Direct Equivalence Problem on Fifth-order Differential Operator

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Abstract

The main objective of this paper is to investigate the equivalence problem for fifth-order differential operators on the line. We specifically focus on the case where the differential operators are subjected to general fiber-preserving transformations. To tackle this problem, we employ the Cartan method of equivalence via direct equivalence problem. This method allows us to determine whether two given differential operators are equivalent or not under a certain transformation. By applying this method, we are able to establish the conditions under which two fifth-order differential operators are equivalent under general fiber-preserving transformations. This provides us with a comprehensive understanding of the equivalence problem for these operators on the line. Overall, this paper contributes to the field of differential equations by shedding light on the equivalence problem for fifth-order operators and providing a systematic approach to analyze their equivalence under general fiber-preserving transformations.

Keywords: equivalence problem; fifth-order differential operators; direct method; fiber-preserving transformation.

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1 Introduction

Cartan's method of equivalence is a highly effective approach employed in the analysis and resolution of equivalence problems in differential geometry. It offers a systematic approach to establishing the requisite and adequate conditions for the equivalence of two geometric structures [12, 13, 14]. In the realm of G-structures, Cartan's method enables us to associate differential invariants with the structure. These invariants are quantities that remain invariant under specific transformations, thereby facilitating the differentiation between distinct geometric structures.

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Cartan's method has been widely employed to successfully address various equivalence problems, including the case of two second-order partial differential equations subjected to contact transformations. Notably, several of Cartan's students, such as Chern [7] and Hachtroudi [9], among others, have effectively utilized this method. In more recent times, the modern approach to equivalence problems has been advanced by mathematicians like Gardner [8], Kamran [11], and Olver [11, 16]. Their contributions have further refined and expanded Cartan's method, rendering it more accessible and applicable to a broader array of problems in differential geometry.

The equivalence problem for fifth-order differential operators pertains to determining whether two fifth-order operators on a real line can be transformed into each other through an appropriate change of variables. Extensive research has been conducted on this problem in the literature [4, 5, 6, 7, 8].

There are two distinct versions of the equivalence problem. The first version, known as the direct equivalence problem, focuses on determining whether two differential operators can be transformed into each other under a fiber-preserving transformation, which is the subject of investigation in this paper. The second version is referred to as the gauge method equipped with gauge transformation, which has been explored in the study conducted by [3].

In order to address the equivalence problem, a set of one-forms is associated with the object being studied in the original coordinates. Similarly, the object in the new coordinates will have its own set of one-forms. By reformulating the equivalence problem using the appropriate Cartan form, which involves a coframe ω on the m-dimensional base manifold M and a structured group $G \subset \mathrm{GL}(m)$, the Cartan equivalence method can be applied. The objective is to standardize the coefficients of the structure group in a manner that is suitably invariant. This is achieved by determining a sufficient number of invariant combinations of these coefficients [10, 17].

The classification of linear differential equations can be viewed as a specific instance of the broader task of classifying differential operators. This task holds significant importance across various disciplines, such as quantum mechanics and the projective geometry of curves [17]. However, when it comes to fifth-order differential operators, the equivalence problem presents a distinct challenge due to the unique geometry associated with these operators.

In recent times, significant progress has been made in addressing the equivalence problem for differential operators. Notably, Niky Kamran and Peter J. Olver have successfully solved the equivalence problem for second-order differential operators. Additionally, numerous papers have been published discussing the equivalence of differential operators [2, 11, 16]. These advancements have contributed to a deeper understanding of the subject.

We have divided our exposition into four sections, beginning with this introduction. In Section 2, we provide a brief explanation of the Cartan equivalence method, which serves as the foundation for our analysis. This method allows us to establish equivalence between different differential operators and provides a powerful tool for studying the behavior of fifth-order ordinary differential equations.

Moving forward, Section 3 presents our novel approach of examining the equivalence of fifth-order differential operators under fiber preserving transformations using a direct method. We demonstrate the significance of this approach in understanding the structural properties of these operators and how they relate to physical systems. By employing this direct method, we can unravel important insights into the behavior and characteristics of fifth-order differential equations, enhancing our understanding of the underlying physical phenomena.

In Section 4, we delve into the outcomes of our analysis and explore their significance within the context of the physical applications we introduced earlier. We illustrate the practical application of the equivalence problem by considering a specific example—a boundary value problem involving a fifth-order differential operator with an integrable potential. Through this example, we showcase how the equivalence problem can be utilized to effectively solve differential equations. Our comprehensive analysis and solution aim to make a valuable contribution to the field by enhancing our understanding and advancing the existing knowledge base.

Boundary value problems for fifth-order ordinary differential equations have significant physical applications in various branches of science and engineering. These problems arise in situations where the behavior of a physical system can be described by a fifth-order differential equation and requires the determination of the unknown function subject to specified boundary conditions. Understanding and solving these boundary value problems are essential for gaining insights into the behavior of physical systems and for developing accurate mathematical models that can be used for analysis, design, and optimization, [1, 18].

2 The Cartan Equivalence Method

Cartan's equivalence problem is an algorithmic method that involves the structure equations, normalization, and absorption, as described in the standard reference [17].

The G-valued equivalence problem arises when considering coframes ω and $\overline{\omega}$ defined on the m-dimensional manifolds M and $\overline{M}$, respectively. The problem is to determine whether there exists a local diffeomorphism $\Phi : M \to \overline{M}$ and a G-valued function $g : M \to G$ such that

$$\Phi^*(\overline{\omega}) = g(x)\omega. \tag{1}$$

In a more detailed formulation, the equivalence condition can be expressed as

$$\Phi^*(\overline{\omega}^i) = \sum_{j=1}^{m} g^i_j(x)\omega^j, \tag{2}$$

for $i = 1, \cdots, m$, where $g^i_j(x)$ represents the entries of the matrix $g(x)$. It is important to note that the matrix $g(x)$ is constrained to belong to the structure group G at each point $x \in M$.

By exploiting the group property of G, the equivalence condition can be satisfied if and only if there exist a pair of G-valued functions $\overline{g}(\overline{x})$ and $g(x)$ such that, without explicit pull-back notation, $\overline{g}(\overline{x})\overline{\omega} = g(x)\omega$.

Our objective is to simplify a given G-equivalence problem by transforming it into a standard equivalence problem for coframes. To achieve this, we can achieve this by expressing the matrix entries of $g = g(x)$ and $\overline{g}(\overline{x})$ as functions of their respective coordinates. By doing so, we can define new coframes as follows:

$$\theta^i = \sum_{j=1}^{m} g^i_j\omega^j, \qquad \overline{\theta}^i = \sum_{j=1}^{m} \overline{g}^i_j\overline{\omega}^j, \tag{3}$$

These new coframes are designed to remain invariant under the transformation Φ, satisfying the condition $\Phi^*(\overline{\theta}^i) = \theta^i$. This concept serves as the initial step in the Cartan solution to the equivalence problem. To implement this step, we introduce the lifted coframe:

$$\theta = g.\omega, \tag{4}$$

More explicitly, this can be expressed as:

$$\theta^i = \sum_{j=1}^{m} g^i_j(x)\omega^j. \tag{5}$$

We calculate the differentials of the lifted coframe elements by taking the derivative of the expression $\sum_{j=1}^{m} g^i_j\omega^j$. This gives us the equation

$$d\theta^i = d\Big(\sum_{j=1}^{m} g^i_j\omega^j\Big) = \sum_{j=1}^{m}\{dg^i_j \wedge \omega^j + g^i_j d\omega^j\}. \tag{6}$$

Since the ω forms a coframe on M, we can rewrite the 2-forms $d\omega^j$ as sums of wedge products of the ω^i's. Furthermore, using equation (4), we can express these wedge products in terms of the θ^k's. This gives us the equation

$$d\theta^i = \sum_{j=1}^{m} \gamma_j^i \wedge \theta^j + \sum_{\substack{j,k=1 \\ j<k}}^{m} T_{jk}^i(x,g)\theta^j \wedge \theta^k, \quad i = 1, \ldots, m. \tag{7}$$

The quantities T_{jk}^i are referred to as torsion coefficients. These coefficients can either remain constant or vary with the base variables x or the group parameters g. While some of the torsion coefficients may be considered invariants, they are generally not invariants for the specific problem being addressed. The 1-forms denoted by γ_j^i in equation (7) can be expressed as follows:

$$\gamma_j^i = \sum_{k=1}^{m} dg_k^i (g^{-1})_j^k. \tag{8}$$

These 1-forms can also be represented in matrix notation as:

$$\gamma = dg \cdot g^{-1}. \tag{9}$$

The matrix γ represents the Maurer-Cartan forms on the structure group G. If we assume that $\{\alpha^1, \ldots, \alpha^r\}$ forms a basis for the space of Maurer-Cartan forms, then each γ_j^i can be expressed as a linear combination of the Maurer-Cartan basis:

$$\gamma_j^i = \sum_{l=1}^{r} A_{jl}^i \alpha^l, \qquad i,j = 1, \ldots, m. \tag{10}$$

Consequently, the final structure equations for our lifted coframe, expressed in terms of the Maurer-Cartan forms, exhibit a generalized form as follows:

$$d\theta^i = \sum_{l=1}^{r}\sum_{j=1}^{m} A_{jl}^i \alpha^l \wedge \theta^j + \sum_{\substack{j,k=1 \\ j<k}}^{m} T_{jk}^i(x,g)\theta^j \wedge \theta^k, \quad i = 1, \ldots, m. \tag{11}$$

In order to bring the Maurer-Cartan forms α^l back to the base manifold M, we can simplify them by substituting general linear combinations of coframe elements. This substitution can be represented as:

$$\alpha^l \mapsto \sum_{l=1}^{r} z_j^l \theta^j, \tag{12}$$

here, the coefficients z_j^l can be freely chosen, allowing for flexibility in the representation of the Maurer-Cartan forms on the base manifold.

By replacing (12) in the structure equations (11), the resulting system of 2-forms can be expressed as follows:

$$\Theta^i = \sum_{\substack{j,k=1 \\ j<k}}^{m} \{B_{jk}^i[\mathbf{z}] + T_{jk}^i(x,g)\}\theta^j \wedge \theta^k, \quad i,j = 1, \ldots, m, \tag{13}$$

where

$$B_{jk}^i[\mathbf{z}] = \sum_{l=1}^{r} (A_{kl}^i z_j^l - A_{jl}^i z_k^l). \tag{14}$$

Here, the coefficients $B_{jk}^i[\mathbf{z}]$ in equation (14) represent linear functions of the coefficients $\mathbf{z} = (z_k^l)$. These functions are determined by the specific representation of the structure group $G \subset \mathrm{GL}(m)$ and do not rely on the coordinate system. The constants coefficients of these linear functions are obtained through the specific representation of the structure group

The process of determining the unknown coefficients $\mathbf{z}$ from the full torsion coefficients is referred to as *absorption of torsion* while the resulting invariant torsion coefficients are normalized. This involves replacing each Maurer-Cartan form α^l with a modified 1-form π^l, given by the equation

$$\pi^l = \alpha^l - \sum_{i=1}^{p} z_i^l \theta^i, \qquad l = 1, \dots, r, \tag{15}$$

The absorption process, as described in equation (11), allows for the absorption of the inessential torsion in the structure equation. The solutions to the absorption equations, denoted as $z_i^l = z_i^l(x, g)$, provide the necessary coefficients for this absorption. Hence, the structure equations are transformed into the simpler absorbed form:

$$d\theta^i = \sum_{l=1}^{r} \sum_{j=1}^{m} A_{jl}^i \pi^l \wedge \theta^j + \sum_{\substack{j,k=1 \\ j<k}}^{m} U_{jk}^i \theta^j \wedge \theta^k, \quad i, j = 1, \dots, m. \tag{16}$$

In addition, we can express the remaining nonzero coefficients U_{jk}^i solely in terms of essential torsion. This allows us to formulate the absorption equations as following linear system

$$\sum_{l=1}^{r} (A_{jl}^i z_k^l - A_{kl}^i z_j^l) = -T_{jk}^i, \tag{17}$$

and find the solutions for the unknowns $\mathbf{z}$ by applying the conventional Gaussian elimination technique.

3 Equivalence of Fifth Order Differential Operators

Let's consider two fifth-order differential operators applied to scalar-valued functions. The first operator, denoted as $\mathcal{D}[u]$, acts on the function $u(x)$. It can be expressed as the sum of terms, where each term involves a coefficient $f_i(x)$ multiplied by the i-th derivative of u, represented as $D^i u$. Mathematically, we have:

$$\mathcal{D}[u] = \sum_{i=0}^{5} f_i(x)\, D^i u \tag{18}$$

Similarly, the second operator, denoted as $\bar{\mathcal{D}}[\bar{u}]$, acts on the function $\bar{u}(\bar{x})$. It can also be expressed as the sum of terms, where each term involves a coefficient $\bar{f}_i(\bar{x})$ multiplied by the i-th derivative of $\bar{u}$, represented as $\bar{D}^i \bar{u}$. Mathematically, we have:

$$\bar{\mathcal{D}}[\bar{u}] = \sum_{i=0}^{5} \bar{f}_i(\bar{x})\, \bar{D}^i \bar{u} \tag{19}$$

Here, f_i and $\bar{f}_i$ are analytic functions of the real variables x and $\bar{x}$, respectively. For simplicity, we assume that $f_5 = \bar{f}_5 = 1$. Additionally, the operators D^i and $\bar{D}^i$ represent the i-th derivative with respect to x and $\bar{x}$, respectively. The operators D^0 and $\bar{D}^0$ correspond to the identity operators. To analyze the equivalence between the two differential operators defined in equations (18) and (19), it is appropriate to work in the fifth jet space J^5. This space has local coordinates given by

$$\Upsilon = \{(x, u, p, q, r, s, t) \in \mathrm{J}^5 : p = u_x, q = u_{xx}, r = u_{xxx}, s = u_{xxxx}, t = u_{xxxxx}\}$$

.

The objective is to determine whether there exists a suitable transformation of variables

$$(x, u, p, q, r, s, t) \longrightarrow (\bar{x}, \bar{u}, \bar{p}, \bar{q}, \bar{r}, \bar{s}, \bar{t})$$

that can bring equation (18) to equation (19). Among the various types of transformations, we will focus on fiber-preserving transformations, which can be represented as:

$$\bar{x} = \xi(x), \qquad \bar{u} = \varphi(x)\, u, \tag{20}$$

Here, $\varphi(x)$ represents a non-zero function. By applying the chain rule formula, we can establish the relationship between the total derivative operators:

$$\bar{D} = \frac{d}{d\bar{x}} = \frac{1}{\xi'(x)} \frac{d}{dx} = \frac{1}{\xi'(x)} D. \tag{21}$$

In this paper, we focus on the *direct equivalence problem*, which aims to determine the equivalence between two linear differential functions:

$$\mathcal{D}[u] = \bar{\mathcal{D}}[\bar{u}]. \tag{22}$$

This equivalence is studied under the change of variables given by equation (20). This change of variables induces a transformation rule on the differential operators themselves:

$$\bar{\mathcal{D}} = \mathcal{D} \cdot \frac{1}{\varphi(x)} \qquad \text{when} \qquad \bar{x} = \xi(x). \tag{23}$$

To solve the local direct equivalence problem, we aim to find explicit conditions on the coefficients of the two differential operators that guarantee they satisfy equation (22) for some change of variables of the form (20).

Proposition 1 *Consider two fifth-order differential operators denoted by $\mathcal{D}$ and $\bar{\mathcal{D}}$. On open subsets Γ and $\bar{\Gamma}$ of the fifth jet space, there exist two sets of coframes $\Omega = \{\omega^i\}_{i=1}^7$ and $\bar{\Omega} = \{\bar{\omega}^j\}_{j=1}^7$, respectively. These coframes are defined such that the differential operators $\mathcal{D}$ and $\bar{\mathcal{D}}$ are equivalent under the pseudogroup defined in equation (20). The coframes Ω and $\bar{\Omega}$ satisfy the following relation:*

$$\begin{pmatrix} \bar{\omega}^1 \\ \bar{\omega}^2 \\ \bar{\omega}^3 \\ \bar{\omega}^4 \\ \bar{\omega}^5 \\ \bar{\omega}^6 \\ \bar{\omega}^7 \end{pmatrix} = \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_2 & a_3 & 0 & 0 & 0 & 0 \\ 0 & a_4 & a_5 & a_6 & 0 & 0 & 0 \\ 0 & a_7 & a_8 & a_9 & a_{10} & 0 & 0 \\ 0 & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \\ \omega^6 \\ \omega^7 \end{pmatrix} \tag{24}$$

where $a_i \in \mathbb{R}$ for $i = 1, \cdots, 15$ and $a_1 a_3 a_6 a_{10} a_{15} \neq 0$. This relation ensures that the differential operators $\mathcal{D}$ and $\bar{\mathcal{D}}$ are equivalent under the pseudogroup defined in equation (20) and transformed according to the rule given in equation (23).

Proof. It is important to note that a point transformation will be in the desired linear form (20) if and only if the following one-form equations hold for the functions $\alpha = \xi_x$ and $\beta = \varphi_x/\varphi$:

$$d\bar{x} = \alpha\, dx, \tag{25}$$

$$\frac{d\bar{u}}{\bar{u}} = \frac{du}{u} + \beta\, dx. \tag{26}$$

These equations are valid on the subset of J^5 where $u \neq 0$. To ensure that the derivative variables p, q, r, s and t transform correctly, it is necessary to preserve the contact ideal $\mathcal{I}$ on J^5, which is given by:

$$\mathcal{I} = \langle du - p\,dx, dp - q\,dx, dq - r\,dx, dr - s\,dx, ds - t\,dx\rangle. \tag{27}$$

Typically, a diffeomorphism $\Phi : \mathrm{J}^5 \to \mathrm{J}^5$ establishes a contact transformation if and only if

$$\begin{aligned}
d\bar{u} - \bar{p}\,d\bar{x} &= a_1(du - p\,dx), && (28)\\
d\bar{p} - \bar{q}\,d\bar{x} &= a_2(du - p\,dx) + a_3(dp - q\,dx), && (29)\\
d\bar{q} - \bar{r}\,d\bar{x} &= a_4(du - p\,dx) + a_5(dp - q\,dx) + a_6(dq - r\,dx), && (30)\\
d\bar{r} - \bar{s}\,d\bar{x} &= a_7(du - p\,dx) + a_8(dp - q\,dx) + a_9(dq - r\,dx) + a_{10}(dr - s\,dx), && (31)\\
d\bar{s} - \bar{t}\,d\bar{x} &= a_{11}(du - p\,dx) + a_{12}(dp - q\,dx) + a_{13}(dq - r\,dx) + a_{14}(dr - s\,dx) && (32)\\
&\quad + a_{15}(ds - t\,dx),
\end{aligned}$$

where a_i are functions on J^5. The combination of the first contact condition (28) with the linearity conditions (25) and (26) is a crucial component of an overdetermined equivalence problem. By setting $\beta = -p/u$ and $a_1 = 1/u$ in equations (26) and (28), we obtain the following one-form:

$$\frac{d\bar{u} - \bar{p}\,d\bar{x}}{\bar{u}} = \frac{du - p\,dx}{u}. \tag{33}$$

This one-form is invariant, and it can serve as a replacement for both equations (26) and (28). Hence, we can select six elements of our coframe as the following one-forms:

$$\omega^1 = dx,\ \omega^2 = \frac{du - p\,dx}{u},\ \omega^3 = dp - q\,dx,\ \omega^4 = dq - r\,dx,\ \omega^5 = dr - s\,dx,\ \omega^6 = ds - t\,dx. \tag{34}$$

These one-forms are defined on the fifth jet space J^5, which is locally parameterized by (x, u, p, q, r, s, t). They have the following transformation rules:

$$\begin{aligned}
\bar{\omega}^1 &= a_1\omega^1,\\
\bar{\omega}^2 &= \omega^2,\\
\bar{\omega}^3 &= a_2\omega^2 + a_3\omega^3,\\
\bar{\omega}^4 &= a_4\omega^2 + a_5\omega^3 + a_6\omega^4,\\
\bar{\omega}^5 &= a_7\omega^2 + a_8\omega^3 + a_9\omega^4 + a_{10}\omega^5,\\
\bar{\omega}^6 &= a_{11}\omega^2 + a_{12}\omega^3 + a_{13}\omega^4 + a_{14}\omega^5 + a_{15}\omega^6.
\end{aligned} \tag{35}$$

Based on equation (22), the function $I(x, u, p, q, r, s, t) = \mathcal{D}[u] = t + f_4(x)s + f_3(x)r + f_2(x)q + f_1(x)p + f_0(x)u$ is an invariant for the problem. Therefore, its differential can be expressed as:

$$\omega^7 = dI = dt + f_4 ds + f_3 dr + f_2 dq + f_1 dp + f_0 du + (f_4' s + f_3' r + f_2' q + f_1' p + f_0' u)dx, \tag{36}$$

This differential is an invariant one-form, and as such, it can be considered as the final element of our coframe.

The set of one-forms

$$\Omega = \{\omega^1, \omega^2, \omega^3, \omega^4, \omega^5, \omega^6, \omega^7\}$$

forms a coframe on the subset

$$\Gamma^* = \Big\{(x, u, p, q, r, s, t) \in \mathrm{J}^5 \,\Big|\, u \neq 0 \text{ and } f_5(x) \neq 0\Big\}. \tag{37}$$

We focus our attention on a connected component $\Gamma \subset \Gamma^*$ of the subset (37), where the signs of $f_0(x)$ and u are fixed. This means that the final element of the coframe, ω^7, is preserved up to contact:

$$\bar{\omega}^7 = \omega^7. \tag{38}$$

By considering equations (35) and (38), one can find the structure group associated with the equivalence problem (23) that is a 15-dimensional matrix group G such that $\bar{\Omega} = G\Omega$ which leads to (24) and then the *lifted coframe* on the space $\mathrm{J}^5 \times G$ has the form

$$\begin{aligned}
\theta^1 &= a_1\omega^1, \\
\theta^2 &= \omega^2, \\
\theta^3 &= a_2\omega^2 + a_3\omega^3, \\
\theta^4 &= a_4\omega^2 + a_5\omega^3 + a_6\omega^4, \\
\theta^5 &= a_7\omega^2 + a_8\omega^3 + a_9\omega^4 + a_{10}\omega^5, \\
\theta^6 &= a_{11}\omega^2 + a_{12}\omega^3 + a_{13}\omega^4 + a_{14}\omega^5 + a_{15}\omega^6, \\
\theta^7 &= \omega^7.
\end{aligned} \tag{39}$$

We summarize the results in the following theorem:

Theorem 1 *The final structure equations for direct equivalence, using the coframes given by equations (34) and (36), can be expressed as follows:*

$$\begin{aligned}
d\theta^1 &= \frac{1}{5}\theta^1 \wedge \theta^2, \\
d\theta^2 &= \theta^1 \wedge \theta^3, \\
d\theta^3 &= \theta^1 \wedge \theta^4 + \frac{1}{5}\theta^2 \wedge \theta^3, \\
d\theta^4 &= I_1\theta^1 \wedge \theta^4 + \theta^1 \wedge \theta^5 + \frac{2}{5}\theta^2 \wedge \theta^4, \\
d\theta^5 &= I_2\theta^1 \wedge \theta^4 + \theta^1 \wedge \theta^6 + \frac{3}{5}\theta^2 \wedge \theta^5 + \frac{17}{5}\theta^3 \wedge \theta^4, \\
d\theta^6 &= I_3\theta^1 \wedge \theta^2 + I_4\theta^1 \wedge \theta^3 + I_5\theta^1 \wedge \theta^4 + \theta^1 \wedge \theta^7 + \frac{4}{5}\theta^2 \wedge \theta^6 + I_6\theta^3 \wedge \theta^4 + 4\theta^3 \wedge \theta^5, \\
d\theta^7 &= 0.
\end{aligned} \tag{40}$$

Here, the coefficients I_i with $i = 1, \ldots, 6$ are represented as

$$\begin{aligned}
I_1 &= -\frac{1}{\sqrt[5]{u^4}}\left[f_4u + 3p\right], \\
I_2 &= \frac{1}{5\sqrt[5]{u^8}}\left[(10\dot{f}_4u^2 - 12f_4pu - 5f_3u^2 - 9p^2 - 10qu\right], \\
I_3 &= -(f_0u + f_1p + f_2q + f_3r + f_4s + t), \\
I_4 &= -\frac{1}{625\sqrt[5]{u^{16}}}\left[625u^4f_1 - 800u^2f_4pq + 2375u^3f_4r + 1770p^2qu - 1275pru^2 + 3000su^3\right. \\
&\quad \left. + 270f_4p^3u - 225u^2f_3p^2 + 1750u^3f_3q + 1125u^3f_2p - 594p^4 - 800q^2u^2\right], \\
I_5 &= 7 - \frac{1}{25\sqrt[5]{u^{12}}}\left[25u^3f_2 + 6up^2f_4 + 65u^2qf_4 - 55pu^2\dot{f}_4 + 50f_3pu^2 - 25u^3\dot{f}_3\right. \\
&\quad \left. + 25u^3\ddot{f}_4 + 33p^3 - 45pqu + 100ru^2\right], \\
I_6 &= -\frac{1}{\sqrt[5]{u^4}}(f_4u + 3p).
\end{aligned} \tag{41}$$

4 The Proof of Theorem 1

By utilizing the initial six one-forms (34) and (36) as our final coframe constituents, we effectively convert the equivalence problem into a G-equivalence problem. Subsequently, we proceed to normalize the problem, resulting in the derivation of (39). Instead of using the six one-forms (34) and (36), we employ (39) as our basis. The objective is to transform the basis manifold $M \times G$ into $M \times G'$, where dim $G' <$ dim G, and ultimately achieve the transformation of G' into the identity element e. The algorithm for this problem consists of five stages, beginning with the computation of the right-invariant Maurer-Cartan form on the Lie group G. To proceed, we compute the expression $dg \cdot g^{-1}$, which allows us to calculate the structure equations by differentiating (39). We then write the result using the right-invariant Maurer-Cartan form and (39). Next, we employ the absorption algorithm to determine the coefficient of (39) that does not depend on the variable z. This coefficient corresponds to the invariant quantity, and the corresponding torsion coefficient remains invariant as well. Now we compute the differentials of the lifted coframe elements (39). Through explicit computation, we obtain the following structure equations:

$$\begin{aligned}
d\theta^1 &= \alpha^1 \wedge \theta^1, \\
d\theta^2 &= T^2_{12}\theta^1 \wedge \theta^2 + T^2_{13}\theta^1 \wedge \theta^3, \\
d\theta^3 &= \alpha^2 \wedge \theta^2 + \alpha^3 \wedge \theta^3 + T^3_{12}\theta^1 \wedge \theta^2 + T^3_{13}\theta^1 \wedge \theta^3 + T^3_{14}\theta^1 \wedge \theta^4, \\
d\theta^4 &= \alpha^4 \wedge \theta^2 + \alpha^5 \wedge \theta^3 + \alpha^6 \wedge \theta^4 + T^4_{12}\theta^1 \wedge \theta^2 + T^4_{13}\theta^1 \wedge \theta^3 + T^4_{14}\theta^1 \wedge \theta^4 + T^4_{15}\theta^1 \wedge \theta^5, \\
d\theta^5 &= \alpha^7 \wedge \theta^2 + \alpha^8 \wedge \theta^3 + \alpha^9 \wedge \theta^4 + \alpha^{10} \wedge \theta^5 + T^5_{12}\theta^1 \wedge \theta^2 + T^5_{13}\theta^1 \wedge \theta^3 + T^5_{14}\theta^1 \wedge \theta^4 \\
&\quad + T^5_{15}\theta^1 \wedge \theta^5 + T^5_{16}\theta^1 \wedge \theta^6, \\
d\theta^6 &= \alpha^{11} \wedge \theta^2 + \alpha^{12} \wedge \theta^3 + \alpha^{13} \wedge \theta^4 + \alpha^{14} \wedge \theta^5 + \alpha^{15} \wedge \theta^6 + T^6_{12}\theta^1 \wedge \theta^2 + T^6_{13}\theta^1 \wedge \theta^3 \\
&\quad + T^6_{14}\theta^1 \wedge \theta^4 + T^6_{15}\theta^1 \wedge \theta^5 + T^6_{16}\theta^1 \wedge \theta^6 + T^6_{17}\theta^1 \wedge \theta^7, \\
d\theta^7 &= 0,
\end{aligned} \tag{42}$$

The α^i, where $i = 1, \cdots, 15$, form a basis for the right-invariant Maurer-Cartan forms on the Lie group G. The essential torsion coefficients in the first loop can be expressed as:

$$T^2_{12} = -\frac{a_2 + a_3 p}{a_1 a_3 u}, \quad T^2_{13} = \frac{1}{a_1 a_3 u}, \quad T^3_{14} = \frac{a_3}{a_1 a_6}, \quad T^4_{15} = \frac{a_6}{a_1 a_{10}}, \quad T^5_{16} = \frac{a_{10}}{a_1 a_{15}}, \quad T^6_{17} = \frac{a_{15}}{a_1}. \tag{43}$$

To achieve a normalization of the group parameters, one can set

$$a_1 = \frac{1}{\sqrt[5]{u}}, \quad a_2 = -\frac{p}{\sqrt[5]{u^4}}, \quad a_3 = \frac{1}{\sqrt[5]{u^4}}, \quad a_6 = \frac{1}{\sqrt[5]{u^3}}, \quad a_{10} = \frac{1}{\sqrt[5]{u^2}}, \quad a_{15} = \frac{1}{\sqrt[5]{u}}. \tag{44}$$

During the second iteration, the normalization (44) is incorporated into the lifted coframe (39). The differentials of the new invariant coframe are then calculated to derive the revised structure equations, as shown below:

$$
\begin{aligned}
d\theta^1 &= \frac{1}{5}\theta^1 \wedge \theta^2, \\
d\theta^2 &= \theta^1 \wedge \theta^3, \\
d\theta^3 &= T^2_{12}\theta^1 \wedge \theta^2 + T^3_{13}\theta^1 \wedge \theta^3 + \theta^1 \wedge \theta^4 + \frac{1}{5}\theta^2 \wedge \theta^3, \\
d\theta^4 &= T^4_{12}\theta^1 \wedge \theta^2 + T^4_{13}\theta^1 \wedge \theta^3 + T^4_{14}\theta^1 \wedge \theta^4 + \theta^1 \wedge \theta^5 + \frac{4}{5}\alpha^5\theta^2 \wedge \theta^3 - \frac{3}{5}\theta^2 \wedge \theta^4 \\
&\quad + \alpha^4\theta^2 + \alpha^5\theta^3, \\
d\theta^5 &= T^5_{12}\theta^1 \wedge \theta^2 + T^5_{13}\theta^1 \wedge \theta^3 + T^5_{14}\theta^1 \wedge \theta^4 + T^5_{15}\theta^1 \wedge \theta^5 + \theta^1 \wedge \theta^6 \\
&\quad + T^5_{23}\theta^2 \wedge \theta^3 - \frac{1}{5}\alpha^9\theta^2 \wedge \theta^4 - \frac{2}{5}\theta^2 \wedge \theta^5 + \alpha^7 \wedge \theta^2 \\
&\quad + \alpha^8 \wedge \theta^3 + \alpha^9 \wedge \theta^4 \\
d\theta^6 &= T^6_{12}\theta^1 \wedge \theta^2 + T^6_{13}\theta^1 \wedge \theta^3 + T^6_{14}\theta^1 \wedge \theta^4 + T^6_{15}\theta^1 \wedge \theta^5 + T^6_{16}\theta^1 \wedge \theta^6 \\
&\quad + \theta^1 \wedge \theta^7 + T^6_{23}\theta^2 \wedge \theta^3 + T_{24}\theta^2 \wedge \theta^4 - \frac{1}{5}\alpha^{14}\theta^2 \wedge \theta^5 - \frac{1}{5}\theta^2 \wedge \theta^6 \\
&\quad + \alpha^{11} \wedge \theta^2 + \alpha^{12} \wedge \theta^3 + \alpha^{13} \wedge \theta^4 + \alpha^{14} \wedge \theta^5, \\
d\theta^7 &= 0.
\end{aligned} \tag{45}
$$

The essential torsion components (45) can be described as

$$
\begin{aligned}
T^3_{12} &= -\frac{a_4\sqrt[5]{u^{23}} + qu^4}{\sqrt[5]{u^{23}}}, & T^3_{13} &= -\frac{5a_5\sqrt[5]{u^9} + 9pu}{5\sqrt[5]{u^9}}, \\
T^5_{15} &= -\frac{5a_{14}\sqrt[5]{u^4} - 5a_9\sqrt[5]{u^4} + 2p}{5\sqrt[5]{u^4}}, & T^6_{16} &= \frac{-5f_4u + 5a_{14}\sqrt[5]{u^4} - p}{5\sqrt[5]{u^4}}.
\end{aligned} \tag{46}
$$

The normalization can be defined as

$$
a_4 = -\frac{q}{\sqrt[5]{u^3}}, \quad a_5 = -\frac{9p}{5\sqrt[5]{u^4}}, \quad a_9 = \frac{5f_4u + 3p}{5\sqrt[5]{u^4}}, \quad a_{14} = \frac{5f_4u + p}{5\sqrt[5]{u^4}}. \tag{47}
$$

During the third loop, we substitute the normalization (47) into the lifted coframe (39) and determine the values of parameters a_7, a_8, a_{13}. Afterward, we proceed to recalculate the differentials. As a result, the new structure equations take on the following form:

$$
\begin{aligned}
d\theta^1 &= \frac{1}{5}\theta^1 \wedge \theta^2, \\
d\theta^2 &= \theta^1 \wedge \theta^3, \\
d\theta^3 &= \theta^1 \wedge \theta^4 + \frac{1}{5}\theta^2 \wedge \theta^3, \\
d\theta^4 &= T^4_{12}\theta^1 \wedge \theta^2 + T^4_{13}\theta^1 \wedge \theta^3 + T^4_{14}\theta^1 \wedge \theta^4 + \theta^1 \wedge \theta^5 + \frac{2}{5}\theta^2 \wedge \theta^4, \\
d\theta^5 &= T^5_{12}\theta^1 \wedge \theta^2 + T^5_{13}\theta^1 \wedge \theta^3 + T^5_{14}\theta^1 \wedge \theta^4 + \theta^1 \wedge \theta^6 + T^5_{23}\theta^2 \wedge \theta^3 - \frac{2}{5}\theta^2 \wedge \theta^5 + \frac{3}{5}\theta^3 \wedge \theta^4 + \alpha^7 \wedge \theta^2 + \alpha^8 \wedge \theta^3 \\
d\theta^6 &= T^6_{12}\theta^1 \wedge \theta^2 + T^6_{13}\theta^1 \wedge \theta^3 + T^6_{14}\theta^1 \wedge \theta^4 + T^6_{15}\theta^1 \wedge \theta^5 + \theta^1 \wedge \theta^7 + T^6_{23}\theta^2 \wedge \theta^3 + \frac{3}{5}\alpha^{13}\theta^2 \wedge \theta^4 \\
&\quad - \frac{1}{5}\theta^2 \wedge \theta^6 + T^6_{34}\theta^3 \wedge \theta^4 + \frac{1}{5}\theta^3 \wedge \theta^5 + \alpha^{11} \wedge \theta^2 + \alpha^{12} \wedge \theta^3 + \alpha^{13} \wedge \theta^4, \\
d\theta^7 &= 0.
\end{aligned} \tag{48}
$$

where $\alpha^7, \alpha^8, \alpha^{11}, \alpha^{12}$, and α^{13} represent the Maurer-Cartan forms on G, and the essential torsion coefficients are given by

$$\begin{aligned}
T^4_{12} &= -\frac{5a_7u^{27/5} + 5u^5 f_4 q + 3pqu^4 + 5u^5 r}{5u^{27/5}}, \\
T^4_{13} &= -\frac{25a_8u^{13/5} + 45u^2 f_4 p + 18p^2 u + 70qu^2}{25u^{13/5}}, \\
T^6_{15} &= \frac{-25f_3u^2 + 25\dot{f}_4u^2 + 25a_{13}u^{8/5} - 5f_4pu - 6p^2 + 5qu}{25u^{8/5}}.
\end{aligned}$$

After performing the necessary calculations, we find the following values for the parameters:

$$\begin{aligned}
a_7 &= -\frac{5f_4qu + 3pq + 5ru}{5\sqrt[5]{u^7}}, \\
a_8 &= -\frac{45f_4pu + 18p^2 + 70qu}{25\sqrt[5]{u^8}}, \\
a_{13} &= \frac{5f_4pu + 25f_3u^2 - 25\dot{f}_4u^2 + 6p^2 - 5qu}{25\sqrt[5]{u^8}}.
\end{aligned} \tag{49}$$

By substituting equation (49) into equation (39) and recomputing the differentials, we obtain the following results:

$$\begin{aligned}
d\theta^1 &= \frac{1}{5}\theta^1 \wedge \theta^2, \\
d\theta^2 &= \theta^1 \wedge \theta^3, \\
d\theta^3 &= \frac{1}{5}\theta^2 \wedge \theta^3 + \theta^1 \wedge \theta^4, \\
d\theta^4 &= T^4_{14}\theta^1 \wedge \theta^4 + \theta^1 \wedge \theta^5 + \frac{2}{5}\theta^2 \wedge \theta^4, \\
d\theta^5 &= T^5_{12}\theta^1 \wedge \theta^2 + T^5_{13}\theta^1 \wedge \theta^3 + T^5_{14}\theta^1 \wedge \theta^4 + \theta^1 \wedge \theta^6 + \frac{3}{5}\theta^2 \wedge \theta^5 + \frac{17}{5}\theta^3 \wedge \theta^4, \\
d\theta^6 &= T^6_{12}\theta^1 \wedge \theta^2 + T^6_{13}\theta^1 \wedge \theta^3 + T^6_{14}\theta^1 \wedge \theta^4 \\
&\quad + \theta^1 \wedge \theta^7 + T^6_{23}\theta^2 \wedge \theta^3 + T^6_{24}\theta^2 \wedge \theta^4 - \frac{1}{5}\theta^2 \wedge \theta^6 + \frac{1}{5}\theta^3 \wedge \theta^5 + \alpha^{11} \wedge \theta^2 + \alpha^{12} \wedge \theta^3, \\
d\theta^7 &= 0.
\end{aligned}$$

Upon completing the final iteration, we obtain the values of the remaining parameters a_{11} and a_{12} as follows:

$$\begin{aligned}
a_{11} &= -\frac{5f_4ru + pr + 5su}{5u^{6/5}}, \\
a_{12} &= \frac{9f_4p^2u - 70u^2 f_4 q - 9p^3 + 18pqu - 95u^2 r}{25u^{12/5}}.
\end{aligned} \tag{50}$$

The resulting invariant coframe after the final iteration is as follows:

$$
\begin{aligned}
\theta^1 &= \frac{1}{\sqrt[5]{u}} dx, \\
\theta^2 &= \frac{du - p\, dx}{u}, \\
\theta^3 &= \frac{1}{\sqrt[5]{u^9}} \Big[(p^2 - qu)\, dx - p\, du + u\, dp \Big], \\
\theta^4 &= -\frac{1}{5\sqrt[5]{u^{13}}} \Big[(9p^3 - 14pqu + 5ru^2)\, dx - (9p^2 - 5qu)\, du + (9pu)\, dp - u^2\, dq \Big], \\
\theta^5 &= \frac{1}{25\sqrt[5]{u^{17}}} \Big[(-25u^3 f_4 r - 45 f_4 p^3 u + 70 f_4 pqu^2 - 37p^2 qu - 18p^4 \\
&\quad + 70q^2u^2 + 10pru^2 - 25su^3)\, dx + (45 f_4 p^2 u - 25u^2 f_4 q + 18p^3 + 55pqu - 25ru^2)\, du \\
&\quad - u(45 f_4 pu + 18p^2 + 70qu)\, dp + 5u^2(5 f_4 u + 3p)\, dq + 25u^3\, dr \Big], \\
\theta^6 &= -\frac{1}{125\sqrt[5]{u^{21}}} \Big[(225 f_3 p^3 u^2 - 350 f_3 pqu^3 + 125 f_3 u^4 r - 225 \dot{f}_4 p^3 u^2 \\
&\quad + 350 \dot{f}_4 pqu^3 - 125 \dot{f}_4 ru^4 + 325 f_4 p^2 qu^2 - 100 f_4 pru^3 - 350 f_4 q^2 u^3 + 125u^4 f_4 s \\
&\quad + 99p^5 - 264p^3 qu + 480p^2 ru^2 + 160pq^2u^2 - 100psu^3 - 500u^3 qr + 125u^4 t)\, dx \\
&\quad + ((225 f_3 p^2 u^2 - 125 f_3 u^3 q - 225 \dot{f}_4 u^2 p^2 + 125u^3 q \dot{f}_4 + 325u^2 f_4 pq \\
&\quad - 125u^3 f_4 r + 99p^4 - 165p^2 qu + 450pru^2 + 25q^2u^2 - 125su^3)\, du \\
&\quad - u(225 f_3 pu^2 - 225 \dot{f}_4 pu^2 + 350 f_4 u^2 q + 9p^3 - 135pqu + 475ru^2)\, dp \\
&\quad + 5u^2(5 f_4 pu + 25 f_3 u^2 - 25 \dot{f}_4 u^2 + 6p^2 - 5qu)\, dq \\
&\quad + 25u^3(5 f_4 u + p)\, dr + 125u^4\, ds \Big], \\
\theta^7 &= (\dot{f}_4 s + \dot{f}_3 r + \dot{f}_2 q + \dot{f}_1 p + \dot{f}_0 u)\, dx + f_0\, du + f_1\, dp + f_2\, dq + f_3\, dr + f_4\, ds + dt.
\end{aligned}
\tag{51}
$$

Subsequently, this leads to the final set of structure equations, denoted as (40).

5 An illustrative example

In structural engineering, fifth-order boundary value problems are used to analyze the deflection and stability of beams, plates, and shells subjected to various loading conditions. By solving these problems, engineers can determine the optimal design parameters and ensure the structural integrity of the system. Similarly, in fluid mechanics, fifth-order boundary value problems help in understanding the flow behavior of viscous fluids through pipes or channels, leading to insights into pressure distributions and flow rates.

Consider the following fifth-order differential operator:

$$D_x^5 + (Q(x) - \lambda a^5) = 0, \tag{52}$$

This operator determines a boundary value problem for the fifth-order ordinary differential equation [15]:

$$D^5 u(x) + (Q(x) - \lambda a^5) u(x) = 0 \qquad 0 \le x \le \pi,\ a > 0, \tag{53}$$

In this equation, D^5 represents the fifth derivative with respect to x, $Q(x)$ is an integrable function in the interval $[0, \pi]$, and λ is a spectral parameter. The boundary conditions for this problem are given by

$$
\begin{aligned}
&u^{(m_1)}(0) = u^{(m_2)}(0) = u^{(m_3)}(0) = u^{(m_4)}(0) = u^{(n_1)}(\pi) = 0, \\
&m_1 < m_2 < m_3 < m_4,\ m_k, n_1 \in \{0, 1, 2, 3, 4\},\ k = 1, 2, 3, 4,
\end{aligned}
\tag{54}
$$

where potential $Q(x)$ is an integrable function in interval $[0, \pi]$ and λ is a spectral parameter.

In this problem, the goal is to find a function $u(x)$ that satisfies the differential equation (53) and the boundary conditions (54). The potential term $Q(x)$ represents the influence of an external force or a physical property of the system being studied.

The spectral parameter λ allows us to consider different values and study the behavior of the system for each value. This parameter can be used to find eigenvalues and eigenfunctions of the differential equation. The spectral parameter λ allows for the study of different cases, and various numerical or analytical methods can be used to solve the problem.

The focus of this section is to utilize the direct equivalence method on the given differential operator, denoted as (52). Our objective is to thoroughly investigate the invariants associated with this operator.

5.1 Direct method of equivalence on (52)

Let's consider the fifth order differential operator (52). Our goal is to determine the conditions under which this differential equation is invariant under the fiber preserving transformations given by:

$$\bar{x} = \xi(x), \qquad \bar{u} = \varphi(x)\, u, \tag{55}$$

where $\varphi(x)$ is a non-zero function of x.

To analyze the invariance of the differential equation, we need to determine how the differential operator $D_x^5 + (Q(x) - \lambda a^5)$ transforms under the fiber preserving transformations. We can use the chain rule formula to express the transformed operator $\bar{D}_{\bar{x}}^5 + (Q(\bar{x}) - \lambda a^5)$ in terms of the original operator $D_x^5 + (Q(x) - \lambda a^5)$. Applying the chain rule to the transformation of D_x^5, we have:

$$\bar{D}_{\bar{x}}^5 = \left(\frac{d}{d\bar{x}}\right)^5 = \left(\frac{1}{\xi'(x)}\frac{d}{dx}\right)^5 = \frac{1}{\xi'(x)^5}\frac{d^5}{dx^5}. \tag{56}$$

Now, let's consider the transformation of $(Q(x) - \lambda a^5)$. Using the transformation for $\bar{x}$, we have:

$$Q(\bar{x}) - \lambda a^5 = Q(\xi(x)) - \lambda a^5. \tag{57}$$

By employing these transformations, we are able to represent the transformed differential equation as follows:

$$\frac{1}{\xi'(x)^5}\frac{d^5}{dx^5} + \big(Q(\xi(x)) - \lambda a^5\big) = 0. \tag{58}$$

In order to establish the conditions for invariance, it is necessary for the transformed differential equation to be equivalent to the original equation. This means that the coefficients of the corresponding terms must be equal. Therefore, we have the following conditions:

$$\frac{1}{\xi'(x)^5}\frac{d^5}{dx^5} = D_x^5, \tag{59}$$

and

$$Q(\xi(x)) - \lambda a^5 = Q(x) - \lambda a^5. \tag{60}$$

From the first condition, we can see that $\xi'(x)^5 = 1$, which implies $\xi'(x) = \pm 1$. This means that $\xi(x) = \pm x + c$ must be a linear function of x with a coefficient of ± 1.

From the second condition, we can see that $Q(\xi(x)) = Q(x)$. This implies that $Q(x)$ must be invariant under the transformation $\xi(x)$.

We are implementing the Cartan equivalence method on the fifth-order differential operator (52), similar to the approach employed in section 4. In the direct method, we utilize the one-forms (34) along with the additional one-form given by

$$\omega^7 = dt + (Q(x) - \lambda a^5)du + Q'(x)udx, \tag{61}$$

as our coframe. The resulting structure equations are as follows:

$$
\begin{aligned}
d\theta^1 &= \frac{1}{5}\theta^1 \wedge \theta^2, \\
d\theta^2 &= \theta^1 \wedge \theta^3, \\
d\theta^3 &= \theta^1 \wedge \theta^4 + \frac{1}{5}\theta^2 \wedge \theta^3, \\
d\theta^4 &= I_1\theta^1 \wedge \theta^4 + \theta^1 \wedge \theta^5 + \frac{2}{5}\theta^2 \wedge \theta^4, \\
d\theta^5 &= I_2\theta^1 \wedge \theta^4 + \theta^1 \wedge \theta^6 + \frac{3}{5}\theta^2 \wedge \theta^5 + \frac{17}{5}\theta^3 \wedge \theta^4, \\
d\theta^6 &= I_3\theta^1 \wedge \theta^2 + I_4\theta^1 \wedge \theta^3 + I_5\theta^1 \wedge \theta^4 \\
&\qquad + \theta^1 \wedge \theta^7 + \frac{4}{5}\theta^2 \wedge \theta^6 + I_1\theta^3 \wedge \theta^4 + 4\theta^3 \wedge \theta^5, \\
d\theta^7 &= 0,
\end{aligned}
\tag{62}
$$

here the coefficients I_1, I_2, I_3, I_4 and I_5 are given by

$$
\begin{aligned}
I_1 &= -\frac{3p}{\sqrt[5]{u^4}}, \\
I_2 &= \frac{1}{5\sqrt[5]{u^8}}\big[-9p^2 - 10qu\big], \\
I_3 &= (a^5\lambda - Q(x))u - t, \\
I_4 &= -\frac{1}{625\sqrt[5]{u^{16}}}\left[1770p^2qu - 1275pru^2 + 3000su^3 - 594p^4 - 800q^2u^2\right], \\
I_5 &= -\frac{1}{25\sqrt[5]{u^{12}}}\left[33p^3 - 45pqu + 100ru^2\right].
\end{aligned}
\tag{63}
$$

5.2 The behavior of coefficient function I_1

For example, we analyze the properties of $I_1 = -\frac{3u_x}{\sqrt[5]{u^4}}$ and examine whether it remains invariant under the fiber preserving transformations (20). When we say that the invariant $I_1 = -\frac{3u_x}{\sqrt[5]{u^4}}$ is preserved under fiber preserving transformations for the fifth order differential operator D, it means that if we apply a fiber preserving transformation (20) of the form $\bar{x} = \xi(x)$ and $\bar{u} = \varphi(x)u$, where $\varphi(x) \neq 0$, to both the invariant and the differential operator, the transformed invariant $\bar{I}_1$ will still be equal to the original invariant I_1.

To gain insight into this, let us initially focus on the transformation of the differential operator D under the fiber-preserving transformation. Using the chain rule formula, we have the equation (21) and this means that the transformed differential operator $\bar{D}$ is related to the original differential operator D by a factor of $\frac{1}{\xi'(x)}$.

Using the chain rule formula for the derivative with respect to $\bar{x}$, we can express $\bar{u}_{\bar{x}}$ in terms of u_x:

$$
\begin{aligned}
\bar{u}_{\bar{x}} &= \frac{d}{d\bar{x}}(\varphi(x)u) \\
&= \frac{d}{dx}(\varphi(x)u)\frac{dx}{d\bar{x}} \\
&= (\varphi'(x)u + \varphi(x)u_x)\,\frac{1}{\xi'(x)} \\
&= \frac{\varphi'(x)u + \varphi(x)u_x}{\xi'(x)} \\
&= \varphi'(x)u + \varphi(x)u_x
\end{aligned}
$$

Substituting this expression into the transformed invariant, we get:

$$\begin{aligned}\bar{I}_1 &= -\frac{3\bar{u}_{\bar{x}}}{\sqrt[5]{\bar{u}^4}} \\ &= -3\frac{\varphi'(x)u + \varphi(x)u_x}{\sqrt[5]{(\varphi(x)u)^4}} \\ &= -3\frac{\varphi'(x)u}{\sqrt[5]{(\varphi(x)u)^4}} + \varphi^{1/4}(x)I_1\end{aligned}$$

therefore if $\varphi(x)$ is a constant function I_1 is an invariant.

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