# One more proof about the spectrum of Transposition graph

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**Abstract**

A *Transposition graph* $T_n$ is defined as a Cayley graph over the symmetric group $Sym_n$ generated by all transpositions. This paper shows how the spectrum of $T_n$ can be obtained using the spectral properties of the Jucys-Murphy elements.

*Keywords:* Transposition graph; integral graph; spectrum;

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## 1. Introduction

*Transposition graph* $T_n$ is defined as a Cayley graph over the symmetric group generated by all transpositions. The *spectrum* of a graph is defined as a multiset of distinct eigenvalues of its adjacency matrix together with their multiplicities. The graph $T_n, n \geqslant 2$, is bipartite, so its spectrum is symmetric about zero. The largest eigenvalue of $T_n$ is equal to $\binom{n}{2}$ which implies that all eigenvalues of $T_n$ lie in the interval $[-\binom{n}{2}, \binom{n}{2}]$. In [7] it was shown that the eigenvalues of $T_n$ correspond to partitions $\lambda \vdash n$, and formulas for multiplicities of these eigenvalues were given. It was also shown in [7, Lemma 3] that $T_n$ is integral which means that its spectrum $Spec(T_n)$ consists of integers. Later and independently in [11, Theorem 2], it was shown that $T_n$ is integral. The obtaining of the spectrum in [7] is based on the Zieschang theorem [15, Theorem 1]. This paper shows how to obtain the spectrum of $T_n$ in an another way, using the spectral properties of the Jucys-Murphy elements. The Jucys-Murphy elements are significant in the study of representation theory related to symmetric groups [13] and have diverse applications. For example, spectral properties of the Jucys-Murphy elements have previously been used to find the spectrum of the Star graph [1]. Also in [3, 4], Jucys-Murphy elements are used to study the spectral properties of this graph.

The paper is organised as follows. First, in Section 2 we give basic facts from the representation theory of the symmetric group that are needed in further

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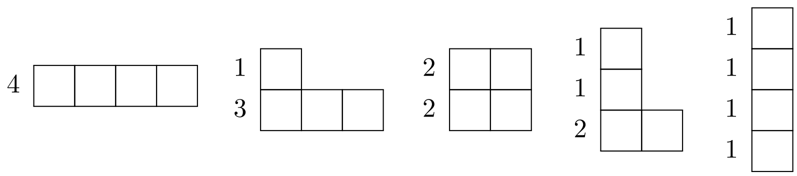


Figure 1: Young diagrams corresponding to all partitions of the number 4.

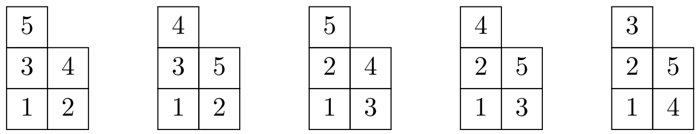


Figure 2: Standard Young tables of the shape $\lambda = (2, 2, 1)$.

proofs. After that in Section 3 we describe the Jucys-Murphy elements and introduce the $J_\Sigma^n$ operator and study it's spectrum. Then in Section 4 we show that the operator $J_\Sigma^n$ is equal to adjacency matrix of $T_n$.

## 2. Preliminaries

### *2.1. Partitions and Young tableaux*

A nonincreasing sequence of positive integers $\lambda = (n_1, \ldots, n_k) \vdash n$, for which $n = \sum_{j=1}^{k} n_j$, is called a *partition* of $n$.

A *Young diagram* is a finite set of cells arranged in left-justified rows, with the lengths of the rows in non-increasing order. It is not hard to see the bijection between Young diagrams and partitions.

We say that a Young diagram has the shape $\lambda$, where $\lambda = (n_1, n_2, \ldots, n_k)$ if the $i$-th row has $n_i$ cells. We use French notation, in which the rows are constructed from bottom to top. Figure 1 shows Young diagrams corresponding to all partitions of $n = 4$.

A *standard Young tableau* of the shape $\lambda \vdash n$ is the filling of a Young diagram of the shape $\lambda$ with numbers from 1 to $n$ such that the numbers in each row and each column are in ascending order. Let $\mathcal{T}_\lambda$ be the set of standard Young tableaux of the shape $\lambda \vdash n$. Figure 2 shows all standard Young tableaux from $\mathcal{T}_{(2,2,1)}$.

For each cell at position $(i, j)$, the *hook* $H_{ij}^\lambda$ is defined as the set of cells $(k, l)$ such that $k = i$ and $l \geqslant j$ or $l = j$ and $k \geqslant i$. The hook length $h_{ij}^\lambda$ is the number of cells in $H_{ij}^\lambda$. The number of standard Young tableaux of the shape $\lambda$ is obtained using the hook-length formula [2, Theorem 1]:

$$f_\lambda = \frac{n!}{\prod_{(ij)} h_{ij}^\lambda}. \tag{1}$$

Consider some standard Young tableau $t_\lambda \in \mathcal{T}_\lambda, \lambda \vdash n$. For each $i \in \{1, \ldots, n\}$ we define $c^t_\lambda(i) = y - x$, where $x$ and $y$ are the abscissa and ordinate of number $i$, respectively. For example, for the first table from Figure 2 we have $c(5) = 3 - 1 = 1, c(4) = 0, c(3) = 1, c(2) = -1, c(1) = 0$.

Note that the sum $c^t_\lambda(i), i \in \{0, \ldots, n\}$ does not depend on how the numbers in the standard Young table $t_\lambda$ are filled. Let us denote by $s_\lambda$ the following sum:

$$s_\lambda = \sum_{i=1}^{n} c_\lambda(i). \tag{2}$$

For example, for the first Young diagram in Figure 1, $s_\lambda = -6$, and for the Young diagram in Figure 2, $s_\lambda = 3$.

Let $\lambda = (n_1, n_2, \ldots, n_k) \vdash n$.

**Lemma 1.**

$$s_\lambda = -\sum_{j=1}^{k} \frac{n_j(n_j - 2j + 1)}{2}. \tag{3}$$

**Proof.**

Let us consider the $j$-th row in the Young tableau of the shape $\lambda$. Elements from this line add the expression $\sum\limits_{i=1}^{n_j}(j - i) = -\frac{n_j(n_j-2j+1)}{2}$ to $s_\lambda$. □

Note that different partitions can correspond to the same value of $s_\lambda$. For example, $s_{(3,3)} = s_{(4,1,1)} = -3$.

The *conjugate of a partition* $\lambda$, denoted by $\lambda'$, is also a partition of $n$ where its parts are the nonincreasing sequence $\lambda' = (n'_1, n'_2, \ldots, n'_\kappa)$, where $n'_j = \sum\limits_{i|n_i \geqslant j} 1$.

**Lemma 2.** $s_\lambda = -s_{\lambda'}$.

**Proof.** The proof is the same as for Lemma 1 if we consider the columns for the Young table corresponding to $\lambda'$.

*2.2. Regular representation*

Let $GL(V)$ represent the collection of all invertible transformations of vector space $V$ onto itself, known as the general linear group of $V$. A representation of a group $G$ on a vector space $V$ over a field $\mathbb{C}$ is a homomorphism $\rho : G \longrightarrow GL(V)$. $V$ is also called *$G$-module*.

Let $W$ be a vector subspace of $V$. $W$ is considered $\rho$-invariant if $\rho(g)(w) \in W$ for any $g \in G$ and $w \in W$. A representation $\rho$ is defined as irreducible if the only invariant subspaces of V are $\{0\}$ and $V$. The irreducible representations of $Sym_n$ are indexed by partitions of $n$. We denote by $V_\lambda$ the irreducible module associated with the partition $\lambda \vdash n$.

Let $G$ be a finite group of order $n$. Consider a vector space $V$ of dimension $n$, having a basis whose elements correspond to the elements of $G$. We denote this basis as $\{e_g\}_{g\in G}$, where $e_g$ is the basis vector corresponding to the element $g$ of the group $G$. The *regular representation* of the group $G$ is a representation $\rho$, defined as follows:

$$\rho(g)(e_h) = e_{gh}$$

for all $g, h \in G$.

The regular representation of $Sym_n$ can be decomposed into irreducible submodules in the following manner [14, Prop. 1.10.1]:

$$\mathbb{C}[Sym_n] = \underbrace{(V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_1})}_{\dim(V_{\lambda_1})} \oplus \cdots \oplus \underbrace{(V_{\lambda_k} \oplus \cdots \oplus V_{\lambda_k})}_{\dim(V_{\lambda_k})} = \bigoplus_{\lambda\vdash n} V_\lambda^{dim(V_\lambda)}, \quad (4)$$

and $dim(V_\lambda) = f_\lambda$.

## 3. Jucys-Murphy elements and spectrum of $J_\Sigma^n$

The *Jucys-Murphy elements* are elements of group algebra $\mathbb{C}[Sym_n]$. These elements were introduced independently by Jucys in 1966 [5] and Murphy in 1981 [12] and are defined as follows:

$$J_i^n = (1,i) + (2,i) + \cdots + (i-1,i) = \sum_{j=1}^{i}(j,i), i = 2, \ldots, n, \quad (5)$$

where by the summation of transpositions means the summation of the matrices corresponding to these transpositions in some representation.

The Jucys-Murphy elements can be viewed as a linear operator acting on the basis vectors of the regular representation as follows:

$$J_i^n(e_{\sigma_k}) = \sum_{j=1}^{i} e_{(j,i)\cdot\sigma_k}.$$

The spectrum of Jucys-Murphy elements is obtained by the following theorem.

**Theorem 1.** [6] *Let $\lambda \vdash n$. Then there exists a basis $(v_t)_{t\in\mathcal{T}_\lambda}$ of irreducible module $V_\lambda$, indexed by standart Young tables of shape $\lambda$, such that for all $i \in \{2, \ldots, n\}$, one has:*

$$J_i^n(v_t) = c_\lambda^t(i)v_t. \quad (6)$$

Let us denote by $J_\Sigma^n$ the following expression:

$$J_\Sigma^n = \sum_{i=2}^{n} J_i^n. \quad (7)$$

Consider the basis $(v_t)_{t\in\mathcal{T}_\lambda}$ from Theorem 1. From the definition of $J^n_\Sigma$ and $s_\lambda$ it follows directly that $\{s_\lambda\}_{\lambda\vdash n}$ are eigenvalues of the operator $J^n_\Sigma$:

$$J^n_\Sigma(v_t) = s_\lambda v_t. \tag{8}$$

The following lemma gives an expression of the multiplicity of the eigenvalue $s_\lambda$ for the operator $J^n_\Sigma$.

**Lemma 3.** $mul(s_\lambda) = \sum\limits_{\eta: s_\alpha = s_\lambda} f^2_\alpha.$

**Proof.** By (4), the number of irreducible modules $V_\lambda$ is $f_\lambda$. In each of $V_\lambda$ the number of eigenvectors $v_t$ corresponding to the eigenvalue $s_\lambda$ is equal to $f_\lambda$ too. Note also that $mul(s_\lambda) = mul(s_{\lambda'})$. □

We denote by $\eta_\lambda = s_{\lambda'} = -s_\lambda$. Then we are able to formulate the following theorem.

**Theorem 2.** *The spectrum of* $J^n_\Sigma$ *consists of the eigenvalues corresponding to the partitions* $\lambda \vdash n$. *Any partition* $\lambda \vdash n$ *corresponds to an eigenvalue* $\eta_\lambda$ *of* $J^n_\Sigma$ *by the following expression:*

$$\eta_\lambda = \sum_{j=1}^{k} \frac{n_j(n_j - 2j + 1)}{2} \tag{9}$$

*with multiplicity* $mul(\eta_\lambda) = \sum\limits_{\alpha: \eta_\alpha = \eta_\lambda} f^2_\alpha.$

## 4. Equivalence of $J^n_\Sigma$ and the adjacency matrix of $T_n$

Let us denote by $A_{T_n}$ the adjacency matrix (or equal linear operator) of $T_n$. Then we can formulate the following lemma.

**Lemma 4.** $A_{T_n} = J^n_\Sigma$.

**Proof.** Let us see how the operator $A_{T_n}$ acts on an arbitrary basis element $e_{\sigma_k}$:

$$A_{T_n}(e_{\sigma_k}) = \sum_{\sigma_l : (\sigma_l, \sigma_k) \in E(T_n)} e_{\sigma_l}. \tag{10}$$

An edge $(\sigma_l, \sigma_k) \in E(T_n)$ if and only if $\sigma_l = (i, j)\sigma_k$ for any transposition $(i, j)$. Therefore, (10) can be rewritten in the following form:

$$A_{T_n}(e_{\sigma_k}) = \sum_{i<j} e_{(j,i)\sigma_k}. \tag{11}$$

Now let us consider the action of $J^n_\Sigma$ on $e_{\sigma_k}$:

$$J^n_\Sigma(e_{\sigma_k}) = \sum_{i=2}^{n} J^n_i(e_{\sigma_k}) = \sum_{i=2}^{n}\sum_{j=1}^{i} e_{(j,i)\sigma_k} = \sum_{j<i} e_{(j,i)\sigma_k}. \tag{12}$$

Thus, the action of operators $J^n_\Sigma$ and $A_{T_n}$ on an arbitrary basis element $e_{\sigma_k}$ is the same, so operators $J^n_\Sigma$ and $A_{T_n}$ are equal to each other. □

Lemma 4 shows that the spectrum of $T_n$ can be expressed using Theorem 2.

## 5. Discussions and further research

In this paper, it was shown how one can use the spectral properties of Jucys-Murphy elements to obtain the spectrum of $T_n$.

However, by looking at the expression ( 9) it is not possible to answer the following questions:

- are all integers from the interval $[0,k], k \in \mathbb{N}$, lie in the spectrum of $T_n$?
- what is the asymptotics for the number of unique values in the spectrum of $T_n$?

Ideally, one would like to be able to answer the question whether a given integer number $k \in [-\binom{n}{2}, \binom{n}{2}]$ lies in the spectrum of $T_n$. The papers [8, 9, 10] partially address these issues.


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