, \{v_2\}, \{v_3\}\}$ and $TCG(C_3, \pi) \cong C_3$. Further assume that C_n is a cycle of order $n \geq 4$. The number of total coalition graphs for cycles of small order can be enumerated by computer. Results of our computer calculations are collected in Table **??**.

TABLE 1. Number of total coalition graphs of C_n.

$TCG(C_n)$	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}	C_{14}	C_{15}	C_{16}	C_{17}	C_{18}
P_3	2	10	42	112	338	882	2350	6072	15638	39130	97762	243040	601218	1476450	3617502
K_2	1	5	16	35	81	180	391	825	1726	3575	7351	15020	30561	61965	125296
K_3	.	5	6	21	24	85	150	341	600	1365	2646	5461	10584	21845	43350
$2K_2$	.	.	.	.	64	.	.	.	1530	.	.	.	28864	.	.
P_4	.	.	.	.	32	.	.	.	390	.	.	.	3392	.	.
C_4	1	.	.	.	1	.	.	.	1	.	.	.	1	.	.

Proposition 1. *The cycle C_n defines the total coalition graphs K_2, P_3, and K_3 for $n \geq 5$.*

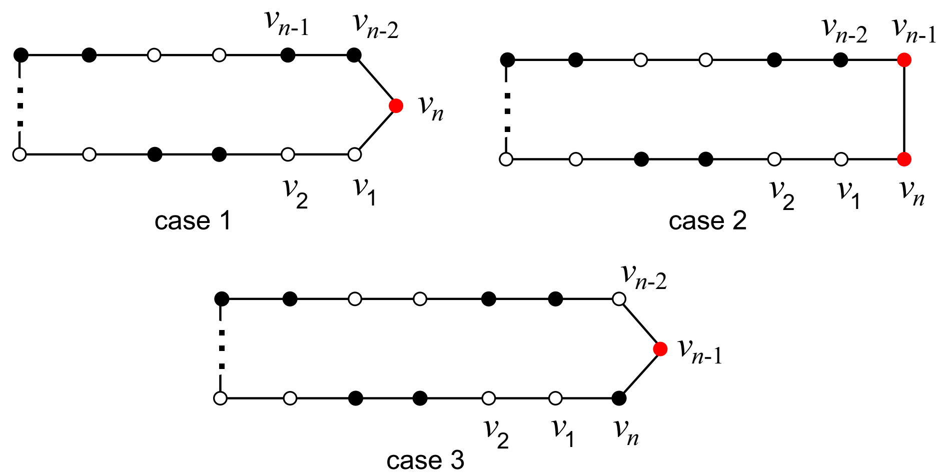

FIG. 1. Total coalition partitions of cycle C_n for graph K_3.

Proof. Let $V(C_n) = (v_1, v_2, \dots, v_n)$. To prove the proposition, we provide three total coalition partitions π_1, π_2, π_3 of $V(C_n)$ whose corresponding total coalition graphs are K_2, P_3 and K_3, respectively. Let $\pi_1 = \{V_1, V_2\}$, such that $V_1 = \{v_1, v_3\}$ and $V_2 = \{v_2, v_4, v_5, \dots, v_n\}$. It can be seen that none of two sets of π_1 is total dominating but whose union of $V_1 \cup V_2$ is a total coalition. Hence, $TCG(C_n, \pi_1) \cong K_2$.

Next assume that $\pi_2 = \{V_1, V_2, V_3\}$, such that $V_1 = \{v_1\}$, $V_2 = \{v_2, v_3\}$ and $V_3 = \{v_4, v_5 \dots v_n\}$. The sets V_1 and V_3 form a total coalition, as do the sets V_2 and V_3 while the sets V_1 and V_2 do not form a total coalition. Hence, $TCG(C_n, \pi_2) \cong P_3$.

Finally, let $\pi_3 = \{V_1, V_2, V_3\}$. We consider the following cases. Total coalition partitions for these cases are depicted in Fig. 1.

Case 1. Suppose that $n \equiv 1 \pmod 4$. Let $V_1 = \cup_{i=1}^{(n-1)/4}\{v_{4i-3}, v_{4i-2}\}$, $V_2 = \cup_{i=1}^{(n-1)/4}\{v_{4i-1}, v_{4i}\}$, and $V_3 = \{v_n\}$.

Case 2. Let $n \equiv 2 \pmod 4$. Suppose that $V_1 = \cup_{i=1}^{(n-2)/4}\{v_{4i-3}, v_{4i-2}\}$, $V_2 = \cup_{i=1}^{(n-2)/4}\{v_{4i-1}, v_{4i}\}$, and $V_3 = \{v_{n-1}, v_n\}$.

Case 3. Let $n \equiv 3 \pmod 4$ and $V_1 = \cup_{i=1}^{(n-3)/4}\{v_{4i-3}, v_{4i-2}\} \cup \{v_{n-2}\}$, $V_2 = \cup_{i=1}^{(n-3)/4}\{v_{4i-1}, v_{4i}\} \cup \{v_n\}$, and $V_3 = \{v_{n-1}\}$.

It not hard to verify that none of the sets of π_3 is a total dominating but each of the sets V_1 and V_2 form a total coalition with V_3. This implies $TCG(C_n, \pi_3) \cong K_3$. □

In the following, for a coalition partition $\{V_1, V_2, ..., V_k\}$, we will label all vertices of the set V_i with integers i for all $i = 1, 2, \dots, k$.

Proposition 2. *The cycle C_{4k} defines the total coalition graphs $2K_2$, P_4, and C_4 for $k \geq 2$.*

Proof. Let $V(C_n) = (v_1, v_2, \ldots, v_n)$. We first establish the total coalition partition π for C_{4k} whose corresponding total coalition graph is $2K_2$. Let $\pi = \{V_1, V_2, V_3, V_4\}$, such that $V_1 = \cup_{i=1}^{k} v_{4i-3}$, $V_2 = \cup_{i=1}^{k} v_{4i-2}$, $V_3 = \cup_{i=1}^{k} v_{4i-1}$ and $V_4 = \cup_{i=1}^{k} v_{4i}$ (see Fig. 2). It is easy to see that only $V_i \cup V_{i+1}$ for $i = 1, 2, 3$ and $V_1 \cup V_4$ form total coalitions. Hence, $TCG(C_{4k}, \pi) \cong C_4$.

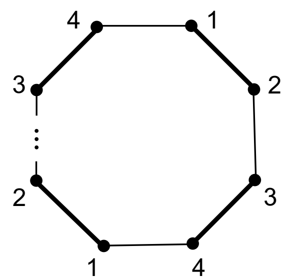

FIG. 2. Total coalition partition of cycle C_{4k} for graph C_4.

We now proceed to prove that C_{4k} generates the total coalition graph C_4. The vertex partition $\pi_1 = \{V_1 = \{v_1, v_2, v_5\}, V_2 = \{v_3, v_4, v_7\}, V_3 = \{v_6\}, V_4 = \{v_8\}\}$ is a total coalition partition of cycle C_8 in which only $V_1 \cup V_3$ and $V_2 \cup V_4$ form total coalitions (see vertex labeling in Fig. 3a). We also observe that each of $V_1 \cup V_2$, $V_1 \cup V_4$, $V_2 \cup V_3$ and $V_3 \cup V_4$ is not a total coalition as they contain a vertex that has no neighbors in its corresponding union. Then $TCG(C_8, \pi_1) \cong 2K_2$.

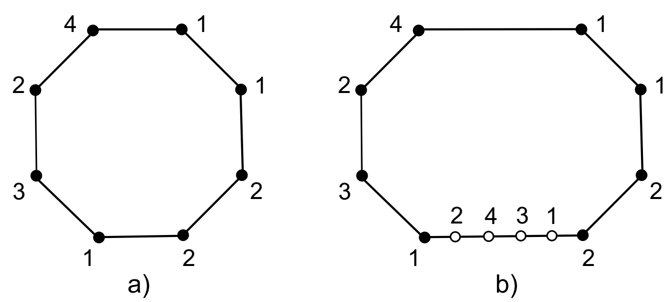

FIG. 3. Total coalition partitions of cycle C_8 (a) and cycle C_{12} (b) for graph $2K_2$.

Now we construct the total coalition partition of cycle C_{12} by adding four new (white) vertices between two vertices of C_8 labeled 1 and 2 as shown in Fig. 3b. Then a total coalition partition π_2 for C_{12} is constructed from π_1 by adding one white vertex to every set of π_1. Note that total dominating sets remain unchanged after adding the white vertices. Consequently, each of $V_1 \cup V_3$ and $V_2 \cup V_4$ are still the total dominating sets of cycle C_{12}. Hence, $TCG(C_{12}, \pi_2) \cong 2K_2$. Analogously, by inserting four new vertices to C_{12}, we get a total coalition partition π_3 of C_{16} and $TCG(C_{16}, \pi_3) \cong 2K_2$. If we continue in this manner, we conclude that $TCG(C_{4k}, \pi_{k-1}) \cong 2K_2$.

Finally, we show that C_{4k} defines the total coalition graph P_4 by applying the approach from the previous case. Let $\pi_1 = \{V_1 = \{v_1, v_2, v_5\}, V_2 = \{v_3\}, V_3 = \{v_4, v_7, v_8\}, V_4 = \{v_6\}\}$ is a total coalition partition of cycle C_8 in which only $V_1 \cup V_3$, $V_2 \cup V_4$ and $V_2 \cup V_3$ form total coalitions (see vertex labeling in Fig. 4a). It is easy to verify that each of $V_1 \cup V_2$, $V_1 \cup V_4$ and $V_3 \cup V_4$ is not a total coalition as they contain a vertex that has no neighbors in its corresponding union. Hence, $TCG(C_8, \pi_1) \cong P_4$.

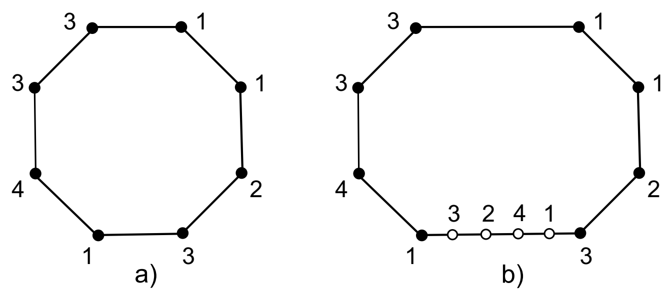

FIG. 4. Total coalition partitions of cycle C_8 (a) and cycle C_{12} (b) for graph P_4.

Now we construct the total coalition partition of cycle C_{12} by adding four new (white) vertices between two vertices of C_8 labeled 1 and 3 as illustrated in Fig. 4b. Note that total dominating sets remain unchanged after adding the white vertices. A total coalition partition π_2 of C_{12} is obtained by adding white vertices to the sets of π_1. Therefore, each of $V_1 \cup V_3$, $V_2 \cup V_4$ and $V_2 \cup V_3$ in π_2 are the total dominating sets of cycle C_{12}. Then, $TCG(C_{12}, \pi_2) \cong P_4$. By continuing this pattern, we infer that $TCG(C_{4k}, \pi_{k-1}) \cong P_4$. □

Propositions 1 and 2 lead to the next corollary.

Corollary 1. *The cycle C_{4k} is a universal total coalition cycle for $k \geq 2$.*

Now we turn our attention to total coalition graph of paths. Their total coalition numbers were determined in [1].

Proposition 3. [1] *For any path P_n of order $n \geq 3$, it holds that*

$$TC(P_n) = \begin{cases} 2, & \text{if } n = 4 \\ 3, & \text{otherwise.} \end{cases}$$

By Lemma 1 and Proposition 3, the possible total coalition graphs of the path P_n are K_2 and P_3 where $n \geq 3$. The number of total coalition graphs for paths of small order are presented in Table 2.

TABLE 2. Number of total coalition graphs of P_n.

$TCG(P_n)$	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}	P_{16}	P_{17}	P_{18}
K_2	1	4	11	23	48	103	217	448	919	1879	3824	7751	15669	31612	63667	128047
P_3	1	.	3	12	30	84	239	620	1564	3976	10033	24948	61622	151844	372851	912084

Proposition 4. *The path P_n defines the coalition graphs K_2 and P_3 for $n \geq 5$.*

Proof. Let $V(P_n) = (v_1, v_2, \ldots, v_n)$. To prove the proposition, we provide two total coalition partitions π_1, π_2 for P_n whose the corresponding total coalition graphs are K_2 and P_3, respectively. Let $\pi_1 = \{A, B\}$, such that $A = \cup_{i=1}^{n} v_{2i-1}$ and $B = \cup_{i=1}^{n} v_{2i}$. We observe that the union of sets A and B forms a total coalition. Hence, $TCG(P_n, \pi_1) \cong K_2$.

Next assume that the partition $\pi_2 = \{C, D, E\}$, such that $C = \{v_1\}$, $D = \{v_3\}$ and $E = \{v_2, v_4, \ldots, v_n\}$. The set E forms a total coalition with the set C or D while the sets C and D do not form a total coalition. Therefore, $TCG(P_n, \pi_2) \cong P_3$. □

As a consequence of Proposition 4, we get the following result.

Corollary 2. *The path P_n is a universal total coalition path for $n \geq 5$.*

In conclusion, we state the following open problem.

Problem 1. *Characterize the total coalition graphs of trees.*

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