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ON A CONJECTURE ON SHARYGIN TRIANGLES

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Abstract: By a simple method we prove the following conjecture on Sharygin triangles: there is only one Sharygin triangle (up to an isometry) whose vertices are chosen from the set of vertices of a regular polygon inscribed in a circle of radius 1.

Keywords: Sharygin triangle, trigonometric diophantine equation.

1 Introduction

In the paper, we will use the same terminology as in [8]. Let us recall the most important definitions.

For a triangle ABC, let A_1, B_1, and C_1 be the intersection points of the *internal bisectors* with opposite sides (here, A_1 lies on BC etc.). We call the triangle $A_1B_1C_1$ the *bisectral triangle* of a given triangle ABC.

Definition. A non-isosceles triangle ABC is called a *Sharygin triangle* if its bisectral triangle $A_1B_1C_1$ is isosceles.

Surprisingly, there are Sharygin triangles. Furthermore, each of them must have an obtuse angle lying in a very small interval between $\approx 102.663°$ and $\approx 104.478°$ (see for more details the solution of Problem 203 in [10, Sec. 2]).

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The most famous example of a Sharygin triangle ABC is the so-called *heptagonal triangle* with angles

$$\angle A = \pi/7, \quad \angle B = 2\pi/7, \quad \angle C = 4\pi/7. \tag{1.1}$$

For this triangle, we have

$$C_1A_1 = C_1B_1. \tag{1.2}$$

The heptagonal triangle was firstly studied in [1]. Later, it has been rediscovered several times (see, for instance, [11]).

It is easy to see that all the vertices of the heptagonal triangle can be placed in some vertices of a regular heptagon. Many different properties of the heptagonal triangle can be found in [6, Ch. III] (see Problems 11, 23, 45 in Sec. 1 and 14 in Sec. 2).

In [8], the authors asked the following natural question: "Are there other regular polygons such that three of its vertices form a Sharygin triangle?". The computer experiments led them to the following conjecture (see Hypothesis 2.2 in [8]) which has been verified for all regular N-gons with N up to 2000.

Conjecture. Suppose that the vertices of a Sharygin triangle coincide with three vertices of a regular polygon. Then this triangle is similar to the heptagonal triangle.

The main aim of our paper is to prove it in the general case. In Section 2, we prove Theorem from which we can very easily deduce the statement of Conjecture (see Corollary in Section 3). We also find two triangles (the so-called pentadecagonal triangles) which have similar properties to the heptagonal triangle. In the Section 4, we give some comments and extra results.

Denote by $\mathbb{N}$ the set of positive integers. For an arbitrary $N \in \mathbb{N}$, let

$$\zeta_N = \exp\left(2\pi\sqrt{-1}/N\right)$$

be the standard primitive Nth root of unity in the field of complex numbers $\mathbb{C}$. Let

$$f(x, y) = 1 + x + y + x^2y^3 + x^3y^2 + x^3y^3$$

be the special symmetric polynomial in x, y mentioned in [8, Sec. 2]. According to the results obtained in [8, Sec. 2], we need to solve the system

$$f(x, y) = 0 \text{ and } x^N = y^N = 1 \text{ for some } N \in \mathbb{N} \tag{1.3}$$

over $\mathbb{C}$. Here, the unknowns complex numbers x and y have the following sense: their absolute values are 1 while their arguments $\arg x$ and $\arg y$ coincide with two acute angles of a Sharygin triangle (recall that its third angle must be obtuse). Therefore, it is enough to prove that there is unique non-ordered pair $\{x, y\}$ satisfying (1.3) for which real and imaginary parts of three numbers x, y, xy are positive. Obviously, this pair $\{x, y\}$ must correspond to the heptagonal triangle that means

$$\{\arg x, \arg y\} = \{\pi/7, 2\pi/7\}.$$

As it turns out later, a simplifying point is that the system (1.3) has only two unknowns. Our idea is to construct a finite set of additional algebraic equations $h_i(x,y)=0$ such that any solution of (1.3) must be a solution of one of the systems

$$f(x,y)=0, \quad h_i(x,y)=0.$$

Then we could just solve all such systems using a *computer algebra system* (in view of *Bezout's theorem* and absolute irreducibility of $f(x,y)$, we can a priori assume that each of them has only a finite set of solutions over $\mathbb{C}$).

2 Main Result

Let $\alpha=\zeta_3$, $\beta=\zeta_5$, $\gamma=\zeta_7$ and define the sets S_0, …, S_3 as follows:

$$\begin{aligned}
S_0 &= \{(1,-1),(-1,1)\},\\
S_1 &= \{(\alpha,\alpha),(\alpha^2,\alpha),(\alpha,\alpha^2),(\alpha^2,\alpha^2),\\
&\qquad (\alpha,-\alpha),(-\alpha,\alpha),(\alpha^2,-\alpha^2),(-\alpha^2,\alpha^2)\},\\
S_2 &= \{(\beta\alpha,\beta\alpha^2),(\beta^2\alpha,\beta^2\alpha^2),(\beta^3\alpha,\beta^3\alpha^2),(\beta^4\alpha,\beta^4\alpha^2),\\
&\qquad (\beta\alpha^2,\beta\alpha),(\beta^2\alpha^2,\beta^2\alpha),(\beta^3\alpha^2,\beta^3\alpha),(\beta^4\alpha^2,\beta^4\alpha)\},\\
S_3 &= \{(\gamma,-\gamma^4),(\gamma^2,-\gamma),(\gamma^3,-\gamma^5),(\gamma^4,-\gamma^2),(\gamma^5,-\gamma^6),(\gamma^6,-\gamma^3),\\
&\qquad (-\gamma,\gamma^2),(-\gamma^2,\gamma^4),(-\gamma^3,\gamma^6),(-\gamma^4,\gamma),(-\gamma^5,\gamma^3),(-\gamma^6,\gamma^5)\}.
\end{aligned}$$

Also, let S be the union of S_0, …, S_3 so that $|S|=30$.

Theorem. The set of all solutions (x,y) of the system (1.3) over $\mathbb{C}$ coincides with S.

Denote by V the set $\{-1,-1/2,-1/4\}$. Let φ and μ be the *Euler totient function* and the *Möbius function*, respectively (the standard properties of these functions can be found, for example, in [7, §A.5, §A.6]). First we prove the following auxiliary result.

Lemma. For any $N\in\mathbb{N}$ and $k\in\mathbb{Z}$, if $M=N/(k,N)$ and $\mu(M)/\varphi(M)\in V$ then there is $t\in\mathbb{Z}$ such that $k=tN/2$ or $k=tN/3$ or $k=tN/5$. Here, (k,N) denotes $\gcd(k,N)$.

PROOF. We have to consider three cases with respect to the special values of $\mu(M)/\varphi(M)$ represented in V.

(a) If $\mu(M)/\varphi(M)=-1$ then $\mu(M)=-1$ and $\varphi(M)=1$. Therefore, $M=2$, $(k,N)=N/2$ and $k=tN/2$ for some $t\in\mathbb{Z}$.

(b) If $\mu(M)/\varphi(M)=-1/2$ then $\mu(M)=-1$ and $\varphi(M)=2$. This implies $M=3$ (indeed, if $\varphi(M)=2$ then $M\in\{3,4,6\}$ and only $M=3$ is suitable). Therefore, $(k,N)=N/3$ and we get $k=tN/3$ for some $t\in\mathbb{Z}$.

(c) If $\mu(M)/\varphi(M)=-1/4$ then $\mu(M)=-1$ and $\varphi(M)=4$. Here, we have $M=5$ (indeed, if $\varphi(M)=4$ then $M\in\{5,8,10,12\}$ and only $M=5$ is suitable). Therefore, $(k,N)=N/5$ and $k=tN/5$ fore some $t\in\mathbb{Z}$.

The proof is completed. □

We can now proceed to prove the main result of the paper.

Proof of Theorem. We can set $x = \zeta_N^a$, $y = \zeta_N^b$ for some integers a, b and $N \in \mathbb{N}$. Then

$$1 + \zeta_N^a + \zeta_N^b + \zeta_N^{2a+3b} + \zeta_N^{3a+2b} + \zeta_N^{3a+3b} = 0. \tag{2.1}$$

Clearly, all summands in the left hand side (2.1) are elements of the *cyclotomic field* $\mathbb{Q}(\zeta_N)$. Now, let us recall some basics facts on cyclotomic fields (see, for instance, [12, Ch. 2]). The *automorphisms* of $\mathbb{Q}(\zeta_N)$ are exactly the maps defined by the rule $\zeta_N \mapsto \zeta_N^j$ where j is a residue modulo N with $\gcd(j, N) = 1$. Denote by R_N^* the set of all such residues j. Then we have

$$[\mathbb{Q}(\zeta_N) : \mathbb{Q}] = |R_N^*| = \varphi(N).$$

This is a direct consequence of the fact that the so-called *cyclotomic polynomial*

$$\Phi_N(x) = \prod_{j \in R_N^*} (x - \zeta_N^j)$$

has rational (even integer) coefficients and is irreducible over $\mathbb{Q}$.

It follows from (2.1) that

$$1 + \zeta_N^{aj} + \zeta_N^{bj} + \zeta_N^{(2a+3b)j} + \zeta_N^{(3a+2b)j} + \zeta_N^{(3a+3b)j} = 0$$

for each $j \in R_N^*$. After summation over $j \in R_N^*$, we obtain

$$\varphi(N) + c_N(a) + c_N(b) + c_N(2a+3b) + c_N(3a+2b) + c_N(3a+3b) = 0, \tag{2.2}$$

where c_N is the *Ramanujan sum* defined by

$$c_N(k) = \sum_{j \in R_N^*} \zeta_N^{kj}.$$

It is a well known fact that

$$c_N(k) = \frac{\varphi(N)\mu(N/(k,N))}{\varphi(N/(k,N))} \tag{2.3}$$

(see, for example, [7, §A.7]).

For a convenient notation, let

$$k_1 = a, \quad k_2 = b, \quad k_3 = 2a+3b, \quad k_4 = 3a+2b, \quad k_5 = 3a+3b$$

so that (2.1) and (2.2) become

$$1 + \sum_{i=1}^{5} \zeta_N^{k_i} = 0, \quad \varphi(N) + \sum_{i=1}^{5} c_N(k_i) = 0,$$

respectively. Applying (2.3) and reducing by $\varphi(N)$, rewrite the last equality as

$$1 + \sum_{i=1}^{5} \frac{\mu(M_i)}{\varphi(M_i)} = 0 \tag{2.4}$$

where $M_i = N/(k_i, N)$ for $i = 1, \dots, 5$.

Note that, for every $M \in \mathbb{N}$, a fraction of the shape $\mu(M)/\varphi(M)$ must be equal to 0, ± 1, or $\pm 1/m$ where $m \in \mathbb{N}$ is even. Based on Lemma, one can deduce the following: if one of the fractions $\mu(M_i)/\varphi(M_i)$ belongs to the set V then the corresponding k_i can be represented as $k_i = tN/2$ or $k_i = tN/3$ or $k_i = tN/5$ with some integer t. It is important to remark that the condition

"one of the fractions $\mu(M_i)/\varphi(M_i)$ is an element of V"

must be fulfilled. Indeed, otherwise there are at most five negative fractions $\mu(M_i)/\varphi(M_i)$ and each of them in absolute value does not exceed 1/6; in this case, the equality (2.4) cannot be true.

Let us consider all indicated cases for k_i's. Introduce the monomials

$$g_1(x,y) = x, \quad g_2(x,y) = y,$$
$$g_3(x,y) = x^2y^3, \quad g_4(x,y) = x^3y^2, \quad g_5(x,y) = x^3y^3$$

and note that for $x = \zeta_N^a$ and $y = \zeta_N^b$ the value of $g_i(x,y)$ is $\zeta_N^{k_i}$ $(i = 1,\dots,5)$. If $k_i = tN/2$ then $\zeta_N^{2k_i} = 1$. This means that we obtain an additional equation $g_i(x,y)^2 = 1$ for some i. Likewise, in two remaining cases $k_i = tN/3$ and $k_i = tN/5$ we get an additional equation $g_i(x,y)^3 = 1$ and $g_i(x,y)^5 = 1$ with some i, respectively.

Thus, it just remains to solve over $\mathbb{C}$ fifteen systems of algebraic equations of the form

$$f(x,y) = 0, \quad g_i(x,y)^l = 1 \tag{2.5}$$

where $i = 1,\dots,5$ and $l \in \{2,3,5\}$. More precisely, we have to determine those solutions (x,y) for which there is $N \in \mathbb{N}$ such that $x^N = y^N = 1$ (let us call such (x,y) the *special solutions*). A priori, N may depend on (x,y) but below we show that one can take $N = 210$ for any special solution (x,y).

For this purpose, we can proceed different well-known techniques. We prefer *Gröbner bases*, but we could just as well use *resultants* (regarding these notions, we refer the reader to any book on algebra or computer algebra, for example [4]). For computing with polynomials, we plan use a computer algebra system (for instance, Maple CAS [14]).

For each of the systems (2.5), we compute Gröbner basis G with respect to *pure lexicographic order* $y \succ x$. The first polynomial in G depends only on x and has rational coefficients since both $f(x,y)$ and $g_i(x,y)$ are in $\mathbb{Q}[x,y]$. Denote this polynomial by $F(x)$. Further, we factorize $F(x)$ over $\mathbb{Q}$ and find out whether one of its irreducible factors $p(x)$ coincides with some cyclotomic polynomial $\Phi_n(x)$.

Eventually, after factorization of all such polynomials $F(x)$ for all systems (2.5), we obtain a finite set K of pairwise distinct monic irreducible factors $p(x)$ so that if $p(x) \in K$ then

$$d = \deg p(x) \in D = \{1,2,4,6,8,12,16,28\}.$$

As an example, represent an irreducible factor of degree $d = 12$ received for the system (2.5) with $i = 2$ and $l = 5$:

$$p_{12}(x) = x^{12} + 2x^{11} + 6x^{10} + 5x^9 - 8x^7 - 11x^6 - 8x^5 + 5x^3 + 6x^2 + 2x + 1.$$

If $p_{12}(x) = \Phi_n(x)$ for some n then $\deg \Phi_n(x) = \varphi(n) = 12$. Therefore, we get $n \in \{13, 21, 26, 28, 36, 42\}$. However, we actually have $p_{12}(x) \neq \Phi_n(x)$ for such n.

In general, for all $p(x) \in K$, a possible coincidence $p(x) = \Phi_n(x)$ for some n implies

$$n \in I_1 = \{1 \text{ through } 10,$$
$$12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 26, 28, 29, 30, 32, 34, 36, 40, 42, 48, 58, 60\}.$$

Let $L = \{\Phi_n(x) : n \in I_1\}$. In fact, we obtain $K \cap L = \{\Phi_n(x) : n \in I_2\}$ with

$$I_2 = \{1, 2, 3, 5, 6, 7, 14, 15\} \subset I_1.$$

Since $\operatorname{lcm}(I_2) = 210$, we arrive at $x^{210} = 1$ for any special solution (x, y). We also have $y^{210} = 1$ due to a symmetry.

We can now determine all solutions $(x, y) = (\zeta_N^a, \zeta_N^b)$ of the system (1.3) just letting $N = 210$ and using brute force search for (a, b). As a result, we obtain the following set P of pairs (a, b): two pairs

$$(70, 70),\ (140, 140)$$

with $a = b$ and twenty eight pairs with $a \neq b$ which are

$$(0, 105),\ (14, 154),\ (15, 30),\ (28, 98),\ (35, 140),\ (45, 90),\ (56, 196),$$
$$(60, 135),\ (70, 140),\ (70, 175),\ (75, 150),\ (112, 182),\ (120, 165),\ (180, 195)$$

together with symmetric ones. Thus, the set of all solutions of the system (1.3) is

$$\{(\zeta_{210}^a, \zeta_{210}^b) : (a, b) \in P\}. \tag{2.6}$$

One can verify that (2.6) is exactly the set S defining before. This completes the proof. □

Remark. Surely, in the final part of the proof, we could solve only five systems (2.5) with a single value l equal to $\operatorname{lcm}(2, 3, 5) = 30$. However, this would make the corresponding sets D, I_1, I_2 longer. Moreover, it would be more difficult to compute Gröbner base G for those monomials $g_i(x, y)$ for which $\deg (g_i(x, y)^{30} - 1)$ admits large values (in this case, we would have to compute some resultants). Anyway, it's clear that this part of the proof is purely technical.

3 Proof of a Conjecture

Based on Theorem, we can now prove the statement of Conjecture.

Corollary. The statement of Conjecture is true.

Proof. We have only to highlight those solutions $(x, y) \in S$ for which real and imaginary parts of three numbers x, y, and xy are positive. A direct checking shows us that this condition is satisfied for only one (up to a symmetry) solution

$$(x, y) = (\gamma, -\gamma^4) \tag{3.1}$$

which lies in S_3. Here, we have $\arg x = 2\pi/7$ and $\arg y = \pi/7$ as desired. As noted before, this leads us to the heptagonal triangle. □

Meanwhile, the following question remains: do the other solutions (x, y) from S make any geometrical sense? It's clear that only the solutions $(x, y) \in S_2$ can be of interest.

Below, we show how the solution

$$(x, y) = (\beta\alpha, \beta\alpha^2) \tag{3.2}$$

might be interpreted geometrically. For this purpose, we interpret the Euclidean plan as the *complex plan* $\mathbb{C}$. Let us define a triangle ABC by the tangency points

$$T_a = z_1, \quad T_b = z_2, \quad T_c = z_3 \tag{3.3}$$

of the *inscribed circle* with the sides BC, CA, AB, respectively. Here, we assume the parameters z_i's to be complex numbers with absolute values 1 so that

$$A = \frac{2z_2z_3}{z_2 + z_3}, \quad B = \frac{2z_3z_1}{z_3 + z_1}, \quad C = \frac{2z_1z_2}{z_1 + z_2}$$

(see [9] for more details about such a parametrization of triangles on Euclidean plan). Due to geometrical reasons, the triangle $T_aT_bT_c$ must be acute-angled. Then we have

$$A_1 = \frac{2z_1z_2z_3}{z_1^2 + z_2z_3}, \quad B_1 = \frac{2z_1z_2z_3}{z_2^2 + z_3z_1}, \quad C_1 = \frac{2z_1z_2z_3}{z_3^2 + z_1z_2}.$$

One can verify that if the triangle ABC is non-isosceles then the condition (1.2) is equivalent to the following equation:

$$z_1^3z_2 + z_2^3z_1 - z_1^3z_3 - z_2^3z_3 + z_1^2z_3^2 + z_2^2z_3^2 = 0.$$

Letting here $z_3 = -1$, $z_1 = x$, $z_2 = y^{-1}$, we get exactly our equation

$$f(x, y) = 0.$$

For the solution (3.1), three points

$$T_a = x, \quad T_b = y^{-1}, \quad T_c = -1$$

actually form an acute-angled triangle. Moreover, a direct checking shows that the corresponding non-isosceles triangle ABC whose vertices are

$$A = -\frac{2}{1 - y}, \quad B = \frac{2x}{1 - x}, \quad C = \frac{2x}{1 + xy}$$

is exactly the heptagonal triangle with angles (1.1).

On the contrary, for the solution (3.2), the triangle $T_aT_bT_c$ is not acute-angled. Nevertheless, the corresponding triangle ABC exists and its angles are

$$\angle A = 11\pi/15, \quad \angle B = \pi/15, \quad \angle C = \pi/5. \tag{3.4}$$

We need to interpret correctly (from a geometrical point of view) the equality (1.2).

For an arbitrary non-isosceles triangle ABC, let A_2, B_2, and C_2 be the intersection points of the *external bisectors* with opposite sides. For the triangle ABC generated by the solution (3.2), the formally internal bisectors AA_1 and BB_1 are actually its external bisectors AA_2 and BB_2, respectively. Thus, for the triangle ABC with angles (3.4), the equality (1.2) must be interpreted as

$$C_1A_2 = C_1B_2.$$

This yields a desired geometrical interpretation of (3.2).

Similarly, the solution

$$(x, y) = (\beta^2\alpha, \beta^2\alpha^2) \tag{3.5}$$

brings us another non-isosceles triangle ABC with angles

$$\angle A = 2\pi/15, \quad \angle B = 7\pi/15, \quad \angle C = 6\pi/15 \tag{3.6}$$

but now the correct way to interpret (1.2) is

$$C_2A_1 = C_2B_2.$$

For a triangle ABC, the corresponding triangles $A_1B_2C_2$, $A_2B_1C_2$, and $A_2B_2C_1$ also will be called the bisectral triangles. As the vertices of the triangles with angles (3.4) and (3.6) can be placed in suitable vertices of a regular pentadecagon, we call them the *pentadecagonal triangles* (the *first* and the *second*, respectively).

4 Concluding Remarks and extra results

Let us comment on the results we have obtained and discuss other ways of proving Theorem. Also, we give some extra results concerning the pentadecagonal triangles.

1. How diverse can the equalities of the form (2.4) be if we go through all thirty solutions (2.6)? A direct computation shows that the complete list of such equalities is the following:

$$1 + 1 - 1 - 1 + 1 - 1 = 0,$$
$$1 - 1 + 1 + 1 - 1 - 1 = 0$$

(without fractions, only ± 1) and

$$1 + \frac{1}{2} - \frac{1}{2} - \frac{1}{2} + \frac{1}{2} - 1 = 0,$$
$$1 - \frac{1}{2} + \frac{1}{2} + \frac{1}{2} - \frac{1}{2} - 1 = 0,$$
$$1 + \frac{1}{6} - \frac{1}{6} - \frac{1}{6} - 1 + \frac{1}{6} = 0,$$

$$1-\frac{1}{6}+\frac{1}{6}-1-\frac{1}{6}+\frac{1}{6}=0,$$
$$1-\frac{1}{2}-\frac{1}{2}-\frac{1}{2}-\frac{1}{2}+1=0,$$
$$1+\frac{1}{8}+\frac{1}{8}-\frac{1}{2}-\frac{1}{2}-\frac{1}{4}=0.$$

One can see that all numbers from the set $V=\{-1,-1/2,-1/4\}$ are actually represented here. Meanwhile, there are other equalities which could to be (2.4), namely

$$\begin{aligned} &1+0-\frac{1}{4}-\frac{1}{4}-\frac{1}{4}-\frac{1}{4}=0,\\ &1-\frac{1}{4}-\frac{1}{4}-\frac{1}{4}-\frac{1}{6}-\frac{1}{12}=0,\\ &1-\frac{1}{4}-\frac{1}{4}-\frac{1}{4}-\frac{1}{8}-\frac{1}{8}=0,\\ &1-\frac{1}{4}-\frac{1}{4}-\frac{1}{6}-\frac{1}{6}-\frac{1}{6}=0. \end{aligned} \tag{4.1}$$

Since the system (1.3) has only two unknowns x and y (and, consequently, there is a simpler way which was chosen by us), we do not need to study all of them.

2. Among the components of all pairs $(x,y)\in S$ there are all primitive nth roots of unity whose order $n\in I_2$ except $n=5$. For instance, we have

$$-\alpha^2=\zeta_6,\quad -\gamma^4=\zeta_{14},\quad \beta^2\alpha^2=\zeta_{15}.$$

In order to clarify I_2, we can modify the proof of Theorem as follows. Let $V'=\{-1,-1/2\}$. If the condition

$$\text{“one of the fractions } \mu(M_i)/\varphi(M_i) \text{ is an element of } V'\text{”} \tag{4.2}$$

is satisfied then we solve ten systems (2.5) with $i=1,\dots,5$ and $l\in\{2,3\}$ as described above. In this case, the corresponding sets D, I_1, I_2 will be shorter, namely

$D=\{1,2,6,8,12\}$,
$I_1=\{1,2,3,4,6,7,9,13,14,15,16,18,20,21,24,26,28,30,36,42\}$,
$I_2=\{1,2,3,6,7,14,15\}$.

Solving all such systems, we arrive at the same set S of solutions (x,y) as earlier. But we have to consider the non-empty case when (4.2) is not satisfied. In this case, the equality (2.4) must be one of the equalities (4.1). Next, we can, for example, solve twenty systems of the form

$$f(x,y)=0,\quad g_{i_1}(x,y)^l=g_{i_2}(x,y)^l=g_{i_3}(x,y)^l=1$$

with $1\leqslant i_1<i_2<i_3\leqslant 5$ and $l\in\{5,7\}$ (see Lemma which needs to be supplemented with the following: if $\mu(M)/\varphi(M)=-1/6$ then $k=tN/7$ where $t\in\mathbb{Z}$). Finally, it occurs that all these systems have no any solutions (x,y).

3. Each pair $(x, y) \in S$ yields a certain identity of the form

$$1 + \sum_{i=1}^{5} \xi_i = 0 \tag{4.3}$$

where $\xi_1 = x$, $\xi_2 = y$, $\xi_3 = x^2y^3$, $\xi_4 = x^3y^2$ and $\xi_5 = x^3y^3$ are some roots of unity. Thus, we have some *vanishing sums* of roots of unity which was studied in [2] systematically. In our case, the equalities (4.3) can be of the following form:

$$(1 - 1) + (1 - 1) + (1 - 1) = 0$$

in the case $(x, y) \in S_0$,

$$(1 - 1) + (\alpha - \alpha) + (\alpha^2 - \alpha^2) = 0, \quad (1 + \alpha + \alpha^2) + (1 + \alpha + \alpha^2) = 0$$

in the case $(x, y) \in S_1$,

$$(1 + \alpha + \alpha^2) + \beta^l(1 + \alpha + \alpha^2) = 0 \quad (l = 1, \dots, 4)$$

in the case $(x, y) \in S_2$, and

$$(1 - 1) + (\gamma^l - \gamma^l) + (\gamma^{\sigma(l)} - \gamma^{\sigma(l)}) = 0 \quad (l = 1, \dots, 6)$$

in the case $(x, y) \in S_3$ (here, σ denotes the permutation $(142)(356)$). This corresponds to Theorem 6 from [2] which provides a description of all *non-empty* vanishing sums of roots of unity of length at most nine. Additionally, Theorem 6 suggests an alternative way to prove our Theorem, but the proof of Theorem 6 itself is quite difficult.

4. The method proposed in the proof of Theorem is more elementary and simpler compared to one from [2]. It can be applied to other *trigonometric diophantine equations* as the *Gordan equation* [2, 5]. This equation in our notation can be written as

$$2 + x + x^{-1} + y + y^{-1} + z + z^{-1} = 0. \tag{4.4}$$

Here, we have three unknowns x, y, z satisfying the additional condition

$$x^N = y^N = z^N = 1 \text{ for some } N \in \mathbb{N}.$$

Also, the method works for a similar equation

$$1 + x + x^{-1} + y + y^{-1} + z + z^{-1} = 0$$

and even for the *Coxeter* (or *Crosby*) *equation* [2, 3]

$$x + x^{-1} + y + y^{-1} + z + z^{-1} = 0$$

which must be rewritten previously as

$$1 + x^2 + xy + xy^{-1} + xz + xz^{-1} = 0.$$

5. As it turns out, the second pentadecagonal triangle is *algebraically conjugate* to the first one that means the following. Letting $\delta = \zeta_{15}$, rewrite the solutions (3.2) and (3.5) as (δ^8, δ^{13}) and (δ^{11}, δ), respectively. Then the automorphism of the field $\mathbb{Q}(\delta)$ defined by $\delta \mapsto \delta^7$ sends the first pair (δ^8, δ^{13})

to the second pair (δ^{11}, δ). However, this idea of "algebraic replication" gives us at most two triangles that are geometrically distinct.

What happens if we apply such an "algebraic replication" to the heptagonal triangle ABC with angles (1.1)? Expectedly, no new triangle will be discovered, but a new isosceles bisectral triangle for ABC will be detected, namely the triangle $A_2B_2C_1$ for which the equality

$$A_2B_2 = A_2C_1$$

holds (in [1] and [6, Ch. III], this fact seems not to be noted).

6. For the first pentadecagonal triangle, we also have

$$AA_2 = BB_2. \tag{4.5}$$

This gives us an example of a non-isosceles triangle with two equal external bisectors.[1] But for the second pentadecagonal triangle, an analogue of (4.5) is

$$AA_1 = BB_2. \tag{4.6}$$

In the class of triangles ABC whose angles are commensurable with π, the equality (4.5) (the equality (4.6), respectively) holds only for the first (the second, respectively) pentadecagonal triangle. The proof of both statements is based on the fact that we are able to solve the Gordan equation (4.4). Below we clarify this for the condition (4.5).

Instead of the standard parametrization (3.3), we can use the following deeper parametrization

$$X_a = w_1, \quad X_b = w_2, \quad X_c = w_3$$

also described in [9]. Here, the points X_a, X_b, X_c are the intersection points of the segments AI, BI, CI with the inscribed circle of the triangle ABC, respectively (I denotes the center of this circle which assumed to be of radius 1). In particular, we obtain

$$T_a = -\frac{w_2w_3}{w_1}, \quad T_b = -\frac{w_3w_1}{w_2}, \quad T_c = -\frac{w_1w_2}{w_3}.$$

Next, we compute all the points involved in (4.5):

$$A = -\frac{2w_1w_2w_3}{w_2^2 + w_3^2}, \quad B = -\frac{2w_1w_2w_3}{w_3^2 + w_1^2},$$

$$A_2 = -\frac{2w_1w_2w_3(w_1^2(w_2^2 + w_3^2) - 2w_2^2w_3^2)}{(w_2^2 + w_3^2)(w_1^2 - w_2w_3)(w_1^2 + w_2w_3)},$$

$$B_2 = -\frac{2w_1w_2w_3(w_2^2(w_3^2 + w_1^2) - 2w_3^2w_1^2)}{(w_3^2 + w_1^2)(w_2^2 - w_3w_1)(w_2^2 + w_3w_1)}.$$

If we set $w_3 = 1$ then the condition (4.5) for a non-isosceles triangle ABC leads to the equation of the form

$$F(w_1, w_2)F(-w_1, w_2) = 0 \tag{4.7}$$

[1]For comparison, we note that there is no non-isosceles triangle with two equal internal bisectors (the so-called *Steiner–Lehmus theorem*).

where

$$F(w_1, w_2) = 2 + \frac{w_1}{w_2} + \frac{w_2}{w_1} + w_1 w_2 + \frac{1}{w_1 w_2} + \frac{w_1^2}{w_2^2} + \frac{w_2^2}{w_1^2}.$$

Letting $x = \varepsilon w_1/w_2$, $y = \varepsilon w_1 w_2$, $z = w_1^2/w_2^2$ where $\varepsilon = 1$ or $\varepsilon = -1$, we can reduce the equation (4.7) to the Gordan equation (4.4). It is well known [5] that all the *non-trivial* solutions (x, y, z) of (4.4) (that is, if each of x, y, z is different from -1) are

$$(x, y, z) \in \{(\zeta_3, \zeta_3, \zeta_4), (\zeta_3, \zeta_5, \zeta_5^2)\}$$

(up to rearranging the components and replacing the component by its inverse). Finally, we go through all the solutions of (4.4) including the trivial ones. This leads us to a (geometrically) unique triangle ABC, namely the first pentadecagonal triangle.

7. The condition (4.5) can be written as

$$(a + b)(ab + c^2) - 3abc - c^3 = 0 \tag{4.8}$$

where a, b, c are the sides of a non-isosceles triangle ABC (namely, $a = BC$ etc.). The equation (4.8) occurs in [10, Sec. 1] while solving Problem 260 (with b replaced by c and vice versa). It is easily to see that this equation has infinitely many real solutions (a, b, c) defining triangles ABC (Sharygin gives $(a, b, c) = ((1 + \sqrt{17})/2, 1, 2)$ as an example).

Following [8], we can ask whether exists any non-isosceles triangle ABC with integer a, b, c satisfying (4.8). This time the answer is "No". Indeed, the equation (4.8) defines an *elliptic curve* whose *minimal Weierstrass equation* is

$$Y^2 + XY + Y = X^3 - X. \tag{4.9}$$

According to the LMFDB Database, the group of rational points of (4.9) is of rank zero (see more details at the link [13]).

References

[1] L. Bankoff, J. Garfunkel, *The Heptagonal Triangle*, Math. Mag., **46**:1 (1973), 7–19.

[2] J. Conway, A. Jones, *Trigonometric diophantine equations (On vanishing sums of roots of unity)*, Acta Arithm., **30** (1976), 229–240.

[3] H. Coxeter, W. Crosby, *Solution to Problem 4136*, Amer. Math. Monthly, **53**:2 (1946), 103–107.

[4] J. von zur Gathen, J. Gerhard, *Modern Computer Algebra*, Cambridge University Press, Cambridge, 2013.

[5] P. Gordan, *Ueber endliche Gruppen linearer Transformationen einer Veränderlichen*, Math. Ann. **12** (1877), 23–46.

[6] P.S. Modenov, *Problems in Geometry*, Mir Publishers, Moscow, 1981.

[7] M.B. Nathanson, *Additive Number Theory The Classical Bases*, Springer, New York, 1996.

[8] I.V. Netay, A.V. Savvateev, *Sharygin Triangles and Elliptic Curves*, Bull. Korean Math. Soc., **54**:5 (2017), 1597–1617.

[9] N.N. Osipov, *Mechanical Proof of Planimetric Theorems of Rational Type*, Programming and Computer Software **40**:2 (2014), 71–78.
[10] I.F. Sharygin, *Problems in Plane Geometry*, Mir Publishers, Moscow, 1988.
[11] S. Tokarev, *Problem M2001*, Kvant **3** (2006), 17.
[12] L.C. Washington, *Introduction to Cyclotomic Fields*, Springer, New York, 1997.
[13] https://www.lmfdb.org/EllipticCurve/Q/14/a/5
[14] https://www.maplesoft.com

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