Collective order convergence and collectively qualified sets of operators

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Abstract

Collective versions of order convergences and corresponding types of collectively qualified sets of operators in vector lattices are investigated. It is proved that collectively order to norm bounded sets are bounded in the operator norm and collectively order continuous sets are collectively order bounded.

Keywords: vector lattice, order convergence, collectively qualified set of operators
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1 Introduction

The study of collectively compact sets of operators between normed spaces was initiated by Anselone and Palmer [6] (for recent advances see [9, 10, 12]). In many cases it is necessary to deal with "uniform" or "collective" properties of a set of operators between vector lattices (shortly, VLs) regarding the order convergence in their domains. Whereas the corresponding technique is rather simple for the norm convergence, the case of order convergence in VLs requires an additional attention. The present note is devoted to investigation of collective order convergence and its applications to collectively qualified sets of operators in VLs.

We abbreviate a normed space (normed lattice, Banach lattice) by NS (NL, BL). Throughout the text, vector spaces are real, operators are linear, symbol $\mathcal{L}(X,Y)$ stands for the space of operators between vector spaces X to Y, symbol $x_\alpha \downarrow 0$ for a decreasing net in a VL such that $\inf\limits_\alpha x_\alpha = 0$, symbol $\text{sol}(A)$ for the solid hull $\bigcup_{a\in A}[-|a|,|a|]$ of a subset A of a VL E, and $\mathcal{L}(E,F)$ ($\mathcal{L}_+(E,F)$, $\mathcal{L}_{ob}(E,F)$, $\mathcal{L}_{oc}(E,F)$) for the space of linear (resp., positive, order bounded, order continuous) operators between VLs E and F.

We shall use the following modes of convergence in VLs.

Definition 1.1. A net (x_α) in a VL E

a) *order converges* to $x \in E$ (briefly, $x_\alpha \xrightarrow{o} x$) if there exists a net (g_β) in E such that $g_\beta \downarrow 0$ and, for each β, there is an α_β with $|x_\alpha - x| \le g_\beta$ for all $\alpha \ge \alpha_\beta$.

b) *unbounded order converges* to $x \in E$ (briefly, $x_\alpha \xrightarrow{\text{uo}} x$) if $|x_\alpha - x| \wedge u \xrightarrow{\text{o}} 0$ for every $u \in E_+$.

c) *relative uniform converges* to $x \in E$ (briefly, $x_\alpha \xrightarrow{\text{ru}} x$) if, for some $u \in E_+$ there exists an increasing sequence (α_n) of indices such that $|x_\alpha - x| \leq \frac{1}{n}u$ for all $\alpha \geq \alpha_n$.

An operator T from a VL E to a VL F is called o-(uo-, ru-)-*continuous* if $Tx_\alpha \xrightarrow{\text{o}} 0$ ($Tx_\alpha \xrightarrow{\text{uo}} 0$, $Tx_\alpha \xrightarrow{\text{ru}} 0$) whenever $x_\alpha \xrightarrow{\text{o}} 0$ (resp., $x_\alpha \xrightarrow{\text{uo}} 0$, $x_\alpha \xrightarrow{\text{ru}} 0$).

The following notions of "collective convergence" are useful in working with families of nets indexed by the same directed set.

Definition 1.2. Let $\mathcal{B} = \{(x^b_\alpha)_{\alpha \in A}\}_{b \in B}$ be a set of nets in a VL E indexed by a directed set A. We say that $\mathcal{B}$

a) *collective* o-*converges* to 0 (briefly, $\mathcal{B} \xrightarrow{\text{c-o}} 0$) if there exists a net $g_\beta \downarrow 0$ such that, for each β, there is an α_β with $|x^b_\alpha| \leq g_\beta$ for $\alpha \geq \alpha_\beta$ and $b \in B$.

b) *collective* uo-*converges* to 0 (briefly, $\mathcal{B} \xrightarrow{\text{c-uo}} 0$) if $\{(|x^b_\alpha| \wedge u)_{\alpha \in A}\}_{b \in B} \xrightarrow{\text{c-o}} 0$ for every $u \in E_+$.

c) *collective* ru-*converges* to 0 (briefly, $\mathcal{B} \xrightarrow{\text{c-ru}} 0$) if, for some $u \in E_+$, there exists an increasing sequence (α_n) of indices such that $|x^b_\alpha| \leq \frac{1}{n}u$ for $\alpha \geq \alpha_n$ and $b \in B$.

For the basic theory of VLs we refer to [3, 4, 13, 14].

2 Main results

2.1 Preliminaries. We begin with the following auxiliary properties of c-o, c-uo, and c-ru convergences.

Proposition 2.1. *Let $\mathcal{B} = \{(x^b_\alpha)\}_{b \in B}$ and $\mathcal{C} = \{(x^c_\alpha)\}_{c \in C}$ be nonempty sets of nets in a VL E indexed by the same directed set A, and let $r \in \mathbb{R}$. The following holds.*

i) *If $\mathcal{B} \xrightarrow{\text{c-o}} 0$ and $\mathcal{C} \xrightarrow{\text{c-o}} 0$ then $r\mathcal{B} = \{(rx^b_\alpha)\}_{\alpha \in A; b \in B} \xrightarrow{\text{c-o}} 0$,*

$$\mathrm{sol}(\mathcal{B}) := \{(x_\alpha) : (\exists b \in B)(\forall \alpha \in A)|x_\alpha| \leq |x^b_\alpha|\} \xrightarrow{\text{c-o}} 0,$$

$\mathcal{B} \cup \mathcal{C} \xrightarrow{\text{c-o}} 0$, $\mathcal{B} + \mathcal{C} = \{(x^b_\alpha + x^c_\alpha) : b \in B, c \in C\} \xrightarrow{\text{c-o}} 0$, and $\mathrm{co}(\mathcal{B}) \xrightarrow{\text{c-o}} 0$, where

$$\mathrm{co}(\mathcal{B}) = \Big\{(x_\alpha) = \Big(\sum_{k=1}^{n} r_k x^{b_k}_\alpha\Big) : b_k \in B, r_k \in \mathbb{R}_+, \sum_{k=1}^{n} r_k = 1\Big\}.$$

ii) *If $\mathcal{B} \xrightarrow{\text{c-uo}} 0$ and $\mathcal{C} \xrightarrow{\text{c-uo}} 0$ then $r\mathcal{B} \xrightarrow{\text{c-uo}} 0$, $\mathrm{sol}(\mathcal{B}) \xrightarrow{\text{c-uo}} 0$, $\mathcal{B} \cup \mathcal{C} \xrightarrow{\text{c-uo}} 0$, $\mathcal{B} + \mathcal{C} \xrightarrow{\text{c-uo}} 0$, and $\mathrm{co}(\mathcal{B}) \xrightarrow{\text{c-uo}} 0$.*

iii) *If $\mathcal{B} \xrightarrow{\text{c-ru}} 0$ and $\mathcal{C} \xrightarrow{\text{c-ru}} 0$ then $r\mathcal{B} \xrightarrow{\text{c-ru}} 0$, $\mathrm{sol}(\mathcal{B}) \xrightarrow{\text{c-ru}} 0$, $\mathcal{B} \cup \mathcal{C} \xrightarrow{\text{c-ru}} 0$, $\mathcal{B} + \mathcal{C} \xrightarrow{\text{c-ru}} 0$, and $\mathrm{co}(\mathcal{B}) \xrightarrow{\text{c-ru}} 0$.*

Moreover, $x_\alpha \xrightarrow{\text{o}} 0 \iff \{(x_\alpha)\} \xrightarrow{\text{c-o}} 0$*;* $x_\alpha \xrightarrow{\text{uo}} 0 \iff \{(x_\alpha)\} \xrightarrow{\text{c-uo}} 0$*; and* $x_\alpha \xrightarrow{\text{ru}} 0 \iff \{(x_\alpha)\} \xrightarrow{\text{c-ru}} 0$.

Proof. *i)* Let $\mathcal{B} \xrightarrow{\text{c-o}} 0$ and $\mathcal{C} \xrightarrow{\text{c-o}} 0$. Take two nets $g_\beta \downarrow 0$ and $p_\gamma \downarrow 0$ in E such that, for every β and γ, there exist indices α_β and α_γ satisfying $|x^b_\alpha| \le g_\beta$ for $\alpha \ge \alpha_\beta$ and $b \in B$, and $|x^c_\alpha| \le p_\gamma$ for $\alpha \ge \alpha_\gamma$ and $c \in C$.

Obviously, $r\mathcal{B} \xrightarrow{\text{c-o}} 0$ and $\text{sol}(\mathcal{B}) \xrightarrow{\text{c-o}} 0$.

In order to show $\mathcal{B} \cup \mathcal{C} \xrightarrow{\text{c-o}} 0$, consider the net $(g_\beta + p_\gamma)_{\beta;\gamma} \downarrow 0$. Since $\mathcal{B} \cup \mathcal{C} = \{(x^d_\alpha)\}_{\alpha \in A; d \in B \cup C}$ then $|x^d_\alpha| \le g_\beta + p_\gamma$ for $\alpha \ge \alpha_\beta, \alpha_\gamma$ and $d \in B \cup C$. For every β and γ find $\alpha_{(\beta,\gamma)} \ge \alpha_\beta, \alpha_\gamma$. Then $|x^d_\alpha| \le g_\beta + p_\gamma$ for $\alpha \ge \alpha_{(\beta,\gamma)}$ and $d \in B \cup C$ as desired.

For proving $\mathcal{B} + \mathcal{C} \xrightarrow{\text{c-o}} 0$, consider $(g_\beta + p_\gamma)_{\beta;\gamma} \downarrow 0$, and pick $\alpha_{(\beta,\gamma)} \ge \alpha_\beta, \alpha_\gamma$. It follows $|x^b_\alpha + x^c_\alpha| \le g_\beta + p_\gamma$ for $\alpha \ge \alpha_{(\beta,\gamma)}$, $b \in B$, and $c \in C$ as desired.

For proving $\text{co}(\mathcal{B}) \xrightarrow{\text{c-o}} 0$, let $b_1, ..., b_n \in B$ and $r_1, ..., r_n \in \mathbb{R}_+$ such that $\sum\limits_{k=1}^{n} r_k = 1$. Since, for $\alpha \ge \alpha_\beta$ we have

$$\Big|\sum_{k=1}^{n} r_k x^{b_k}_\alpha\Big| \le \sum_{k=1}^{n} r_k |x^{b_k}_\alpha| \le \sum_{k=1}^{n} r_k g_\beta = g_\beta,$$

it follows $\text{co}(\mathcal{B}) \xrightarrow{\text{c-o}} 0$.

ii) It is a direct consequence of *i)*.

iii) Let $\mathcal{B} \xrightarrow{\text{c-ru}} 0$ and $\mathcal{C} \xrightarrow{\text{c-ru}} 0$. Obviously, $r\mathcal{B} \xrightarrow{\text{c-ru}} 0$ and $\text{sol}(\mathcal{B}) \xrightarrow{\text{c-ru}} 0$.

Take $u, w \in E_+$ and two increasing sequences (α'_n) and (α''_n) of indices such that $|x^b_\alpha| \le \frac{1}{n}u$ for $\alpha \ge \alpha'_n$ and $b \in B$, and $|x^c_\alpha| \le \frac{1}{n}w$ for $\alpha \ge \alpha''_n$ and $c \in C$. For each n find some $\alpha_n \ge \alpha'_n, \alpha''_n$. Since $|x^d_\alpha| \le \frac{1}{n}(u + w)$ for $\alpha \ge \alpha_n$ and $d \in B \cup C$ then $\mathcal{B} \cup \mathcal{C} \xrightarrow{\text{c-ru}} 0$.

Since $|x^b_\alpha + x^c_\alpha| \le \frac{1}{n}(u + w)$ for all $\alpha \ge \alpha_n$, $b \in B$, and $c \in C$, it follows $\mathcal{B} + \mathcal{C} \xrightarrow{\text{c-ru}} 0$.

In order to show $\text{co}(\mathcal{B}) \xrightarrow{\text{c-ru}} 0$, let $b_1, ..., b_n \in B$ and $r_1, ..., r_n \in \mathbb{R}_+$ satisfy $\sum_{k=1}^{n} r_k = 1$. Since

$$\Big|\sum_{k=1}^{n} r_k x^{b_k}_\alpha\Big| \le \sum_{k=1}^{n} r_k |x^{b_k}_\alpha| \le \sum_{k=1}^{n} r_k \frac{1}{n} u = \frac{1}{n} u$$

for all $\alpha \ge \alpha'_n$, we conclude $\text{co}(\mathcal{B}) \xrightarrow{\text{c-ru}} 0$.

The remaining part of the proof is obvious. □

Definition 1.2 gives rise to the following notions.

Definition 2.1. Let $\mathcal{T} \subseteq \mathcal{L}(E, F)$.

If E and F are VLs, the set $\mathcal{T}$ is

a) *collectively order continuous* (briefly, $\mathcal{T} \in \mathbf{L}_{oc}(E, F)$) if $\mathcal{T}(x_\alpha) = \{(Tx_\alpha)\}_{T \in \mathcal{T}} \xrightarrow{\text{c-o}} 0$ whenever $x_\alpha \xrightarrow{\text{o}} 0$.

b) *collectively* uo-*continuous* (briefly, $\mathcal{T} \in \mathbf{L}_{uoc}(E,F)$) if $\mathcal{T}(x_\alpha) \xrightarrow{\text{c-uo}} 0$ whenever $x_\alpha \xrightarrow{\text{uo}} 0$.

c) *collectively* ru-*continuous* (briefly, $\mathcal{T} \in \mathbf{L}_{rc}(E,F)$) if $\mathcal{T}(x_\alpha) \xrightarrow{\text{c-ru}} 0$ whenever $x_\alpha \xrightarrow{\text{ru}} 0$.

d) *collectively order bounded* (briefly, $\mathcal{T} \in \mathbf{L}_{ob}(E,F)$) if the set $\mathcal{T}[0,b] = \{Tx : T \in \mathcal{T}; x \in [0,b]\}$ is order bounded for every $b \in E_+$.

If E is a VL and F is a NS, the set $\mathcal{T}$ is

e) *collectively order to norm bounded* (briefly, $\mathcal{T} \in \mathbf{L}_{onb}(E,F)$) if $\mathcal{T}[0,b]$ is bounded for every $b \in E_+$.

Clearly, an operator T is order bounded (order to norm bounded, o-, uo-, ru-continuous) iff the set $\{T\}$ is collectively order bounded (resp., collectively order to norm bounded, o-, uo-, ru-continuous). The following example shows that collectively o-bounded and/or collectively compact sets of o-continuous (or, uo-continuous) functionals need not to be o-continuous (uo-continuous) collectively.

Example 2.1. *Consider a sequence (f_n) of fuctionals on ℓ^2*

$$f_n \in (\ell^2)' : \quad f_n(\mathbf{x}) = x_n,$$

for $\mathbf{x} = \sum_{k=1}^{\infty} x_k \mathbf{e}_k \in \ell^2$, where $\mathbf{e}_k$ is the k-th unit vector of ℓ^2. Then (f_n) is a collectively o-*bounded collectively compact sequence of* o- *and* uo-*continuous fuctionals that is neither* o- *nor* uo-*continuous collectively.*

2.2 On continuity of order to norm bounded operators.

It is well known that every order bounded operator from a BL to a NL is continuous (cf., [1, Theorem 1.31]). We extend this fact in the following strong collective way.

Theorem 2.1. *Every collectively order to norm bounded set of operators from a BL to a NS is bounded in the operator norm.*

Proof. Let E be a BL, Y a NS, and $\mathcal{T} \in \mathbf{L}_{onb}(E,Y)$. Assume in contrary, $\mathcal{T}$ is not bounded in the operator norm. Then the set $\mathcal{T}(B_E)$, and hence the set $\mathcal{T}((B_E)_+)$ is not norm bounded. Thus, there exist sequences (x_n) in $(B_E)_+$ and (T_n) in $\mathcal{T}$ satisfying $\|T_n x_n\| \geq n^3$ for all $n \in \mathbb{N}$. Take $x := \|\cdot\|\text{-}\sum\limits_{n=1}^{\infty} \frac{x_n}{n^2} \in E_+$. Since $\mathcal{T} \in \mathbf{L}_{onb}(E,Y)$, there exists $M \in \mathbb{R}_+$ such that $\mathcal{T}([0,x]) \subseteq MB_Y$. Since $\frac{x_n}{n^2} \in [0,x]$ then $\|T_n(\frac{x_n}{n^2})\| \leq M$ for every $n \in \mathbb{N}$. We get a contradiction:

$$M \geq \left\| T_n\Big(\frac{x_n}{n^2}\Big) \right\| \geq n \qquad (\forall n \in \mathbb{N}).$$

Therefore, the set $\mathcal{T}$ is norm bounded. □

It is worth noting the same argument combined with the Krein – Smulian theorem (cf., [5, Theorem 2.37]) proves norm boundedness of every collective order to norm bounded set of linear operators from an ordered Banach space with a closed generating cone to a normed space.

Corollary 2.1. *Every linear operator from a BL to a NS that maps order intervals to norm bounded sets is bounded.*

2.3 Further elementary properties of collectively qualified sets.

Proposition 2.2. *Let E and F be VLs, and $r_1, r_2 \in \mathbb{R}$.*

i) If $\mathcal{T}_1$ and $\mathcal{T}_2$ are nonempty subsets of $\mathbf{L}_{ob}(E,F)$, then $r_1\mathcal{T}_1 + r_2\mathcal{T}_2 = \{r_1T_1 + r_2T_2 : T_1 \in \mathcal{T}_1, T_2 \in \mathcal{T}_2\} \in \mathbf{L}_{ob}(E,F)$.

ii) If $\mathcal{T}_1$ and $\mathcal{T}_2$ are nonempty subsets of $\mathbf{L}_{oc}(E,F)$ (resp., $\mathbf{L}_{uoc}(E,F)$, $\mathbf{L}_{rc}(E,F)$) then $\mathcal{T}_1 \cup \mathcal{T}_2$ and $r_1\mathcal{T}_1 + r_2\mathcal{T}_2$ are subsets of $\mathbf{L}_{oc}(E,F)$ (resp., of $\mathbf{L}_{uoc}(E,F)$, $\mathbf{L}_{rc}(E,F)$).

Proof. It follows directly from Proposition 2.1. □

For each element x of a VL E, we define the set $\mathcal{T}_x := \{y^\sim : |y| \le |x|\}$ of functionals on the order dual $E^\sim$ of E, where $y^\sim(f) = f(y)$ for each $f \in E^\sim = \mathcal{L}_{ob}(E,\mathbb{R})$.

Proposition 2.3. *Let E be a VL and $x \in E$. Then $\mathcal{T}_x \in \mathbf{L}_{oc}(E^\sim,\mathbb{R})$.*

Proof. In order to show that $\mathcal{T}_x$ is collectively order continuous, let $f_\alpha \xrightarrow{\text{o}} 0$ in $E^\sim$ and pick $g_\beta \downarrow 0$ in $E^\sim$ so that, for each β there is an α_β with $|f_\alpha| \le g_\beta$ for $\alpha \ge \alpha_\beta$. By the Riesz-Kantorovich formula, $|y^\sim(f_\alpha)| = |f_\alpha(y)| \le |f_\alpha|(|y|) \le |f_\alpha|(|x|) \le g_\beta(|x|)$ for all $|y| \le |x|$ and $\alpha \ge \alpha_\beta$. Since $g_\beta(|x|) \downarrow 0$, then $\mathcal{T}_x(f_\alpha) = \{(y^\sim(f_\alpha)) : |y| \le |x|\} \xrightarrow{\text{c-o}} 0$. As $f_\alpha \xrightarrow{\text{o}} 0$ is arbitrary, we conclude $\mathcal{T}_x \in \mathbf{L}_{oc}(E^\sim,\mathbb{R})$. □

Let E^δ be a Dedekind completion of a VL E, F be a Dedekind complete VL, and $T \in (\mathcal{L}_{oc})_+(E,F)$. Define $T^\delta \in \mathcal{L}_{oc}(E^\delta,F)$ as the unique linear extension of an additive map $E_+ \ni y \to \sup\limits_{E^\delta_+ \ni y \ge x \in E} Tx$. The mapping $(\mathcal{L}_{oc})_+(E,F) \ni T \to T^\delta$ has a unique extension to a Riesz isomorphism of $\mathcal{L}_{oc}(E,F)$ onto $\mathcal{L}_{oc}(E^\delta,F)$ (see, for example, [3, Theorem 1.84], [13, Theorem 3.2.3]).

Proposition 2.4. *Let a VL E be Archimedean, a VL F Dedekind complete, and $\mathcal{T} \subseteq \mathcal{L}_+(E,F)$. Then $\mathcal{T}^\delta \in \mathbf{L}_{oc}(E^\sim,F) \Longleftrightarrow \mathcal{T} \in \mathbf{L}_{oc}(E,F)$.*

Proof. The regularity of E in E^δ (cf., [3, Theorems 1.23 and 1.41]) provides the implication $\mathcal{T}^\delta \in \mathbf{L}_{oc}(E^\sim,F) \Longrightarrow \mathcal{T} \in \mathbf{L}_{oc}(E,F)$.

Now let $\mathcal{T} \in \mathbf{L}_{oc}(E,F)$ and $x_\alpha \xrightarrow{\text{o}} 0$ in E^δ. Then, there is $y_\beta \downarrow 0$ in E^δ such that, for each β there exists α_β with $|x_\alpha| \le y_\beta$ for all $\alpha \ge \alpha_\beta$. Since E is majorizing in E^δ (cf., [3, Theorem 1.41]), we may assume that (y_β) is contained in E. Trivially, $y_\beta \downarrow 0$ in E. Since $\mathcal{T} \in \mathbf{L}_{oc}(E,F)$, we have $\{(Ty_\beta)\}_{T\in\mathcal{T}} \xrightarrow{\text{c-o}} 0$, and hence there is a net $g_\gamma \downarrow 0$ in F s.t., for each γ, there is β_γ with $|Ty_\beta| \le g_\gamma$ for all $\beta \ge \beta_\gamma$ and $T \in \mathcal{T}$. As $\mathcal{T} \subseteq \mathcal{L}_+(E,F)$, we conclude $|T^\delta x_\alpha| \le T^\delta|x_\alpha| \le T^\delta y_{\beta_\gamma} = Ty_{\beta_\gamma} \le g_\gamma$ for all $\alpha \ge \alpha_{\beta_\gamma}$ and $T \in \mathcal{T}$. Thus $\{(T^\delta x_\alpha)\}_{T\in\mathcal{T}} \xrightarrow{\text{c-o}} 0$, and hence $\mathcal{T}^\delta \in \mathbf{L}_{oc}(E^\sim,F)$. □

2.4 Conditions for collectively order boundedness. It is long known that every o-continuous operator between VLs is order bounded (see, e.g., [3, Lemma 1.72]) whenever the order convergence is understood in the sense of [3, Definition 1.12] i.e., $x_\alpha \xrightarrow{o_1} x$ in a VL E if there exists a net $g_\alpha \downarrow 0$ in E s.t. $|x_\alpha - x| \le g_\alpha$ for all α. Abramovich and Sirotkin proved in [2] that every o-continuous operator from an Archimedean VL to a VL is order bounded also if the order convergence is understood in the sense of Definition 1.1. The Archimedean assumption is essential in [2, Theorem 2.1] (see [11, Example 2.1]).

A short looking at the proof of [2, Theorem 2.1] tell us that Abramovich and Sirotkin proved indeed that every r-continuous operator between arbitrary two VLs E and F is order bounded. Conversely, each $T \in \mathcal{L}_{ob}(E,F)$ is r-continuous directly by Definition 1.1 *c*) because, for every $u \in E_+$ there is $w \in F$ with $T[-u,u] \subseteq [-w,w]$, and hence $|x_\alpha| \le \frac{1}{n}u \Longrightarrow |Tx_\alpha| \le \frac{1}{n}w$.

The fact that $T \in \mathcal{L}_{ob}(E,F)$ iff T is r-continuous was independently rediscovered by Taylor and Troitsky in [15, Theorem 5.1]. It is worth mentioned that this fact can also be derived from the nonstandard criteria of order boundednees [7, Theorem 1.7.2] (cf., [8, Theorem 4.6.2]).

We extend the Abramovich–Sirotkin–Taylor–Troitsky result as follows with the key idea of proof coming from [2].

Theorem 2.2. *Let E and F be VLs with E Archimedean. The following statements hold.*

i) *Let $\mathcal{T} \subseteq \mathcal{L}(E,F)$. If $\mathcal{T}x_\alpha \xrightarrow{\text{c-o}} 0$ whenever $x_\alpha \downarrow 0$, then $\mathcal{T} \in \mathbf{L}_{ob}(E,F)$. In particular, $\mathbf{L}_{oc}(E,F) \subseteq \mathbf{L}_{ob}(E,F)$.*

ii) *If $\dim(F) < \infty$ then $\mathbf{L}_{uoc}(E,F) \subseteq \mathbf{L}_{ob}(E,F)$.*

Moreover, $\mathbf{L}_{ob}(E,F) = \mathbf{L}_{rc}(E,F)$ for all VLs E and F.

Proof. i) Let $[0,b]$ be an order interval in E. Like in the proof of [2, Theorem 2.1], let $\mathcal{I} = \mathbb{N} \times [0,b]$ be a set directed with the lexicographical order: $(m,z) \ge (n,y)$ iff either $m > n$ or $m = n$ and $z \ge y$. Let $x_{(k,y)} = \frac{1}{k}y \in [0,b]$. Since $0 \le x_{(m,y)} = \frac{1}{m}y \le \frac{1}{n}b = x_{(n,b)}$ for $(m,y) \ge (n,b)$ then $x_{(m,y)} \xrightarrow{\text{ru}} 0$, and hence $x_{(m,y)} \downarrow 0$ because E is Archimedean.

By the assumption, there exists a net $g_\beta \downarrow 0$ such that, for every β there exists (m_β, y_β) satisfying $|Tx_{(m,y)}| \le g_\beta$ for all $(m,y) \ge (m_\beta, y_\beta)$ and $T \in \mathcal{T}$. Pick any g_{β_0}. Since $(m_{\beta_0}+1, y) \ge (m_{\beta_0}, y_{\beta_0})$, it follows

$$\left|T\Big(\frac{y}{m_{\beta_0}+1}\Big)\right| = \left|Tx_{(m_{\beta_0}+1,y)}\right| \le g_{\beta_0} \quad (y \in [0,b], T \in \mathcal{T}).$$

Then $|Ty| \le (m_{\beta_0}+1)g_{\beta_0}$ for $y \in [0,b]$ and $T \in \mathcal{T}$. Since $[0,b]$ is arbitrary, the set $\mathcal{T}$ is collective order bounded.

ii) Let $[0,b] \subset E$. Take a directed set $\mathcal{I}$ as in the proof of *i*), and consider a net $x_{(k,y)} = \frac{1}{k}y$ in $[0,b]$. Then $x_{(m,y)} \xrightarrow{\text{ru}} 0$, and hence $x_{(m,y)} \xrightarrow{\text{uo}} 0$ as E is Archimedean. Because of $\dim(F) < \infty$, F has a strong unit, say w. Since $\mathcal{T} \in \mathbf{L}_{uoc}(E,F)$, there exists a net $g_\beta \downarrow 0$ in F such that, for every β, there exists (m_β, y_β) with $|Tx_{(m,y)}| \wedge w \le g_\beta$ for all $(m,y) \ge (m_\beta, y_\beta)$ and $T \in \mathcal{T}$. As $\dim(F) < \infty$, by passing to a tail, we may assume $g_\beta \le \frac{w}{2}$ for all β. So, $|Tx_{(m,y)}| \le g_\beta$ for $(m,y) \ge (m_\beta, y_\beta)$ and $T \in \mathcal{T}$. Pick any β_0. Since $(m_{\beta_0}+1, y) \ge (m_{\beta_0}, y_{\beta_0})$,

$$\Big|T\Big(\frac{y}{m_{\beta_0}+1}\Big)\Big| = \big|Tx_{(m_{\beta_0}+1,y)}\big| \le g_{\beta_0} \quad (y \in [0,b], T \in \mathcal{T}).$$

Thus, $|T(y)| \le (m_{\beta_0}+1)g_{\beta_0}$ for all $y \in [0,b]$ and $T \in \mathcal{T}$, and hence $\mathcal{T} \in \mathbf{L}_{ob}(E,F)$.

For the rest of the proof, firstly let $\mathcal{T} \in \mathbf{L}_{ob}(E,F)$ and $x_\alpha \xrightarrow{\text{ru}} 0$. Then, for some $u \in E_+$, there exists an increasing sequence (α_n) of indices such that $n|x_\alpha| \le u$, and hence $0 \le u + nx_\alpha \le 2u$ for all $\alpha \ge \alpha_n$. Since $\mathcal{T} \in \mathbf{L}_{ob}(E,F)$, there exist $w_1, w_2 \in F_+$ satisfying $|Tu| \le w_1$ for all $T \in \mathcal{T}$, and $|T(u+nx_\alpha)| \le w_2$ for all $\alpha \ge \alpha_n$ and $T \in \mathcal{T}$. Then $n|Tx_\alpha| \le |Tu| + w_2 \le w_1 + w_2$, and hence $|Tx_\alpha| \le \frac{w_1+w_2}{n}$ all $\alpha \ge \alpha_n$ and $T \in \mathcal{T}$. Consequently, $\mathcal{T} \in \mathbf{L}_{or}(E,F)$.

Finally, let $\mathcal{T} \in \mathbf{L}_{rc}(E,F)$ and $[0,b] \subset E$. Take a directed set $\mathcal{I}$ and a net $x_{(k,y)} \xrightarrow{\text{ru}} 0$ as in the proof of *i*). Then, for some $u \in E_+$ there exists an increasing sequence (k_n, y_n) in $\mathcal{I}$ with $\big|T\big(\frac{y}{k}\big)\big| = |Tx_{(k,y)}| \le \frac{1}{n}u$ for all $(k,y) \ge (k_n, y_n)$ and $T \in \mathcal{T}$. In particular, $\big|T\big(\frac{y}{k_1+1}\big)\big| = \big|Tx_{(k_1+1,y)}\big| \le u$ for all $y \in [0,b]$ and all $T \in \mathcal{T}$. This implies that $|Ty| \le (k_1+1)u$ for for all $y \in [0,b]$ and all $T \in \mathcal{T}$. Since $[0,b]$ is arbitrary, $\mathcal{T} \in \mathbf{L}_{ob}(E,F)$. ☐

It is an open question, whether the assumption $\dim(F) < \infty$ can be dropped in Theorem 2.2 *ii*). In favor of dropping this assumption, we have a relatively easy observation that each of non order bounded operators in the textbooks [3], [4], and [16] (see, [3, Exercise 30; p. 48], [4, Examples 2.38 and 4.73], [16, Exercise 98.7]) is not uo-continuous.

From the other hand, it is well known that a positive functional need not to be uo-continuous; e.g., $f \in (\ell^1)'$, $f(\mathbf{x}) = \sum_{k=1}^\infty x_k$, for $\mathbf{x} = \sum_{k=1}^\infty x_k \mathbf{e}_k \in \ell^1$, where $\mathbf{e}_k$ is the k-th unit vector of ℓ^1, is not uo-continuous as $f(\mathbf{e}_n) \equiv 1$ despite $\mathbf{e}_n \xrightarrow{\text{uo}} 0$ in ℓ^1.

In the Banach lattice setting, Theorem 2.2 has the following application to extension of the well known fact that each o-continuous operator from a Banach lattice to a normed lattice is continuous.

Theorem 2.3. *Let $\mathcal{T} \in \mathbf{L}_{oc}(E,F)$, were E is a Banach lattice and F a normed lattice. Then the set $\mathcal{T}$ is norm bounded.*

Proof. By Theorem 2.2, $\mathcal{T} \in \mathbf{L}_{ob}(E,F)$, and hence $\mathcal{T} \in \mathbf{L}_{onb}(E,F)$. An application of Theorem 2.1 completes the proof. ☐

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