Δ-continuous and Δ-irresolute functions in topological spaces

Mohammad A. A. Marabeh

[a]*Department of Applied Mathematics, Palestine Technical University-Kadoorie (PTUK), Tulkarem, P.O Box (7), Palestine*
[b]*Email: mohammad.marabeh@ptuk.edu.ps, ORCID: 0000-0003-0644-8183,*

Abstract

In this study, we explore the concepts of Δ-continuous, Δ-irresolute, Δ-open, and Δ-closed functions. It is established that every continuous function is inherently Δ-irresolute, and in turn, every Δ-irresolute function is Δ-continuous. However, the reverse implications do not necessarily hold true. This marks a departure from traditional continuous functions, where the composition of two Δ-continuous functions may not retain Δ-continuity. We introduce several methodologies for constructing new Δ-continuous (or Δ-irresolute) functions from existing ones. Notably, we provide pasting-type lemmas specifically for Δ-continuous and Δ-irresolute functions. The discussion is enriched with several counterexamples to illustrate these concepts.

Keywords: open set, closed set, symmetric difference, Δ-open set, Δ-closed set, continuous function, Δ-continuous function, Δ-irresolute function, Δ-open function, Δ-closed function
2020 MSC: 54A05, 54C08, 54C10

---

1. Introduction

Open-like and closed-like sets within topological spaces have garnered significant interest from researchers over the past few decades. Similarly, numerous notions of continuous-like functions in topological spaces have been introduced and explored. The pioneering work was by N. Levine, who introduced the concepts of *semi-open sets* and *semi-continuous functions* in topological spaces [13]. A set S in a topological space X is defined as semi-open if $S \subseteq Cl(Int(S))$ [13], where $Cl(A)$ and $Int(A)$ denote the closure and the interior of a set A in X, respectively. A function is semi-continuous if

the inverse image of any open set is semi-open [13]. Following this, S. Crossley and S. Hildebrand introduced *irresolute functions*, characterized by the property that the inverse image of any semi-open set is semi-open as well [7].

In 1965, Olav Njåstad introduced α-*sets* [21], where a set S in a topological space X is an α-set if $S \subseteq Int(Cl(Int(S)))$ [21]. Mappings for which the inverse image of each α-set is an α-set were termed α-irresolute mappings [15], paving the way for the definition and exploration of α-continuous functions [18]. Subsequently, concepts such as θ-open and δ-open sets were developed [30], alongside the notions of *pre-open sets* and *pre-continuous mappings* [17]. A set S in a topological space X is pre-open if $S \subseteq Int(Cl(S))$ [17], and a semi-preopen set if $S \subseteq Cl(Int(Cl(S)))$ [2]. The exploration continued with the study of semi-precontinuous mappings, semi-preopen mappings, semi-preclosed mappings, semi-preirresolute mappings, pre-semipreopen mappings, and pre-semi-preclosed mappings, along with their characterizations in topological spaces [20]. M. Veera Kumar further contributed by constructing ψ-continuous and ψ-irresolute functions through a novel class of closed-like sets termed ψ-closed sets [29].

The exploration of new classes of continuous-like functions through open-like and closed-like sets in topological spaces continues to captivate the interest of many researchers. Notable contributions include [1], [3], [5], [6], [9], [10], [11], [12], [24], [25], and [26].

A set in a topological space is termed Δ-*open* if it constitutes the symmetric difference of two open sets. This notion, first appearing in [22] and [8], was credited to a preprint by M. Veera Kumar. Conversely, the complement of a Δ-open set is defined as Δ-*closed*. Both Δ-open and Δ-closed sets, along with related concepts, were extensively studied by the author in [16].

Section 2 consolidates the essential notions and results prerequisite for the discussions that follow.

Section 3 introduces the concept of a Δ-continuous function, as defined in Definition 3.1. It elaborates on some properties of Δ-continuous functions in Theorem 3.2 and outlines necessary conditions for Δ-continuity in Propositions 3.1, 3.2, and 3.3. Furthermore, Example 3.3 demonstrates that the composition of two Δ-continuous functions may not be Δ-continuous, although the composition of a continuous function with a Δ-continuous function results in a Δ-continuous function, as stated in Proposition 3.5. This section also presents methods for constructing Δ-continuous functions in Proposition 3.6 and offers a pasting-type lemma for Δ-continuous functions in Theorem 3.4. It concludes with Proposition 3.7, asserting that a Δ-continuous function

into a product space $X \times Y$ implies Δ-continuity of its coordinate functions.

In Section 4, the concept of a Δ-irresolute function is defined in Definition 4.1. It is noted that every Δ-irresolute function is Δ-continuous, but Example 4.1 shows that the converse is not universally applicable. This section mirrors Section 3 in its formulation of results concerning Δ-irresolute functions.

Section 5 delineates the definitions of Δ-open and Δ-closed functions. Through various examples, it is illustrated that these concepts are independent of each other and also distinct from the notion of Δ-continuous functions. The section further provides results pertaining to Δ-open and Δ-closed functions.

2. Preliminaries

In this section we provide basic notions and results related to Δ-open and Δ-closed sets. These results will be used and applied in the subsequent sections.

Recall that for sets A and B their *symmetric difference* is given as $A \Delta B := (A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.

Definition 2.1. ([22] and [8]) A set A in a topological space (X, τ) is called Δ-*open* if there are open sets O_1 and O_2 so that $A = O_1 \Delta O_2$.

For an open set O in a topological space (X, τ), it is obvious that $O = O \Delta \phi$, so every open set is Δ-open. However, the set $(0, 1] \cup [2, 3) = (0, 2) \Delta (1, 3)$ is Δ-open set which is not open in the standard topology on $\mathbb{R}$. The complement of a Δ-open set is called Δ-*closed*. We recall a characterization of Δ-open sets.

Theorem 2.1. *([16]) A set A in a topological space (X, τ) is Δ-open if and only if there is an open set O and a closed set C such that $A = O \cap C$.*

Corollary 2.1. *([16]) A set B in a topological space (X, τ) is Δ-closed if and only if there is an open set O and a closed set C such that $B = O \cup C$.*

It follows from Theorem 2.1 that any open set and any closed set is Δ-open. Moreover, a finite intersection of Δ-open sets is Δ-open. However, union of two Δ-open sets need not be Δ-open and arbitrary intersection of Δ-open sets need not be Δ-open.

In the same manner, it follows from Corollary 2.1 that any open set and

3

any closed set is Δ-closed. Additionally, a finite union of Δ-closed sets is Δ-closed. Nonetheless, intersection of two Δ-closed sets need not be Δ-closed and arbitrary union of Δ-closed sets need not be Δ-closed.

Example 2.1. ([16]) Let $X = \{a, b, c, d, e\}$ with a topology

$$\tau = \{\phi, X, \{a, b, c\}, \{a, b, c, d\}\}.$$

The collection of all Δ-open sets in X is

$$\tau_{\Delta o} = \{\phi, X, \{a, b, c\}, \{a, b, c, d\}, \{d, e\}, \{d\}, \{e\}\}.$$

Clearly, the sets $\{a, b, c\}$ and $\{e\}$ are Δ-open, whereas their union is not.

It is worth noting from the previous example that the collection of all Δ-open sets does not form a topology in general.

Example 2.2. ([16]) Let $\mathbb{Q} = \bigcup_{n=1}^{\infty} \{r_n\}$ be an enumeration of the rationals. For each $n \in \mathbb{N}$, let $S_n = \mathbb{R} - \{r_1, r_2, ..., r_n\}$, then considering $\mathbb{R}$ under the standard topology, each S_n is open set and so it is Δ-open. Yet, $\bigcap_{n=1}^{\infty} S_n = \mathbb{R} - \mathbb{Q}$ is not Δ-open.

It should be pointed out that the open set O and the closed set C in Corollary 2.1 can be chosen to be disjoint. Indeed, if $B = O \cup C$, with O is open and C is closed, then $B = O \cup (C - O)$ where $C - O$ is closed.

Definition 2.2. ([16]) Let (X, τ) be a topological space and $x \in X$. A Δ-open set containing x is called Δ-*neighborhood*. We write $\Delta N(x)$.

Definition 2.3. ([16]) Let (X, τ) be a topological space and $A \subseteq X$.
(1) The union of all Δ-open sets contained in A is said to be the Δ-*interior of* A and is denoted by $\Delta Int(A)$.
(2) The intersection of all Δ-closed sets containing A is said to be the Δ-*closure of* A and is denoted by $\Delta Cl(A)$.

Clearly, $\Delta Int(A)$ need not be Δ-open and $\Delta Cl(A)$ need not be Δ-closed. It should be also noted that if A is Δ-open, then $\Delta Int(A) = A$, and if A is Δ-closed, then $\Delta Cl(A) = A$. In either case the converse is not true.

Example 2.3. ([16]) Let $\mathbb{Q} = \bigcup_{n=1}^{\infty} \{r_n\}$ be an enumeration of the rationals. For each $n \in \mathbb{N}$, let $O_n = (-n, n)$ and $C_n = \{r_1, r_2, ..., r_n\}$. Then $A_n = O_n \cap C_n$ is Δ-open set in $\mathbb{R}$ under the standard topology. Let $A = \bigcup_{n=1}^{\infty} A_n$ then $Int(A) = A$, nevertheless $A = \mathbb{Q}$ is not Δ-open.

It is important to note that in the proof of [22, Theorem 3], the authors assumed that the Δ-interior of a set is Δ-open. However, the above example shows this is not the case.

Because each open set is Δ-open and each closed set is Δ-closed, the following result follows directly.

Lemma 2.2. *([16]) Let (X, τ) be a topological space and $A \subseteq X$. Then*
(1) $\Delta Int(A) \subseteq \Delta Cl(A)$.
(2) $Int(A) \subseteq \Delta Int(A) \subseteq A$.
(3) $A \subseteq \Delta Cl(A) \subseteq Cl(A)$.

Proposition 2.1. ([16]) Let (X, τ) be a topological space and $A \subseteq X$. Then $x \in \Delta Int(A)$, if and only if, there is a $\Delta N(x)$, such that $\Delta N(x) \subseteq A$.

Basic properties of Δ-interior are summarized in the next proposition.

Proposition 2.2. ([16]) Let (X, τ) be a topological space and $A, B \subseteq X$. Then
(1) If $A \subseteq B$, then $\Delta Int(A) \subseteq \Delta Int(B)$.
(2) $\Delta Int(A) \cup \Delta Int(B) \subseteq \Delta Int(A \cup B)$.
(3) $\Delta Int(A \cap B) = \Delta Int(A) \cap \Delta Int(B)$.

Proposition 2.3. ([16]) Let (X, τ) be a topological space and $Y \subseteq X$. Then, S is Δ-open in Y if and only if there is a Δ-open set A in X such that $S = Y \cap A$.

Proposition 2.4. ([16]) Let (X, τ) be a topological space and $Y \subseteq X$. Then, S is Δ-closed in Y if and only if there is a Δ-closed set B in X such that $S = Y \cap B$.

Proposition 2.5. ([16]) Let (X, τ) be a topological space and $Y \subseteq X$. If S is Δ-open in Y and Y is Δ-open in X, then S is Δ-open in X.

3. Δ-continuous functions

Given topological spaces X and Y. Recall that a function $f : X \to Y$ is called *continuous* if for any open set O in Y, $f^{-1}(O)$ is open in X. A function $f : X \to Y$ is called *semi-continuous* if for any open set O in Y, $f^{-1}(O)$ is semi-open in X; [13]. In the same manner we propose the following notion.

Definition 3.1. Let X and Y be topological spaces. A function $f : X \to Y$ is said to be Δ-*continuous* if for any open set O in Y, $f^{-1}(O)$ is Δ-open in X.

Clearly, every continuous function is Δ-continuous. However, the converse need not be true.

Example 3.1. Let $X = \{a, b, c\}$. Let $\tau_1 = \{\phi, X, \{a\}\}$ and $\tau_2 = \{\phi, X, \{a, b\}\}$ be two topologies on X. Define a function $f : (X, \tau_1) \to (X, \tau_2)$ by $f(a) = c, f(b) = b$, and $f(c) = a$. The collection of all Δ-open sets in (X, τ_1) is

$$\tau_{1\Delta o} = \{\phi, X, \{a\}, \{b, c\}\}$$

Clearly that f is Δ-continuous. Yet $f^{-1}(\{a, b\}) = \{b, c\}$ is not open in (X, τ_1). So, f is not continuous.

We provide the following characterization for Δ-continuous functions.

Theorem 3.1. *Let X and Y be topological spaces; let $f : X \to Y$. Then f is Δ-continuous if and only if for each closed set C in Y, the set $f^{-1}(C)$ is Δ-closed in X.*

Proof. ($\Longrightarrow$) Assume $f : X \to Y$ is Δ-continuous. Let C be a closed set in Y. Then $Y - C$ is open in Y. So, $f^{-1}(Y - C) = X - f^{-1}(C)$ is Δ-open in X. That is, $f^{-1}(C)$ is Δ-closed in X.
($\Longleftarrow$) Let O be an open set in Y. Then $Y - O$ is closed set in Y. So, $f^{-1}(Y - O) = X - f^{-1}(O)$ is Δ-closed in X. That is, $f^{-1}(O)$ is Δ-open in X. Hence, f is Δ-continuous. $\square$

Theorem 3.2. *Let X and Y be topological spaces; let $f : X \to Y$. The Δ-continuity of f implies each of the following statements:*
(1) For each $x \in X$ and for each neighborhood of $f(x)$, $N(f(x))$ in Y, there is a Δ-neighborhood of x, $\Delta N(x)$ in X such that $f(\Delta N(x)) \subseteq N(f(x))$.
(2) For each subset A of X, we have $f(\Delta Cl(A)) \subseteq Cl(f(A))$.
(3) For each subset B of Y, we have $f^{-1}(Int(B)) \subseteq \Delta Int(f^{-1}(B))$.

Proof. (1) Let $x \in X$ and $N(f(x))$ be a neighborhood of $f(x)$ in Y. Then $f^{-1}(N(f(x)))$ is Δ-open in X and $x \in f^{-1}(N(f(x)))$. So, take $\Delta N(x) = f^{-1}(N(f(x)))$. Hence,

$$f(\Delta N(x)) = f\left(f^{-1}(N(f(x)))\right) \subseteq N(f(x))$$

6

(2) Let $A \subseteq X$. Then $Cl\big(f(A)\big)$ is closed in Y. It follows from Theorem 3.1 that $f^{-1}\Big(Cl\big(f(A)\big)\Big)$ is Δ-closed in X. Moreover, $A \subseteq f^{-1}\Big(Cl\big(f(A)\big)\Big)$. As a result, $\Delta Cl(A) \subseteq f^{-1}\Big(Cl\big(f(A)\big)\Big)$. Therefore, $f\big(\Delta Cl(A)\big) \subseteq Cl\big(f(A)\big)$.

(3) Let $B \subseteq Y$. Then $Int(B)$ is open in Y and so $f^{-1}\big(Int(B)\big)$ is Δ-open in X. Furthermore, $f^{-1}\big(Int(B)\big) \subseteq f^{-1}(B)$. Therefore, $f^{-1}\big(Int(B)\big) \subseteq \Delta Int\big(f^{-1}(B)\big)$. $\qquad\square$

None of the statements in Theorem 3.2 assures Δ-continuity.

Example 3.2. Let $X = \{a, b, c\}$ with topologies $\tau_1 = \{\phi, X, \{a\}, \{a, b\}\}$ and $\tau_2 = \{\phi, X, \{a\}\}$. Define a function $f : (X, \tau_1) \to (X, \tau_2)$ by $f(a) = a$, $f(b) = b$, and $f(c) = a$. The collection of all Δ-open sets in (X, τ_1) is

$$\tau_{1\Delta o} = \{\phi, X, \{a\}, \{a, b\}, \{b, c\}, \{c\}, \{b\}\}$$

The set $\{a\}$ is open in (X, τ_2), but $f^{-1}(\{a\}) = \{a, c\}$ is not Δ-open in (X, τ_1). Thus, f is not Δ-continuous.

However, it is easy to check that statement (1) in Theorem 3.2 is satisfied. Furthermore, in (X, τ_1) it is easy to see that $\Delta Cl(A) = \Delta Int(A) = A$ for any subset A of X. So statements (2) and (3) in Theorem 3.2 are obviously satisfied as well.

An extra condition is needed for each statement in Theorem 3.2 to guarantee the Δ-continuity.

Proposition 3.1. Let X and Y be topological spaces; let $f : X \to Y$. Assume for each $x \in X$ and for each neighborhood of $f(x)$, $N(f(x))$ in Y, there is a Δ-neighborhood of x, $\Delta N(x)$ in X such that $f\big(\Delta N(x)\big) \subseteq N(f(x))$. Assume further for any subset A of X the $\Delta Int(A)$ is Δ-open in X. Then f is Δ-continuous.

Proof. Let O be an open set in Y. Let $x \in f^{-1}(O)$. Then O is a neighbourhood of $f(x)$. So, there is a Δ-neighborhood of x, $\Delta N(x)$ in X such that $f\big(\Delta N(x)\big) \subseteq O$. Hence, $\Delta N(x) \subseteq f^{-1}(O)$. Consequently,

$$f^{-1}(O) = \bigcup_{x \in O} \Delta N(x) = \Delta Int\big(f^{-1}(O)\big)$$

Therefore, $f^{-1}(O)$ is Δ-open in X and so f is Δ-continuous. $\qquad\square$

Proposition 3.2. Let X and Y be topological spaces; let $f : X \to Y$. Assume for each subset A of X, we have $f\big(\Delta Cl(A)\big) \subseteq Cl\big(f(A)\big)$. Assume further for any subset A of X the $\Delta Cl(A)$ is Δ-closed in X. Then f is Δ-continuous.

Proof. Let C be a closed set in Y. Let $A = f^{-1}(C)$.Then, $A \subseteq X$ and $f\big(\Delta Cl(A)\big) \subseteq Cl\big(f(A)\big)$. That is,

$$f\Big(\Delta Cl\big(f^{-1}(C)\big)\Big) \subseteq Cl\Big(f\big(f^{-1}(C)\big)\Big) \subseteq Cl(C) = C.$$

Hence,

$$\Delta Cl\big(f^{-1}(C)\big) \subseteq f^{-1}\Big(f\big(\Delta Cl(f^{-1}(C))\big)\Big) \subseteq f^{-1}(C).$$

Thus, $f^{-1}(C) = \Delta Cl\big(f^{-1}(C)\big)$ is Δ-closed in X. Theorem 3.1 implies that f is Δ-continuous. $\square$

Proposition 3.3. Let X and Y be topological spaces; let $f : X \to Y$. Assume for each subset B of Y, we have $f^{-1}\big(Int(B)\big) \subseteq \Delta Int\big(f^{-1}(B)\big)$. Assume further for any subset A of X the $\Delta Int(A)$ is Δ-open in X. Then f is Δ-continuous.

Proof. Let O be an open subset in Y. Then, $f^{-1}(O) = f^{-1}\big(Int(O)\big) \subseteq \Delta Int\big(f^{-1}(O)\big)$. Consequently, $f^{-1}(O) = \Delta Int\big(f^{-1}(O)\big)$ is Δ-open in X. Therefore, f is Δ-continuous. $\square$

Next we consider some methods of constructing Δ-continuous functions, but first we recall the following result on continuous functions in topological spaces.

Proposition 3.4. ([19]) Let X and Y be topological spaces. Then
(1) If $f : X \longrightarrow Y$ is given by $f(x) = y_0$ where $y_0 \in Y$ is a fixed element, then f is continuous.
(2) If A is a subspace of X, the inclusion function $i_A : A \longrightarrow X$ is continuous.
(3) Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be projections onto the first and second factors, respectively. Then π_1 and π_2 are continuous.

The composition of two Δ-continuous functions need not be Δ-continuous as illustrated in the following example.

Example 3.3. Let $X = \{a, b, c, d\}$ with a topology

$$\tau = \{\phi, X, \{a\}, \{a, b\}, \{a, b, c\}\}$$

The collection of all Δ-open sets in X is

$$\tau_{\Delta o} = \{\phi, X, \{a\}, \{a, b\}, \{a, b, c\}, \{b, c, d\}, \{c, d\}, \{d\}, \{b\}, \{c\}, \{b, c\}\}$$

Define a function $f : X \to X$ by $f(a) = d, f(b) = b, f(c) = a$, and $f(d) = c$. It is easy to check that f is Δ-continuous. Nevertheless, $f \circ f$ is not Δ-continuous. Indeed, $(f \circ f)^{-1}(\{a, b, c\}) = f^{-1}(f^{-1}(\{a, b, c\})) = f^{-1}(\{b, c, d\}) = \{a, b, d\}$ which is not Δ-open in X.

Next we show that the composition of a Δ-continuous function and a continuous function produces a Δ-continuous function.

Proposition 3.5. Let X, Y, Z, and W be topological spaces. Let $f : X \to Y, g : Y \to Z$, and $h : Z \to W$.
(1) If f is continuous and g is Δ-continuous, then $g \circ f$ is Δ-continuous.
(2) If g is Δ-continuous and h is continuous, then $h \circ g$ is Δ-continuous.

Proof. (1) Assume f is continuous and g is Δ-continuous. Let G be an open set in Z. As g is Δ-continuous, we get $g^{-1}(G)$ is Δ-open in Y. Theorem 2.1 assures that $g^{-1}(G) = O \cap C$ for some open set O in Y and closed set C in Y. Continuity of f implies that $f^{-1}(O)$ is open in X and $f^{-1}(C)$ is closed in X. Therefore, $(g \circ f)^{-1}(G) = f^{-1}(g^{-1}(G)) = f^{-1}(O \cap C) = f^{-1}(O) \cap f^{-1}(C)$ is Δ-open in X by Theorem 2.1. Hence, $g \circ f$ is Δ-continuous.
(2) Suppose g is Δ-continuous and h is continuous. Let G be an open set in W. As h is continuous, we have $h^{-1}(G)$ is open in Z. Being g is Δ-continuous, we obtain $g^{-1}(h^{-1}(G))$ is Δ-open in Y. So, $(h \circ g)^{-1}(G) = g^{-1}(h^{-1}(G))$ is Δ-open in Y. Thus, $h \circ g$ is Δ-continuous. $\square$

Proposition 3.6. Let X and Y be topological spaces.
(1) If $f : X \to Y$ is Δ-continuous and A is a subspace of X, then the restriction of f to A, $f|_A : A \to Y$ is Δ-continuous.
(2) Let $f : X \to Y$ be Δ-continuous. If B is a subspace of Y such that $f(X) \subseteq B$, then the function $f : X \to B$ is Δ-continuous.
(3) Let $f : X \to Y$ be Δ-continuous. If Z is a topological space having Y as a subspace, then the function $h : X \to Z$ obtained by extending the codomain of f is Δ-continuous.

9

Proof. (1) The function $f|_A$ equals to $f \circ i_A$, where i_A is the inclusion function $i_A : A \to X$. A combination of Proposition 3.4(2) and Proposition 3.5(1) yields that $f|_A$ is Δ-continuous.

(2) Suppose $f : X \to Y$ is Δ-continuous and B is a subspace of Y such that $f(X) \subseteq B$. Let G be an open set in B. Then there is an open set O in Y such that $G = B \cap O$. Then $f^{-1}(G) = f^{-1}(B \cap O) = f^{-1}(B) \cap f^{-1}(O) = X \cap f^{-1}(O) = f^{-1}(O)$ is Δ-open in X. Thus, $f : X \to B$ is Δ-continuous.

(3) Assume $f : X \to Y$ is Δ-continuous and Z is a topological space having Y as a subspace. Let $h : X \to Z$ be the function obtained by extending the codomain of f. Then $h = i_Y \circ f$, where i_Y is the inclusion function $i_Y : Y \to Z$. Proposition 3.4(2) and Proposition 3.5(2) assure that h is Δ-continuous. $\qquad\square$

We recall the usual *pasting lemma* for continuous functions in topological spaces

Theorem 3.3. *([19]) Let X and Y be topological spaces. Let $X = A \cup B$, where A and B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. Assume $f(x) = g(x)$ for each $x \in A \cap B$ and let $h : X \to Y$ be a function defined by $h(x) = f(x)$ for $x \in A$, and $h(x) = g(x)$ for $x \in B$. Then, h is continuous.*

Proof. Let C be a closed subset of Y. Then $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$. Since f is continuous, $f^{-1}(C)$ is closed in A, and so it is closed in X. Similarly, $g^{-1}(C)$ is closed in B, and so in X. Hence, $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$ is closed in X. Therefore, h is continuous. $\qquad\square$

The proof of Theorem 3.3 utilizes the following result: *Let Y be a subspace of X. If C is closed in Y and Y is closed in X, then C is closed in X.* However, such conclusion does not hold in the sense of Δ-closedness.

Example 3.4. Let $X = \{a, b, c, d, e\}$ with a topology

$$\tau = \{\phi, X, \{a, b, c\}, \{a, b, c, d\}\}.$$

The collection of all Δ-open sets in X is

$$\tau_{\Delta o} = \{\phi, X, \{a, b, c\}, \{a, b, c, d\}, \{d, e\}, \{d\}, \{e\}\}.$$

The collection of all Δ-closed sets in X is

$$\tau_{\Delta c} = \{\phi, X, \{a, b, c\}, \{a, b, c, d\}, \{a, b, c, e\}, \{d, e\}, \{e\}\}.$$

Let $Y = \{a, b, c, d\}$ and $C = \{d\}$. Obviously, Y is Δ-closed in X. Note that $C = Y \cap \{d, e\}$, so Proposition 2.4 assures that C is Δ-closed set in Y. Yet, C is not Δ-closed in X.

Motivated by Theorem 3.3 and Example 3.4 we propose the following pasting-type lemma for Δ-continuous functions.

Theorem 3.4. *Let X and Y be topological spaces. Let $X = A \cup B$, where A and B are Δ-open in X. Let $f : A \to Y$ and $g : B \to Y$ be Δ-continuous. Assume $f(x) = g(x)$ for each $x \in A \cap B$ and let $h : X \to Y$ be a function defined by $h(x) = f(x)$ for $x \in A$, and $h(x) = g(x)$ for $x \in B$. Assume further, the union of any two Δ-open sets in X is again Δ-open. Then, h is Δ-continuous.*

Proof. Let O be an open subset of Y. Then $h^{-1}(O) = f^{-1}(O) \cup g^{-1}(O)$. Since f is Δ-continuous, $f^{-1}(O)$ is Δ-open in A, and so it is Δ-open in X by Proposition 2.5. Similarly, $g^{-1}(O)$ is Δ-open in B, and so in X. Hence, $h^{-1}(O) = f^{-1}(O) \cup g^{-1}(O)$ is Δ-open in X. Therefore, h is Δ-continuous. $\square$

The condition in Theorem 3.4 that union of any two Δ-open sets is again Δ-open can not be removed.

Example 3.5. Let $X = \{a, b, c, d, e\}$ with a topology

$$\tau = \{\phi, X, \{a, b, c\}, \{a, b, c, d\}\}.$$

The collection of all Δ-open sets in X is

$$\tau_{\Delta o} = \{\phi, X, \{a, b, c\}, \{a, b, c, d\}, \{d, e\}, \{d\}, \{e\}\}.$$

Clearly, the sets $\{a, b, c\}$ and $\{e\}$ are Δ-open, whereas their union is not. Let $A = \{a, b, c\}$ and $B = \{d, e\}$. Let τ_A and τ_B be the induced topologies on the sets A and B, respectively. Let $f : (A, \tau_A) \to (X, \tau)$ be given as $f(a) = a, f(b) = b$, and $f(c) = c$. Clearly, the function f is continuous so it is Δ-continuous. Define a function $g : (B, \tau_B) \to (X, \tau)$ as $g(d) = e$ and $g(e) = d$. Then g is Δ-continuous.
Let $h : (X, \tau) \to (X, \tau)$ be as stated in Theorem 3.4. For the open set $O = \{a, b, c, d\}$, $h^{-1}(O) = f^{-1}(O) \cup g^{-1}(O) = \{a, b, c\} \cup \{e\} = \{a, b, c, e\}$ is not Δ-open. Thus, h is not Δ-continuous.

Proposition 3.7. Let X, Y, and Z be topological spaces. Let $f_1 : X \to Y$ and $f_2 : X \to Z$. Let $f : X \to Y \times Z$ be given as $f(x) = (f_1(x), f_2(x))$. If f is Δ-continuous on X, then f_1 and f_2 are Δ-continuous on X.

Proof. Let $\pi_1 : Y \times Z \to Y$ and $\pi_2 : Y \times Z \to Z$ be projections onto the first and second factors, respectively. Then $f_1(x) = \pi_1(f(x)) = (\pi_1 \circ f)(x)$ and $f_2(x) = \pi_2(f(x)) = (\pi_2 \circ f)(x)$. Combining Proposition 3.4(3) and Proposition 3.5(2) gives that f_1 and f_2 are Δ-continuous. $\square$

It is well known that open and closed sets are preserved under homoeomorphisms. Similar conclusion holds for Δ-open and Δ-closed sets.

Proposition 3.8. Let $f : X \to Y$ be a homeomorphism. Then
(1) The image of a Δ-open set is Δ-open.
(2) The inverse image of a Δ-open set is Δ-open.
(3) The image of a Δ-closed set is Δ-closed.
(4) The inverse image of a Δ-closed set is Δ-closed.

Proof. (1) Let A be Δ-open in X. Theorem 2.1 assures that $A = O \cap C$ for some an open set O in X and a closed set C in X. As f is one-to-one, we get $f(A) = f(O \cap C) = f(O) \cap f(C)$. Being f a homeomorphism we get $f(O)$ is open in Y and $f(C)$ is closed in Y. Therefore, $f(A)$ is Δ-open in Y by Theorem 2.1.
(2) Let B be Δ-open in Y. There are an open set G in Y and a closed set F in Y satisfying $B = G \cap F$; Theorem 2.1. So, $f^{-1}(B) = f^{-1}(G \cap F) = f^{-1}(G) \cap f^{-1}(F)$. Being f a homeomorphism we get $f^{-1}(G)$ is open in X and $f^{-1}(F)$ is closed in X. Therefore, $f^{-1}(B)$ is Δ-open in X by Theorem 2.1.
With the aid of Corollary 2.1, parts (3) and (4) are proved similarly. $\square$

4. Δ-irresolute functions

Given topological spaces X and Y. A function $f : X \to Y$ is called *irresolute* if for any semi-open set O in Y, $f^{-1}(O)$ is semi-open in X; [7]. In a similar way we propose the following notion in terms of Δ-open sets.

Definition 4.1. Let X and Y be topological spaces. A function $f : X \to Y$ is said to be Δ-*irresolute* if for any Δ-open set S in Y, $f^{-1}(S)$ is Δ-open in X.

Combining Definition 4.1 and Proposition 3.8, we get that each homeomorphism is Δ-irresolute. Evidently each Δ-irresolute function is Δ-continuous. The converse need not be true.

Example 4.1. Let $X = \{a, b, c\}$ with topologies $\tau_1 = \{\phi, X, \{a\}, \{a, b\}\}$ and $\tau_2 = \{\phi, X, \{b\}, \{b, c\}\}$. Define a function $f : (X, \tau_1) \to (X, \tau_2)$ by $f(a) = c$, $f(b) = b$, and $f(c) = a$. The collection of all Δ-open sets in (X, τ_1) is

$$\tau_{1\Delta o} = \{\phi, X, \{a\}, \{a, b\}, \{b, c\}, \{c\}, \{b\}\}$$

The collection of all Δ-open sets in (X, τ_2) is

$$\tau_{2\Delta o} = \{\phi, X, \{b\}, \{b, c\}, \{a, c\}, \{a\}, \{c\}\}$$

Clearly, f is Δ-continuous. Yet, the set $\{a, c\}$ is Δ-open in (X, τ_2), but $f^{-1}(\{a, c\}) = \{a, c\}$ is not Δ-open in (X, τ_1). Thus, f is not Δ-irresolute.

We show that each continuous function is Δ-irresolute.

Proposition 4.1. Let X and Y be topological spaces. If $f : X \to Y$ is continuous, then it is Δ-irresolute.

Proof. Let S be a Δ-open set in Y. By Theorem 2.1 there are an open set O and a closed set C in Y such that $S = O \cap C$. So, $f^{-1}(S) = f^{-1}(O \cap C) = f^{-1}(O) \cap f^{-1}(C)$ which is Δ-open in X by the continuity of f and Theorem 2.1. $\square$

Example 3.1 provides a Δ-irresolute function that is not continuous. We have seen that Δ-continuity is characterized in terms of closed and Δ-closed sets; Theorem 3.1. In the same manner, we have the following result for Δ-irresolute functions.

Theorem 4.1. *Let X and Y be topological spaces; let $f : X \to Y$. Then f is Δ-irresolute if and only if for each Δ-closed set C in Y, the set $f^{-1}(C)$ is Δ-closed in X.*

Proof. ($\Longrightarrow$) Assume $f : X \to Y$ is Δ-irresolute. Let C be a Δ-closed set in Y. Then $Y - C$ is Δ-open in Y. So, $f^{-1}(Y - C) = X - f^{-1}(C)$ is Δ-open in X. That is, $f^{-1}(C)$ is Δ-closed in X.
($\Longleftarrow$) Let O be a Δ-open set in Y. Then $Y - O$ is Δ-closed set in Y. So, $f^{-1}(Y - O) = X - f^{-1}(O)$ is Δ-closed in X. That is, $f^{-1}(O)$ is Δ-open in X. Hence, f is Δ-irresolute. $\square$

By similar arguments as Theorem 3.2, Proposition 3.1, Proposition 3.2, and Proposition 3.3, one can prove the next results on Δ-irresolute functions.

Proposition 4.2. Let X and Y be topological spaces; let $f : X \to Y$. If f is Δ-irresolute, then for each $x \in X$ and for each Δ-neighborhood of $f(x)$, $\Delta N(f(x))$ in Y, there is a Δ-neighborhood of x, $\Delta N(x)$ in X such that $f(\Delta N(x)) \subseteq \Delta N(f(x))$.

Proposition 4.3. Let X and Y be topological spaces; let $f : X \to Y$. If f is Δ-irresolute and for each subset B of Y, the $\Delta Cl(B)$ is Δ-closed in Y, then for each subset A of X, we have $f(\Delta Cl(A)) \subseteq \Delta Cl(f(A))$.

Proposition 4.4. Let X and Y be topological spaces; let $f : X \to Y$. If f is Δ-irresolute and for each subset B of Y, the $\Delta Int(B)$ is Δ-open in Y, then for each subset B of Y, we have $f^{-1}(\Delta Int(B)) \subseteq \Delta Int(f^{-1}(B))$.

Proposition 4.5. Let X and Y be topological spaces; let $f : X \to Y$. Assume for each $x \in X$ and for each Δ-neighborhood of $f(x)$, $\Delta N(f(x))$ in Y, there is a Δ-neighborhood of x, $\Delta N(x)$ in X such that $f(\Delta N(x)) \subseteq \Delta N(f(x))$. Assume further for any subset A of X the $\Delta Int(A)$ is Δ-open in X. Then f is Δ-irresolute.

Proposition 4.6. Let X and Y be topological spaces; let $f : X \to Y$. Assume for each subset A of X, we have $f(\Delta Cl(A)) \subseteq \Delta Cl(f(A))$. Assume further for any subset A of X the $\Delta Cl(A)$ is Δ-closed in X. Then f is Δ-irresolute.

Proposition 4.7. Let X and Y be topological spaces; let $f : X \to Y$. Assume for each subset B of Y, we have $f^{-1}(\Delta Int(B)) \subseteq \Delta Int(f^{-1}(B))$. Assume further for any subset A of X the $\Delta Int(A)$ is Δ-open in X. Then f is Δ-irresolute.

The proof of the following theorem is straight forward and so it is omitted.

Theorem 4.2. *Let X, Y, and Z be topological spaces.*
(1) If $f : X \to Y$ is homeomorphism, then f is Δ-irresolute.
(2) If $f : X \to Y$ is Δ-irresolute and $g : Y \to Z$ is Δ-irresolute, then $g \circ f : X \to Z$ is Δ-irresolute.
(3) If $f : X \to Y$ is Δ-irresolute and A is a subspace of X, then the restriction of f to A, $f|_A : A \to Y$ is Δ-irresolute.
(4) Let $f : X \to Y$ be Δ-irresolute. If B is a subspace of Y such that

$f(X) \subseteq B$, then the function $f : X \to B$ is Δ-irresolute.

(5) Let $f : X \to Y$ be Δ-irresolute. If Z is a topological space having Y as a subspace, then the function $h : X \to Z$ obtained by extending the codomain of f is Δ-irresolute.

(6) Let $f_1 : X \to Y$ and $f_2 : X \to Z$. Let $f : X \to Y \times Z$ be given as $f(x) = (f_1(x), f_2(x))$. If f is Δ-irresolute, then f_1 and f_2 are Δ-irresolute.

In connection with Theorem 3.4 the coming result is easily proved.

Theorem 4.3. *Let X and Y be topological spaces. Let $X = A \cup B$, where A and B are Δ-open in X. Let $f : A \to Y$ and $g : B \to Y$ be Δ-irresolute. Assume $f(x) = g(x)$ for each $x \in A \cap B$ and let $h : X \to Y$ be a function defined by $h(x) = f(x)$ for $x \in A$, and $h(x) = g(x)$ for $x \in B$. Assume further, the union of any two Δ-open sets in X is again Δ-open. Then, h is Δ-irresolute.*

5. Δ-open and Δ-closed functions

Recall that a function is called *open* (*closed*) if the image of each open (closed) set is again an open (a closed) set. Similar notions are given in terms of Δ-open and Δ-closed sets.

Definition 5.1. Let X and Y be topological spaces. A function $f : X \to Y$ is said to be Δ-*open* if for any Δ-open set S in X, $f(S)$ is Δ-open in Y.

Definition 5.2. Let X and Y be topological spaces. A function $f : X \to Y$ is said to be Δ-*closed* if for any Δ-closed set S in X, $f(S)$ is Δ-closed in Y.

In connection with Proposition 3.8, it is evident that a homeomorphism is both a Δ-open and a Δ-closed function. For the function f in Example 3.3 the set $S = \{c, d\}$ is both Δ-open and Δ-closed; but $f(S) = \{a, c\}$ is neither Δ-open nor Δ-closed. This gives an example of a function that is Δ-continuous but it is neither Δ-open nor Δ-closed.

Example 5.1. Let $X = \{a, b, c\}$ with topology $\tau_1 = \{\phi, X, \{a, b\}\}$. Let $Y = \{a, b, c, d\}$ with topology $\tau_2 = \{\phi, Y, \{b, d\}, \{c, d\}, \{d\}, \{b, c, d\}\}$. Define a function $f : (X, \tau_1) \to (Y, \tau_2)$ by $f(a) = b$, $f(b) = a$, and $f(c) = d$. The collection of all Δ-open sets in (X, τ_1) is the same as the collection of all Δ-closed sets in (X, τ_1), which is

$$\tau_{1\Delta o} = \{\phi, X, \{a, b\}, \{c\}\}$$

It is easy to see that f is both Δ-open and Δ-closed function. Yet, the set $\{b, d\}$ is open in (Y, τ_2), but $f^{-1}(\{b, d\}) = \{a, c\}$ is not Δ-open in (X, τ_1). Thus, f is not Δ-continuous. Hence, f is neither Δ-irresolute nor continuous.

Next we give an example of a Δ-open function that is not Δ-closed and vice versa.

Example 5.2. Let $X = \{a, b, c, d\}$ with topology $\tau = \{\phi, X, \{a, c, d\}, \{c, d\}\}$. The collection of all Δ-open sets in (X, τ) is

$$\tau_{\Delta o} = \{\phi, X, \{a, c, d\}, \{c, d\}, \{b\}, \{a, b\}, \{a\}\}$$

On one hand, define a function $f : X \to X$ by $f(a) = f(b) = f(c) = f(d) = a$. It is easy to see that f is Δ-open but not Δ-closed. On the other hand, define a function $g : X \to X$ by $g(a) = g(b) = b$, $g(c) = c$, and $g(d) = d$. Then g is Δ-closed. However, the set $\{a, c, d\}$ is Δ-open, yet $g(\{a, c, d\}) = \{b, c, d\}$ is not Δ-open. Thus, g is not Δ-open.

It is clear that the composition of two Δ-open (Δ-closed) functions is again a Δ-open (Δ-closed) function.

Theorem 5.1. *Let $f : X \to Y$ and $g : Y \to Z$ be functions. Assume that $g \circ f : X \to Z$ is Δ-open (Δ-closed).*
(1) If g is Δ-irresolute injection, then f is Δ-open (Δ-closed).
(2) If f is Δ-irresolute surjection, then g is Δ-open (Δ-closed).

Proof. We only consider the case that $g \circ f$ is Δ-open. The other case is similar.
(1) Let S be a Δ-open set in X. Then $(g \circ f)(S) = g(f(S))$ is Δ-open in Z. As g is Δ-irresolute, $g^{-1}(g(f(S)))$ is Δ-open in Y. But g is injection, so $g^{-1}(g(f(S))) = f(S)$. Hence, f is Δ-open.
(2) Let S be a Δ-open set in Y. Since f is Δ-irresolute, $f^{-1}(S)$ is Δ-open in X. In the view of the fact that $g \circ f$ is Δ-open, $(g \circ f)(f^{-1}(S))$ is Δ-open in Z. As f is surjection, $(g \circ f)(f^{-1}(S)) = g(f(f^{-1}(S))) = g(S)$. Hence, g is Δ-open. $\square$

Proposition 5.1. Let X and Y be topological spaces. If $f : X \to Y$ is Δ-open, then $f\big(\Delta Int(A)\big) \subseteq \Delta Int\big(f(A)\big)$ for any subset A of X.

Proof. Assume f is Δ-open. For a subset A of X, let $y \in f\big(\Delta Int(A)\big)$. Then there is $x \in \Delta Int(A)$ such that $y = f(x)$. Proposition 2.1 assures that $x \in \Delta N(x) \subseteq A$ for some Δ-neighbourhood $\Delta N(x)$ of x. So, $f(x) \in f\big(\Delta N(x)\big) \subseteq f(A)$. Since, f is Δ-open we get $f\big(\Delta N(x)\big)$ is Δ-open in Y. Hence, $f\big(\Delta N(x)\big) \subseteq \Delta Int\big(f(A)\big)$. Therefore, $y = f(x) \in \Delta Int\big(f(A)\big)$. $\square$

The coming result is a partial converse of the above proposition.

Proposition 5.2. Let X and Y be topological spaces and $f : X \to Y$. Suppose for each subset A of X, $f\big(\Delta Int(A)\big) \subseteq \Delta Int\big(f(A)\big)$ and for each subset B of Y the $\Delta Int(B)$ is Δ-open in Y. Then f is Δ-open.

Proof. Let A be a Δ-open set in X. Then $\Delta Int(A) = A$ and so $f(A) = f\big(\Delta Int(A)\big) \subseteq \Delta Int\big(f(A)\big)$. However, $\Delta Int\big(f(A)\big) \subseteq f(A)$ and hence $f(A) = \Delta Int\big(f(A)\big)$. By assumption $\Delta Int\big(f(A)\big) = f(A)$ is Δ-open in Y. Thus, f is Δ-open. $\square$

Proposition 5.3. Let X and Y be topological spaces. If $f : X \to Y$ is Δ-closed, then $\Delta Cl\big(f(A)\big) \subseteq f\big(\Delta Cl(A)\big)$ for any subset A of X.

Proof. Assume f is Δ-closed and A is a subset of X. Then for any Δ-closed set C in X with $C \supseteq A$ we have $f(C)$ is Δ-closed in Y and clearly $f(C) \supseteq f(A)$. Therefore,

$$\Delta Cl\big(f(A)\big) = \bigcap_{\substack{K \supseteq f(A) \\ K \ \Delta\text{-closed}}} K \subseteq \bigcap_{\substack{C \supseteq A \\ C \ \Delta\text{-closed}}} f(C) \subseteq f\left(\bigcap_{\substack{C \supseteq A \\ C \ \Delta\text{-closed}}} C \right) = f\big(\Delta Cl(A)\big).$$

$\square$

The coming result is a partial converse of the above proposition.

Proposition 5.4. Let X and Y be topological spaces and $f : X \to Y$. Suppose for each subset A of X, $\Delta Cl\big(f(A)\big) \subseteq f\big(\Delta Cl(A)\big)$ and for each subset B of Y the $\Delta Cl(B)$ is Δ-closed in Y. Then f is Δ-closed.

Proof. Let A be a Δ-closed set in X. Then $\Delta Cl(A) = A$ and so $\Delta Cl\big(f(A)\big) \subseteq f\big(\Delta Cl(A)\big) = f(A)$. However, $f(A) \subseteq \Delta Cl\big(f(A)\big)$ and hence $f(A) = \Delta Cl\big(f(A)\big)$. By assumption $\Delta Cl\big(f(A)\big) = f(A)$ is Δ-closed in Y. Thus, f is Δ-closed. $\square$

Proposition 5.5. Let X and Y be topological spaces and $f : X \to Y$ be Δ-open. Then for each subset B of Y and each Δ-closed set H in X containing $f^{-1}(B)$ there exists a Δ-closed set K in Y such that $K \supseteq B$ and $f^{-1}(K) \subseteq H$.

Proof. Let $B \subseteq Y$ and H be a Δ-closed set in X with $H \supseteq f^{-1}(B)$. Then $X - H$ is Δ-open in X and hence $f(X - H)$ is Δ-open in Y. Take $K = Y - f(X - H)$. Clearly, K is Δ-closed in Y, and

$$
\begin{aligned}
H \supseteq f^{-1}(B) \;\Leftrightarrow\;& X - H \subseteq X - f^{-1}(B) \\
\Rightarrow\;& f(X - H) \subseteq f\big(X - f^{-1}(B)\big) \\
\Leftrightarrow\;& K = Y - f(X - H) \supseteq Y - f\big(X - f^{-1}(B)\big) \\
\Rightarrow\;& K \supseteq Y - f\big(X - f^{-1}(B)\big) \supseteq B \\
\Rightarrow\;& K \supseteq B.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
f^{-1}(K) \;=\;& f^{-1}\big(Y - f(X - H)\big) \\
=\;& f^{-1}(Y) - f^{-1}\big(f(X - H)\big) \\
=\;& X - f^{-1}\big(f(X - H)\big) \\
\subseteq\;& X - (X - H) = H.
\end{aligned}
$$

$\square$

Proposition 5.6. Let X and Y be topological spaces and $f : X \to Y$ be Δ-closed. Then for each subset B of Y and each Δ-open set U in X containing $f^{-1}(B)$ there exists a Δ-open set V in Y such that $V \supseteq B$ and $f^{-1}(V) \subseteq U$.

Proof. Let $B \subseteq Y$ and U be a Δ-open set in X with $U \supseteq f^{-1}(B)$. Then $X - U$ is Δ-closed in X and hence $f(X - U)$ is Δ-closed in Y. Take $V = Y - f(X - U)$. Clearly, V is Δ-open in Y, and

$$
\begin{aligned}
U \supseteq f^{-1}(B) \;\Leftrightarrow\;& X - U \subseteq X - f^{-1}(B) \\
\Rightarrow\;& f(X - U) \subseteq f\big(X - f^{-1}(B)\big) \\
\Leftrightarrow\;& V = Y - f(X - U) \supseteq Y - f\big(X - f^{-1}(B)\big) \\
\Rightarrow\;& V \supseteq Y - f\big(X - f^{-1}(B)\big) \supseteq B \\
\Rightarrow\;& V \supseteq B.
\end{aligned}
$$

Furthermore,

$$\begin{aligned}
f^{-1}(V) &= f^{-1}\big(Y - f(X - U)\big) \\
&= f^{-1}(Y) - f^{-1}\big(f(X - U)\big) \\
&= X - f^{-1}\big(f(X - U)\big) \\
&\subseteq X - (X - U) = U.
\end{aligned}$$

$\square$

Conclusion

In this work, we introduced four distinct classes of functions within topological spaces, characterized by their interactions with Δ-open and Δ-closed sets.

The inaugural class, Δ-continuous functions, is defined by the criterion that the inverse image of any open set must be Δ-open. A comprehensive characterization of Δ-continuous functions, involving closed and Δ-closed sets, is encapsulated in Theorem 3.1.

Subsequently, we explored Δ-irresolute functions, identified by the property that the inverse image of each Δ-open set remains Δ-open. Theorem 4.1 offers a pivotal characterization of Δ-irresolute functions, leveraging Δ-closed sets.

The implications

continuous function $\Rightarrow$ Δ-irresolute function $\Rightarrow$ Δ-continuous function

underscore the hierarchical relationships among these classes. Nonetheless, as demonstrated by Example 3.1 and Example 4.1, these implications are not bi-directional.

The third and fourth classes—Δ-open and Δ-closed functions, respectively—further enrich our taxonomy. A Δ-open function ensures the Δ-openness of images of Δ-open sets, whereas a Δ-closed function guarantees the Δ-closed nature of images of Δ-closed sets. The distinctions among Δ-open, Δ-closed, and Δ-continuous functions, highlighted through Examples 5.1 and 5.2, reveal the nuanced interplay of these concepts.

Looking ahead, we aim to extend our inquiry into the realms of Δ-*compactness*, Δ-*connectedness*, and Δ-*separation axioms* within topological spaces.

Acknowledgment

Funding

Conflict of Interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

Compliance with Ethical Standards

not applicable.

Data Availability Statement

No data was generated or analyzed in the preparation of this manuscript. Therefore, data sharing is not applicable to this article.

References

[1] Alqahtani, M. H., & Saleh, H. Y. (2023). A Novel Class of Separation Axioms, Compactness, and Continuity via C-Open Sets. *Mathematics, 11*(23), 4729.

[2] D. Andrijevićc, Semi-preopen sets, Matematički Vesnik 38 (93) (1986) 24-32.

[3] Añora, L. A., Paran, C., Butanas, L. L., & Labendia, M. (2023). θ^{ω}-Open Set and its Corresponding Topological Concepts. *The Mindanawan Journal of Mathematics, 5*(2), 75-90.

[4] P. Bhattacharyya, Semi-generalized closed sets in topology, Indian J. Math. 29 (3) (1987) 375-382.

[5] Boonpok, C., & Srisarakham, N. (2023). Weak forms of (Λ, b)-open sets and weak (Λ, b)-continuity. *European Journal of Pure and Applied Mathematics, 16*(1), 29-43.

[6] Boonpok, C., & Viriyapong, C. (2023). On some forms of closed sets and related topics. *European Journal of Pure and Applied Mathematics, 16*(1), 336-362.

[7] Crossley, S., & Hildebrand, S. (1972). Semi-topological properties. *Fundamenta Mathematicae*, 74(3), 233-254.

[8] S. Ganesan, On α-Δ-open sets and generalized Δ-closed sets in topological spaces, International Journal of Analytical and Experimental Model Analysis 12 (2020) 213-239.

[9] Khattabomar, O. Y. (2023). Expansion $\widetilde{gis}$-closed & its lower separation axioms. *Desimal: Jurnal Matematika, 6*(1), 19-28.

[10] Kocaman, A. H. (in press). On some strong irresolute functions defined by betaopen sets. *Annals of Fuzzy Mathematics and Informatics*.

[11] Ladsungnern, S., & Jongrak, A. (2023). A Characterization of S_β-continuous Fixed Point Property. *Progress in Applied Science and Technology, 13*(2), 9-16.

[12] Latif, R. M. (2023). $G^{**}\beta$-Continuous and $G^{**}\beta$-Irresolute Mappings in Topological Spaces. *The Eurasia Proceedings of Science Technology Engineering and Mathematics, 25,* 168-186.

[13] N. Levine, Semi-open sets and semi-continuity in topological spaces, The American mathematical monthly 70 (1) (1963) 36-41.

[14] N. Levine, Generalized closed sets in topology, Rendiconti del Circolo Matematico di Palermo 19 (1) (1970) 89-96.

[15] Maheshwari, S. N., & Thakur, S. S. (1980). On α-irresolute mappings. *Tamkang J. Math*, 11, 209-214.

[16] Marabeh, M. A. A. (2023). Topological-like notions via Δ-open sets. *Results in Nonlinear Analysis*, 6(1), 12-23.

[17] A. Mashhour, On preconlinuous and weak precontinuous mappings, in: Proc. Math. Phys. Soc. Egypt., Vol. 53, 1982, pp. 47-53.

[18] Mashhour, A. S., Hasanein, I. A., & El-Deeb, S. N. (1983). α-continuous and α-open mappings. *Acta Mathematica Hungarica*, 41(3-4), 213-218.

[19] Munkres, J. R. (2014). *Topology* . Pearson New International Edition.

[20] Navalagi, G. B. (2002). Semi-precontinuous functions and properties of generalized semi-preclosed sets in topological spaces. *International Journal of Mathematics and Mathematical Sciences*, 29(2), 85-98.

[21] O. Njåstad, On some classes of nearly open sets, Pacific Journal of Mathematics 15 (3) (1965) 961-970.

[22] T. Nour, A. M. Jaber, Semi Δ-open sets in topological spaces, International Journal of Mathematics Trends and Technology (IJMTT) 66 (8) (2020) 139-143.

[23] N. Palaniappan, K. C. Rao, Regular generalized closed sets, Kyungpook Mathematical Journal 33 (2) (1993) 211-219.

[24] Rajab, A. M., Ali, D. Z., & Hadi, O. A. (2023). Decomposition of Pre-β-Irresolute Maps and g-Closed Sets in Topological Space. *International Journal of Research and Review, 10*(7).

[25] Rajab, A. M., Hamd, H. S. A., & Hameed, E. N. (2023). Properties and Characterizations of k-Continuous Functions and k-Open Sets in Topological Spaces. *International Journal of Science and Healthcare Research, 8*(3).

[26] RAMACHANDRAM, V. V., & Nagapurnima, D. (2023). Irresolute maps in topological ordered spaces. *Journal of Contemporary Technology and Applied Engineering, 2*(2), 1-5.

[27] J. A. Sasam, M. Labendia, θ_{sw}-continuity of maps in the product space and some versions of separation axioms, The Mindanawan Journal of Mathematics 4 (1) (2022) 1-12.

[28] J. S. Shuwaie, A. K. Hussain, Topological spaces F_1 and F_2, Wasit Journal of Computer and Mathematics Sciences 1 (2) (2022) 62-70.

[29] M. Veera Kumar, Between semi-closed sets and semi-pre-closed set, Rend. Istit. Mat. Univ. Trieste, (ITALY) XXXXII, (2000) 25-41.

[30] N. Velicko, H-closed topological spaces, Amer. Math. Soc. Transl. 78 (1968) 103-118.