The Ricci bi-conformal vector fields on Robinson-Trautman spacetimes

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Abstract

The purpose of this paper is to find Ricci bi-conformal vector fields on Robinson-Trautman spacetime. At first, we introduce some necessary definations, then we calculate the Lie derivative of the metric and the Lie derivative of the Ricci tensor. We categorize the Ricci bi-conformal vector fields on this spacetime. Finally, we show it is a Killing vector field and also, it is a gradient vector field.

Keywords: Spacetime, Ricci bi-conformal vector fields, conformal vector fields, Robinson-Trautman spacetime.

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1 Introduction

The relationship between geometry and physics of spacetime in the general theory of relativity, Einstein field equations (EFEs) plays a vital duty. EFEs are extremely nonlinear, and it is involved to detect their correct answers. Wherefore to understand the answers of EFEs, it is essential to inflict some symmetry limitations on spacetimes. These spacetime symmetries help in finding the accurate answers of EFEs, and some of them show constant bases for classifying spacetime, (see [1, 2]). These symmetry restrictions are said to Killing vector fields (KVFs) and by using them one can provide conservation laws, and they play an important role in physics because if physics values can be made invariant under a certain transformation physicists can make simpler their problems. An example for the existence of time-like smmetry is $E = \tau\xi$ where τ and ξ indicate a time-like KVF and momentum of the partical, respectively, which ensures saving of energy of a partical. Also, symmetry restrictions are effective in the perusal of gravitational waves and solve the problems of localization of energy momenyum in GR. Some of the most fundamental symmetries of the spacetimes can be named Killing, conformal, and homothetic vector fields. A conformal symmetry is more important among other known spacetime symmetries. A conformal vector field (CVF) X is defined by

$$\mathcal{L}_X g = 2\varphi g, \tag{1}$$

where φ is the smooth conformal function. In this equation, $\mathcal{L}_X g$ denots Lie derivative of g in direction vector field X. In equation (1), if φ is a steady, then the vector field represents a homothetic vector field (HVF), also when it is zero, then X is called KVFs. It is called proper conformal vector field, if the vector field X is not homothetic but conformal. For more study see [7, 8, 11, 12]. A vector field X is called a bi-conformal vector field (BCVF) if we have the equations mentioned below

$$\mathcal{L}_X g = \alpha g + \beta h, \tag{2}$$

and

$$\mathcal{L}_X S = \alpha h + \beta g, \tag{3}$$

where h is a symmetric square root of g and α, β are smooth functions, (see[9]). The vector field X is said to be a Ricci bi-conformal vector field (RCBVF) when the following equations are deduced for any vector fields Y, Z, and some smooth functions α and β :

$$(\mathcal{L}_X g)(Y, Z) = \alpha g(Y, Z) + \beta S(Y, Z), \tag{4}$$

and

$$(\mathcal{L}_X S)(Y, Z) = \alpha S(Y, Z) + \beta g(Y, Z), \tag{5}$$

where S is the Ricci tensor of M. Note, that Garcia-Parrado and Senovilla introduced BCVFs [9], then De et al. defined RBCVFs in [6]. In [3], [4] and [5] have been studied RBCVFs on Lorentzian five-dimensional two-step nilpotent Lie groups, Siklos spacetimes, and homogeneous Gödel-type spacetimes, respectively. In 1960, Robinson-Trautman (RT) spacetime [14] was introduced as a EFE solution that has a repeated principal null direction along with a shearless, expanding, and non-twisting null geodesic congruence. By using RT spacetime, the mass loss in the final phase during the collision of two black holes can be obtained. Also, the model of gravitational radiation output from spatially limited sources can be represented by RT space-time. In2018, several properties of RT spacetime curvature were investigated by Shaikh et al. [15] and it was shown that it is a Roter type manifold and that the curvature tensor admits curves such as Ricci pseudosymmetric, Weyl conformal curvature tensor, Riemann compactible Ricci tensor, and pseudosymmetric energy momentum tensor.

In this paper, we investigate the existence of RBCVFs on Robinson-Trautman spacetimes and we show it is a KVF and also, it is a gradient vector field (GVF).

2 Main results

In this section, we describe Robinson-Trautman spacetimes, and investigate the Lie derivatives and the Lie derivative of the Ricci tensor on it, and then by solving the equations, we investigate the existence of RBCVF and its results on this type. Notice that we use the Maple software for exam our results.

2.1 Robinson-Trautman spacetimes

Consider RT spacetimes in the usual coordinates (t, r, x, y) as follows (see [10, 13, 14, 16])

$$g = -2(a - 2br - \frac{q}{r})dt^2 + 2dtdr - \frac{r^2}{f^2}(dx^2 + dy^2) \tag{6}$$

where f is nowhere vanishing function of x, y, and a, b, q are constants. Let $e_1 = \partial_t, e_2 = \partial_r, e_3 = \partial_x$, and $e_4 = \partial_y$. The covariant derivatives of the coordinate fields are calculated by

$$\nabla = \begin{pmatrix} A & B & 0 & 0 \\ B & 0 & \frac{1}{r}\partial_x & \frac{1}{r}\partial_y \\ 0 & \frac{1}{r}\partial_x & C & D \\ 0 & \frac{1}{r}\partial_y & D & E \end{pmatrix}. \tag{7}$$

where $A = -\frac{(2br^2-q)}{r^2}\partial_t - \frac{2(ar-2br^2-q)(2br^2-q)}{r^3}\partial_r$,

$B = \frac{(2br^2-q)}{r^2}\partial_r$,

$C = \frac{r}{f^2}\partial_t + \frac{2(ar-2br^2-q)}{f^2}\partial_r - \frac{f_x}{f}\partial_x + \frac{f_y}{f}\partial_y$,

$D = -\frac{f_y}{f}\partial_x - \frac{f_x}{f}\partial_y$,

and $E = \frac{r}{f^2}\partial_t + \frac{2(ar-2br^2-q)}{f^2}\partial_r + \frac{f_x}{f}\partial_x - \frac{f_y}{f}\partial_y$. As well, the Ricci tensor S along to the basis of coordinate vector fields $\partial_t, \partial_x, \partial_y, \partial_z$ is described by

$$S = \begin{pmatrix} F & \frac{4b}{r} & 0 & 0 \\ \frac{4b}{r} & 0 & 0 & 0 \\ 0 & 0 & G & 0 \\ 0 & 0 & 0 & G \end{pmatrix}, \tag{8}$$

where $F = -\frac{8(ar-2br^2-q)b}{r^2}$, and $G = \frac{2a-8br-f_x^2+f_{xx}f-f_y^2+f_{yy}f}{f^2}$. Let $X = X^1\partial_t + X^2\partial_r + X^3\partial_x + X^4\partial_y$ be an optional vector field on concircular vector fields where $X^i = X^i(t, r, x, y)$ are smooth functions on M for $i = 1, 2, 3, 4$.

For any vector field $X = X^k e_k$ by $(\mathcal{L}_X g)(e_i, e_j) = g(\nabla_{e_i}X, e_j) + g(e_i, \nabla_{e_j}X)$ the Lie derivative of the metric g along to the vector field X, is given by

$$\begin{aligned}(\mathcal{L}_X g)_{11} &= \tfrac{-2}{r^2}(-\partial_t X^2 r^2 + 2\partial_t X^1 r^2 a - 4\partial_t X^1 r^3 b \\ &\quad -2\partial_t X^1 rq - 2X^2 br^2 + X^2 q), \\ (\mathcal{L}_X g)_{12} &= \tfrac{-1}{r}(-\partial_r X^2 r + 2\partial_r X^1 ra - 4\partial_r X^1 r^2 b \\ &\quad -2\partial_r X^1 q - \partial_t X^1 r),\end{aligned}$$

$$
\begin{aligned}
(\mathcal{L}_X g)_{13} &= \tfrac{-1}{rf^2}(\partial_t X^3 r^3 - f^2\partial_x X^2 r + 2f^2\partial_x X^1 ar \\
&\quad -4f^2\partial_x X^1 br^2 - 2f^2\partial_x X^1 q), \\
(\mathcal{L}_X g)_{14} &= \tfrac{-1}{rf^2}(\partial_t X^4 r^3 - f^2\partial_y X^2 r + 2f^2\partial_y X^1 ar \\
&\quad -4f^2\partial_y X^1 br^2 - 2f^2\partial_y X^1 q), \\
(\mathcal{L}_X g)_{22} &= 2\partial_r X^1, \\
(\mathcal{L}_X g)_{23} &= \tfrac{1}{f^2}(-\partial_r X^3 r^2 + \partial_x X^1 f^2), \\
(\mathcal{L}_X g)_{24} &= \tfrac{1}{f^2}(-\partial_r X^4 r^2 + \partial_y X^1 f^2), \\
(\mathcal{L}_X g)_{33} &= \tfrac{-2r}{f^3}(\partial_x X^3 rf + X^2 f - X^3 r\partial_x f - X^4 r\partial_y f), \\
(\mathcal{L}_X g)_{34} &= \tfrac{-r^2}{f^2}(\partial_x X^4 + \partial_y X^3), \\
(\mathcal{L}_X g)_{44} &= \tfrac{-2r}{f^3}(\partial_y X^4 rf + X^2 f - X^3 r\partial_x f - X^4 r\partial_y f).
\end{aligned} \tag{9}
$$

Further, using the formula $(\mathcal{L}_X S)(e_i, e_j) = X(S(e_i, e_j)) - S(\mathcal{L}_X e_i, e_j) - S(e_i, \mathcal{L}_X e_j)$ the Lie derivative of the Ricci tensor along X is determined by

$$
\begin{aligned}
(\mathcal{L}_X S)_{11} &= \frac{-8b}{r^3}(-\partial_t X^2 r^2 + 2\partial_t X^1 r^2 a - 4\partial_t X^1 r^3 b \\
&\quad - 2\partial_t X^1 rq - X^2 ar + 2X^2 q), \\
(\mathcal{L}_X S)_{12} &= \frac{-4b}{r^2}(-\partial_r X^2 r + 2\partial_r X^1 ra - 4\partial_r X^1 r^2 b \\
&\quad - 2\partial_r X^1 q - \partial_t X^1 r + X^2), \\
(\mathcal{L}_X S)_{13} &= \frac{-1}{r^2 f^2}(-2\partial_t X^3 r^2 a + 8\partial_t X^3 r^3 b + \partial_t X^3 r^2 \partial_x f^2 \\
&\quad - \partial_t X^3 r^2 \partial_{xx} ff + \partial_t X^3 r^2 \partial_y f^2 - \partial_t X^3 r^2 \partial_{yy} ff - 4bf^2\partial_x X^2 r \\
&\quad + 8bf^2\partial_x X^1 ar - 16b^2 f^2 \partial_x X^1 r^2 - 8bf^2\partial_x X^1 q), \\
(\mathcal{L}_X S)_{14} &= \frac{-1}{r^2 f^2}(-2\partial_t X^4 r^2 a + 8\partial_t X^4 r^3 b + \partial_t X^4 r^2 \partial_x f^2 \\
&\quad - \partial_t X^4 r^2 \partial_{xx} ff + \partial_t X^4 r^2 \partial_y f^2 - \partial_t X^4 r^2 \partial_{yy} ff - 4bf^2\partial_y X^2 r \\
&\quad + 8bf^2\partial_y X^1 ar - 16b^2 f^2 \partial_y X^1 r^2 - 8bf^2\partial_y X^1 q), \\
(\mathcal{L}_X S)_{22} &= \frac{8}{r}\partial_r X^1 b, \\
(\mathcal{L}_X S)_{23} &= \frac{1}{rf^2}(2\partial_r X^3 ra - 8\partial_r X^3 r^2 b \\
&\quad - \partial_r X^3 r\partial_x f^2 + \partial_r X^3 r\partial_{xx} ff - \partial_r X^3 r\partial_y f^2 \\
&\quad + \partial_r X^3 r\partial_{yy} ff + 4\partial_x X^1 bf^2), \\
(\mathcal{L}_X S)_{24} &= \frac{1}{rf^2}(2\partial_r X^4 ra - 8\partial_r X^4 r^2 b \\
&\quad - \partial_r X^4 r\partial_x f^2 + \partial_r X^4 r\partial_{xx} ff - \partial_r X^4 r\partial_y f^2 \\
&\quad + \partial_r X^4 r\partial_{yy} ff + 4\partial_y X^1 bf^2),
\end{aligned}
$$

$$
\begin{aligned}
(\mathcal{L}_X S)_{33} = &\frac{1}{f^3}(4\partial_x X^3 f a - 2\partial_x X^3 f \partial_x f^2 \\
&+ 2\partial_x X^3 f^2 \partial_{xx} f - 2\partial_x X^3 f \partial_y f^2 + 2\partial_x X^3 f^2 \partial_{yy} f \\
&- 8X^2 b f + X^3 \partial_{xxx} f^2 + X^3 \partial_{xyy} f f^2 \\
&- 4X^3 \partial_x f a + 2X^3 \partial_x f \partial_y f^2 + X^4 \partial_{xxy} f f^2 \\
&+ X^4 \partial_{yyy} f f^2 - 4X^4 \partial_y f a + 2X^4 \partial_y f \partial_x f^2 \\
&- 16\partial_x X^3 f b r + 16X^4 \partial_y f b r - 3X^3 f \partial_x f \partial_{xx} f \\
&- 2X^3 f \partial_y f \partial_{xy} f - X^3 f \partial_{yy} f \partial_x f + 16X^3 \partial_x f b r \\
&- 2X^4 f \partial_x f \partial_{xy} f - X^4 f \partial_{xx} f \partial_y f - 3X^4 f \partial_y f \partial_{yy} f \\
&+ 2X^3 \partial_x f^3 + 2X^4 \partial_y f^3), \qquad (10)
\end{aligned}
$$

$$
\begin{aligned}
(\mathcal{L}_X S)_{34} = &\frac{1}{f^2}(2a - 8br - \partial_x f^2 + \partial_{xx} f f - \partial_y f^2 \\
&+ \partial_{yy} f f)(\partial_x X^4 + \partial_y X^3), \\
(\mathcal{L}_X S)_{44} = &\frac{1}{f^3}(-8X^2 b f + X^3 \partial_{xxx} f f^2 \\
&+ X^3 f^2 \partial_{xyy} f - 4X^3 \partial_x f a + 2X^3 \partial_x f \partial_y f^2 \\
&+ X^4 \partial_{xxy} f f^2 + X^4 \partial_{yyy} f f^2 - 4X^4 \partial_y f a \\
&+ 2X^4 \partial_y f \partial_x f^2 + 4\partial_y X^4 f a - 2\partial_y X^4 f \partial_x f^2 \\
&+ 2\partial_y X^4 f^2 \partial_{xx} f - 2\partial_y X^4 f \partial_y f^2 + 2\partial_y X^4 f^2 \partial_{yy} f \\
&+ 16X^4 \partial_y f b r - 3X^3 f \partial_x f \partial_{xx} f - 2X^3 f \partial_y f \partial_{xy} f \\
&- X^3 f \partial_{yy} f \partial_x f + 16X^3 \partial_x f b r - 2X^4 f \partial_x f \partial_{xy} f \\
&- X^4 f \partial_{xx} f \partial_y f - 3X^4 f \partial_y f \partial_{yy} f - 16\partial_y X^4 f b r \\
&+ 2X^3 \partial_x f^3 + 2X^4 \partial_y f^3).
\end{aligned}
$$

3 Ricci bi-conformal vector fields on some spacetimes

In this part, we solve the equations (4) and (5) on RT spacetimes. By applying (6), (8), and (9) in (4), we get

$$\tfrac{1}{f^2}(-\partial_r X^3 r^2 + \partial_x X^1 f^2) = 0, \qquad (11)$$

$$\tfrac{1}{f^2}(-\partial_r X^4 r^2 + \partial_y X^1 f^2) = 0, \qquad (12)$$

$$
\begin{aligned}
&\tfrac{1}{r}(-4\beta b - \alpha r - \partial_r X^2 r - 2\partial_r X^1 \alpha r \\
&+4\partial_r X^1 b r^2 + 2\partial_r X^1 q + \partial_t X^1 r) = 0, \qquad (13)
\end{aligned}
$$

$$
\begin{aligned}
&\tfrac{1}{f^2 r}(-\partial_t X^3 r^3 + f^2 \partial_x X^2 r - 2f^2 \partial_x X^1 a r \\
&+4f^2 \partial_x X^1 b r^2 + 2f^2 \partial_x X^1 q) = 0, \qquad (14)
\end{aligned}
$$

$$2\partial_r X^1 = 0, \tag{15}$$

$$\begin{aligned}&\tfrac{1}{f^3}(2\beta f a - 8\beta f b r - \beta f \partial_x f^2 + \beta f^2 \partial_{xx} f\\&-\beta f \partial_y f^2 + \beta f^2 \partial_{yy} f - \alpha r^2 f + 2\partial_x X^3 r^2 f\\&+2X^2 r f - 2X^3 r^2 \partial_x f - 2X^4 r^2 \partial_y f) = 0,\end{aligned} \tag{16}$$

$$\begin{aligned}&\tfrac{1}{f^3}(2\beta f a - 8\beta f b r - \beta f \partial_x f^2 + \beta f^2 \partial_{xx} f\\&-\beta f \partial_y f^2 + \beta f^2 \partial_{yy} f - \alpha r^2 f + 2\partial_x X^4 r^2 f\\&+2X^2 r f - 2X^3 r^2 \partial_x f - 2X^4 r^2 \partial_y f) = 0,\end{aligned} \tag{17}$$

$$\begin{aligned}&\tfrac{1}{r^2}(8\beta b a r - 16\beta b^2 r^2 - 8\beta b q + 2\alpha r^2 a - 4\alpha r^3 b\\&-2\alpha r q + 2\partial_t X^2 r^2 - 4\partial_t X^1 r^2 a + 8\partial_t X^1 r^3 b\\&+4\partial_t X^1 r q + 4X^2 b r^2 - 2X^2 q) = 0,\end{aligned} \tag{18}$$

$$-\tfrac{1}{f^2} r^2(\partial_x X^4 + \partial_y X^3) = 0, \tag{19}$$

$$\begin{aligned}&\tfrac{1}{f^2 r}(-\partial_t X^4 r^3 + f^2 \partial_y X^2 r - 2f^2 \partial_y X^1 a r\\&+4f^2 \partial_y X^1 b r^2 + 2f^2 \partial_y X^1 q) = 0.\end{aligned} \tag{20}$$

Also, substituting (6), (8), and (10) in (5), we obtain

$$\begin{aligned}&\tfrac{1}{f^3}(-8X^2 b f + X^3 \partial_{xxx} f f^2\\&+X^3 \partial_{xyy} f f^2 - 4X^3 \partial_x f a + 2X^3 \partial_x f \partial_y f^2\\&+X^4 \partial_{xxy} f^2 + X^4 \partial_{yyy} f^2 - 4X^4 \partial_y f a\\&+2X^4 \partial_y f \partial_x f^2 + 4\partial_y X^4 f a - 2\partial_y X^4 f \partial_x f^2\\&+2\partial_y X^4 f^2 \partial_{xx} f - 2\partial_y X^4 f \partial_y f^2 + 2\partial_y X^4 f^2 \partial_{yy} f\\&\beta r^2 f - 2\alpha f a + \alpha f \partial_x f^2 - \alpha f^2 \partial_{xx} f\\&+\alpha f \partial_y f^2 - \alpha f^2 \partial_{yy} f + 16X^4 \partial_y f b r\\&-3X^3 f \partial_x f \partial_{xx} f - 2X^3 f \partial_y f \partial_{xy} f - X^3 f \partial_{yy} f \partial_x f\\&+16X^3 \partial_x f b r - 2X^4 f \partial_x f \partial_{xy} f - X^4 f \partial_{xx} f \partial_y f\\&-3X^4 f \partial_y f \partial_{yy} f - 16\partial_y X^4 f b r + 8\alpha f b r\\&+2X^3 \partial_x f^3 + 2X^4 \partial_y f^3) = 0,\end{aligned} \tag{21}$$

$$\begin{aligned}&\tfrac{1}{f^2}(2a - 8br - \partial_x f^2 + \partial_{xx} f f - \partial_y f^2\\&+\partial_{yy} f f)(\partial_x X^4 + \partial_y X^3) = 0,\end{aligned} \tag{22}$$

$$\begin{aligned}&\tfrac{1}{r f^2}(2\partial_r X^3 r a - 8\partial_r X^3 r^2 b\\&-\partial_r X^3 r \partial_x f^2 + \partial_r X^3 r \partial_{xx} f f - \partial_r X^3 r \partial_y f^2\\&+\partial_r X^3 r \partial_{yy} f f + 4\partial_x X^1 b f^2) = 0,\end{aligned} \tag{23}$$

$$\begin{aligned}&\tfrac{1}{r f^2}(2\partial_r X^4 r a - 8\partial_r X^4 r^2 b\\&-\partial_r X^4 r \partial_x f^2 + \partial_r X^4 r \partial_{xx} f f - \partial_r X^4 r \partial_y f^2\\&+\partial_r X^4 r \partial_{yy} f f + 4\partial_y X^1 b f^2) = 0,\end{aligned} \tag{24}$$

$$\tfrac{1}{r^2 f^2}(2\partial_t X^3 r^2 a - 8\partial_t X^3 r^3 b - \partial_t X^3 r^2 \partial_x f^2$$

$$+\partial_t X^3 r^2 \partial_{xx} f f - \partial_t X^3 r^2 \partial_y f^2 + \partial_t X^3 r^2 \partial_{yy} f f$$
$$+4bf^2 \partial_x X^2 r - 8bf^2 \partial_x X^1 ar + 16b^2 f^2 \partial_x X^1 r^2$$
$$+8bf^2 \partial_x X^1 q) = 0, \tag{25}$$
$$\frac{-1}{r^2}(\beta r^2 + 4\alpha br - 4b\partial_r X^2 + 8b\partial_r X^1 ra$$
$$-16b^2 \partial_r X^1 r^2 - 8b\partial_r X^1 q - 4b\partial_t X^1 r + 4bX^2) = 0, \tag{26}$$
$$\frac{1}{r^3}(2\beta r^3 a - 4\beta r^4 b - 2\beta r^2 q$$
$$+8b\alpha r^2 a - 16\alpha r^3 b^2 - 8b\alpha rq$$
$$+8b\partial_t X^2 r^2 - 16b\partial_t X^1 r^2 a + 32b^2 \partial_t X^1 r^3$$
$$+16b\partial_t X^1 rq + 8bX^2 ar - 16bX^2 q) = 0, \tag{27}$$
$$\frac{-1}{f^3}(-4\partial_x X^3 fa + 2\partial_x X^3 f\partial_x f^2$$
$$-2\partial_x X^3 f^2 \partial_{xx} f + 2\partial_x X^3 f\partial_y f^2 - 2\partial_x X^3 f^2 \partial_{yy} f$$
$$+8X^2 bf - X^3 \partial_{xxx} f^2 - X^3 \partial_{xyy} f f^2$$
$$+4X^3 \partial_x fa - 2X^3 \partial_x f \partial_y f^2 - X^4 \partial_{xxy} f f^2$$
$$-X^4 \partial_{yyy} f f^2 + 4X^4 \partial_y fa - 2X^4 \partial_y f \partial_x f^2$$
$$-\beta r^2 f + 2\alpha fa - \alpha f\partial_x f^2$$
$$+\alpha f^2 \partial_{xx} f - \alpha f \partial_y f^2 + \alpha f^2 \partial_{yy} f$$
$$+16\partial_x X^3 fbr - 16X^4 \partial_y fbr + 3X^3 f\partial_x f \partial_{xx} f$$
$$+2X^3 f\partial_y f \partial_{xy} f + X^3 f\partial_{yy} f \partial_x f - 16X^3 \partial_x fbr$$
$$+2X^4 f\partial_x f \partial_{xy} f + X^4 f\partial_{xx} f \partial_y f + 3X^4 f\partial_y f \partial_{yy} f$$
$$-8\alpha fbr - 2X^3 \partial_x f^3 - 2X^4 \partial_y f^3) = 0, \tag{28}$$
$$\frac{1}{r} 8\partial_r X^1 b = 0, \tag{29}$$
$$\frac{-1}{r^2 f^2}(-2\partial_t X^4 r^2 a + 8\partial_t X^4 r^3 b + \partial_t X^4 r^2 \partial_x f^2$$
$$-\partial_t X^4 r^2 \partial_{xx} f f + \partial_t X^4 r^2 \partial_y f^2 - \partial_t X^4 r^2 \partial_{yy} f f - 4bf^2 \partial_y X^2 r$$
$$+8bf^2 \partial_y X^1 ar - 16b^2 f^2 \partial_y X^1 r^2 - 8bf^2 \partial_y X^1 q) = 0. \tag{30}$$

Therefore, by solving these equations, $X^1, X^2, X^3, X^4, \alpha$, and β obtained as follows

$$X^1 = c_1, X^2 = X^3 = X^4 = \alpha = \beta = 0, \tag{31}$$

for some constant c_1. Now, the following theorem is stated:
Theorem 1. *The vector field $X = X^1\partial_t + X^2\partial_r + X^3\partial_x + X^4\partial_y$ on RT spacetimes where g given by (6), is RBCVF if and only if $X^1 = c_1, X^2 = X^3 = X^4 = \alpha = \beta = 0$.*

So, the following corollary is stated:
Corollary 1. *Any RBCVF X on RT spacetimes is a KVF.*

Now, assume $X = \nabla h$ on RT spacetimes with potential function h. Therefore,

$$\nabla h = (\partial_t h)\partial_r + (\partial_r h)\partial_t + 2(a - 2br - \tfrac{q}{r})(\partial_r h)\partial_r - \tfrac{f^2}{r^2}(\partial_x h)\partial_x - \tfrac{f^2}{r^2}(\partial_y h)\partial_y.$$

From Theorem 2.4, the RBCVF X on RT spacetimes is GVF as ∇h if and only if

$$\begin{aligned} \partial_t h &= -2c_1(a - 2br - \frac{q}{r}), \\ \partial_r h &= c_1, \\ \partial_x h &= 0, \\ \partial_y h &= 0. \end{aligned} \tag{32}$$

Taking derivation of the second equation of the last system along to t, we get $\partial_t \partial_r h = 0$. The first equation of (32) implies that $\partial_r \partial_t h = 4c_1 b - 2c_1 q r^{-2}$. Therefore, $4c_1 b - 2c_1 q r^{-2} = 0$. Thus, $q = b = 0$ or $c_1 = 0$. Therefore, we get

$$h(t, r, x, y) = c_1 r - 2c_1 at + c_2,$$

for some constant c_2. Thus, the following theorem is stated:
Theorem 2. *Any RBCVF X on RT spacetimes is a GVF with potential function f if and only if $h(t, r, x, y) = c_1 r - 2c_1 at + c_2$.*

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Not applicable.

Consent for publication

Not applicable.

Availability of data and material

All data generated or analysed during this study are included in this published article.

Author's contributions

All authors contributed equally in the preparation of this manuscript.

References

[1] A.T. Ali, A.K. Yadav, and S.R. Mahmoud, Some plane symmetric inhomogeneous cosmological models in the scalar-tensor theory of graviation, Astrophys. Space Sci. 349(1) (2014) 539.

[2] S.K. Attallah, M.F. El-Sbbagh and A.T.Ali, Isovector fields and similarity solutions of Einstein vacuum equations for rotating fields, Commun. Nonlinear Sci. Numer. Simulat. 12(7) (2007) 1153.

[3] S. Azami, and U. C. De, Ricci bi-conformal vector fields on Lorentzian five-dimensional two-step nilpotent Lie groups, Hacet. J. Math. Stat. (2023), 10.15672/hujms.1294973.

[4] S. Azami, and G. Fasihi-Ramandi, Ricci bi-conformal vector fields on Siklos spacetimes, Mathematics. Interdisciplinary research, 9(1) (2024), 45-76.

[5] S. Azami, and M. Jafari, Ricci bi-conformal vector fields on homogeneous Gödel-type spacetimes, J. nonlinear Math. Phys. 30 (2023), 1700-1718.

[6] U.C. De, A. Sardar, and A. Sarkar, Some conformal vector fields and conformal Ricci solitons on $N(k)$-contact metric manifolds, AUT J. Math. Com., 2 (1) (2021), 61-71.

[7] S. Deshmukh, Geometry of conformal vector fields, Arab. J. Math., 23(1) (2017), 44-73.

[8] S. Deshmukh and F. R. Al-Solamy, Conformal vector fields on a Riemannian manifold, Balkan Journal of Geometry and its Application, 19(2) (2014), 86-93.

[9] A. Garcia-Parrado and J.M.M. Senivilla, Bi-conformal vector fields and their applications, Classical and Quantum Gravity, 21 (8) (2004), 2153-2177.

[10] J.B Griffiths and J. Podolsky, Exact space-times in Einstein's general relativity, Cambridge University Press, (2009).

[11] S. Jamal, A.H. Kara, and R. Narain, Wave equations in Bianchi space-times, J. Appl. Math. (2012), 765361.

[12] S. Jamal, A.H. Kara, R. Narain, and G. Shabbir, Symmetry structure of a wave equation on some classes of Bianchi cosmological models, Indian J. Phys. 89, 411 (2015).

[13] I. Robinson and A. Trautman, Some spherical gravitational waves in general relativity, Proc. R. Soc. London, Ser. A, 265 (1962), 463-473.

[14] I. Robinson and A. Trautman, Some spherical gravitational waves, Phys. Rev. Lett., 4(8) (1960), 431-432.

[15] A. A. Shaikh, M. Ali, and Z. Ahsan, Curvature properties of Robinson-Trautman metric, J. Geom., 109(38) (2018).

[16] H. Stephani, D. Kramer, M.A.H. MacCullam, C. Hoenselears, and E. Herlt, Exact Solutions of Einstein's Field Equations, Cambridge Univ. Press, UK, (2003).