Collective order convergence and collectively qualified set of operators

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Abstract

Collective versions of order convergences and corresponding types of collectively qualified sets of operators in vector lattices are investigated. It is proved that every collectively order continuous set of operators between Archimedean vector lattices is collectively order bounded.

Keywords: Vector lattice, order convergence, unbounded order convergence, collectively qualified set of operators

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1 Introduction

The study of collectively compact sets of operators between normed spaces was initiated by P. M. Anselone and T. W. Palmer [4]. In many cases it is necessary to deal with "uniform" or "collective" properties of a set of operators between fixed vector lattices (shortly, VLs) regarding the order convergent nets in their domains. Whereas the corresponding technique is rather simple for the norm convergence, the case of order convergence in VLs requires an additional attention. The present note is devoted to investigation of collective order convergence and its applications to collectively qualified sets of operators in VLs. We recall the following.

Definition 1.1. A net (x_α) in a VL E

a) *order converges* to $x \in E$ (briefly, $x_\alpha \xrightarrow{\text{o}} x$) if there exists a net (g_β) in E such that $g_\beta \downarrow 0$ and, for each β, there is an α_β with $|x_\alpha - x| \le g_\beta$ for all $\alpha \ge \alpha_\beta$.

b) *unbounded order converges* to $x \in E$ (briefly, $x_\alpha \xrightarrow{\text{uo}} x$) if $|x_\alpha - x| \wedge u \xrightarrow{\text{o}} 0$ for every $u \in E_+$.

c) *relative uniform converges* to $x \in E$ (briefly, $x_\alpha \xrightarrow{\text{r}} x$) if, for some $u \in E_+$, there exists an increasing sequence (α_n) of indices such that $|x_\alpha - x| \le \frac{1}{n}u$ for all $\alpha \ge \alpha_n$.

Working with families of nets indexed by the same directed set requires the following notion of "collective convergence" and "collective (eventual) order boundedness".

Definition 1.2. Let $\mathcal{B} = \{(x^b_\alpha)_{\alpha\in A}\}_{b\in B}$ be a set of nets in a VL E indexed by a directed set A. Then $\mathcal{B}$

a) *collective* o-*converges* to 0 (briefly, $\mathcal{B} \xrightarrow{\text{c-o}} 0$) if there exists a net $g_\beta \downarrow 0$ such that, for each β, there is an α_β with $|x^b_\alpha| \le g_\beta$ for $\alpha \ge \alpha_\beta$ and $b \in B$.

b) *collective* uo-*converges* to 0 (briefly, $\mathcal{B} \xrightarrow{\text{c-uo}} 0$) if $\{(|x^b_\alpha| \wedge u)_{\alpha\in A}\}_{b\in B} \xrightarrow{\text{c-o}} 0$ for every $u \in E_+$.

c) *collective* r-*converges* to 0 (briefly, $\mathcal{B} \xrightarrow{\text{c-r}} 0$) if, for some $u \in E_+$, there exists an increasing sequence (α_n) of indices such that $|x^b_\alpha| \le \frac{1}{n}u$ for $\alpha \ge \alpha_n$ and $b \in B$.

Throughout the text, all VLs are real, operators are linear, $\text{sol}(A) = \bigcup_{a\in A}[-|a|,|a|]$ denotes the solid hull of a subset A of a VL E, and $\mathcal{L}(E,F)$ (resp., $\mathcal{L}_+(E,F)$, $\mathcal{L}_{ob}(E,F)$, $\mathcal{L}_{oc}(E,F)$) the space of linear (resp., positive, regular, order bounded, order continuous) operators between VLs E and F. For terminology and elementary properties of VLs not explained in this note, we refer the reader to [2, 3, 5, 6].

2 Main results

We begin with some properties concerning c-o, c-uo, and c-r convergences.

Proposition 2.1. *Let $\mathcal{B} = \{(x^b_\alpha)\}_{b\in B}$ and $\mathcal{C} = \{(x^c_\alpha)\}_{c\in C}$ be two nonempty set of nets in a VL E indexed by the same directed set A, and let $r \in \mathbb{R}$. The following holds.*

i) If $\mathcal{B} \xrightarrow{\text{c-o}} 0$ and $\mathcal{C} \xrightarrow{\text{c-o}} 0$ then $r\mathcal{B} = \{(rx^b_\alpha)\}_{\alpha\in A;b\in B} \xrightarrow{\text{c-o}} 0$,

$$\text{sol}(\mathcal{B}) = \{(x_\alpha) : (\exists b \in B)(\forall \alpha \in A)|x_\alpha| \le |x^b_\alpha|\} \xrightarrow{\text{c-o}} 0,$$

$\mathcal{B}\cup\mathcal{C} \xrightarrow{\text{c-o}} 0$, $\mathcal{B}+\mathcal{C} = \{(x^b_\alpha + x^c_\alpha)\}_{b\in B;c\in C} \xrightarrow{\text{c-o}} 0$, and $\text{co}(\mathcal{B}) \xrightarrow{\text{c-o}} 0$, where

$$\text{co}(\mathcal{B}) = \Big\{(x_\alpha) : (\exists b_1,...,b_n \in B; r_1,...,r_n \in \mathbb{R}_+)\Big[\sum_{k=1}^n r_k = 1 \text{ and } (x_\alpha) = \Big(\sum_{k=1}^n r_k x^{b_k}_\alpha\Big)\Big]\Big\}.$$

ii) If $\mathcal{B} \xrightarrow{\text{c-uo}} 0$ and $\mathcal{C} \xrightarrow{\text{c-uo}} 0$ then $\text{sol}(\mathcal{B}) \xrightarrow{\text{c-uo}} 0$, $r\mathcal{B} \xrightarrow{\text{c-uo}} 0$, $\mathcal{B}\cup\mathcal{C} \xrightarrow{\text{c-uo}} 0$, $\mathcal{B}+\mathcal{C} \xrightarrow{\text{c-uo}} 0$, and $\text{co}(\mathcal{B}) \xrightarrow{\text{c-uo}} 0$.

iii) If $\mathcal{B} \xrightarrow{\text{c-r}} 0$ and $\mathcal{C} \xrightarrow{\text{c-r}} 0$ then $\text{sol}(\mathcal{B}) \xrightarrow{\text{c-r}} 0$, $r\mathcal{B} \xrightarrow{\text{c-r}} 0$, $\mathcal{B}\cup\mathcal{C} \xrightarrow{\text{c-r}} 0$, $\mathcal{B}+\mathcal{C} \xrightarrow{\text{c-r}} 0$, and $\text{co}(\mathcal{B}) \xrightarrow{\text{c-r}} 0$.

Moreover, $x_\alpha \xrightarrow{\text{o}} 0$ (resp., $x_\alpha \xrightarrow{\text{uo}} 0$, $x_\alpha \xrightarrow{\text{r}} 0$) iff $\{(x_\alpha)\} \xrightarrow{\text{c-o}} 0$ (resp., $\{(x_\alpha)\} \xrightarrow{\text{c-uo}} 0$, $\{(x_\alpha)\} \xrightarrow{\text{c-r}} 0$).

Proof. *i)* Let $\mathcal{B} \xrightarrow{\text{c-o}} 0$ and $\mathcal{C} \xrightarrow{\text{c-o}} 0$. So, take two nets $g_\beta \downarrow 0$ and $p_\gamma \downarrow 0$ in E such that, for every β and γ, there exist indices α_β and α_γ satisfying $|x^b_\alpha| \le g_\beta$ for $\alpha \ge \alpha_\beta$ and $b \in B$, and $|x^c_\alpha| \le p_\gamma$ for $\alpha \ge \alpha_\gamma$ and $c \in C$.

Obviously, $r\mathcal{B} \xrightarrow{\text{c-o}} 0$ and $\text{sol}(\mathcal{B}) \xrightarrow{\text{c-o}} 0$.

In order to show $\mathcal{B}\cup\mathcal{C} \xrightarrow{\text{c-o}} 0$, consider $(g_\beta+p_\gamma)_{\beta;\gamma} \downarrow 0$. Since $\mathcal{B}\cup\mathcal{C} = \{(x^d_\alpha)\}_{\alpha\in A;d\in B\cup C}$ then $|x^d_\alpha| \le g_\beta + p_\gamma$ for $\alpha \ge \alpha_\beta, \alpha_\gamma$ and $d \in B\cup C$. Pick an $\alpha_{(\beta,\gamma)} \ge \alpha_\beta, \alpha_\gamma$. Then $|x^d_\alpha| \le g_\beta + p_\gamma$ for $\alpha \ge \alpha_{(\beta,\gamma)}$ and $d \in B \cup C$ as desired.

For $\mathcal{B}+\mathcal{C} \xrightarrow{\text{c-o}} 0$, we consider $(g_\beta + p_\gamma)_{\beta;\gamma} \downarrow 0$, and pick an $\alpha_{(\beta,\gamma)} \ge \alpha_\beta, \alpha_\gamma$. This implies that, for $\alpha \ge \alpha_{(\beta,\gamma)}$, $b \in B$, and $c \in C$, we have $|x^b_\alpha + x^c_\alpha| \le g_\beta + p_\gamma$ as desired.

For $\text{co}(\mathcal{B}) \xrightarrow{\text{c-o}} 0$, let $b_1,...,b_n \in B$ and $r_1,...,r_n \in \mathbb{R}_+$ with $\sum_{k=1}^n r_k = 1$. Since, for $\alpha \ge \alpha_\beta$,

$$\Big|\sum_{k=1}^n r_k x^{b_k}_\alpha\Big| \le \sum_{k=1}^n r_k |x^{b_k}_\alpha| \le \sum_{k=1}^n r_k g_\beta = g_\beta,$$

and $b_1,...,b_n \in B$ and $r_1,...,r_n \in \mathbb{R}_+$ are arbitrary, we conclude $\text{co}(\mathcal{B}) \xrightarrow{\text{c-o}} 0$.

ii) It follows directly from *i)*.

iii) Let $\mathcal{B} \xrightarrow{\text{c-r}} 0$ and $\mathcal{C} \xrightarrow{\text{c-r}} 0$. Obviously, $r\mathcal{B} \xrightarrow{\text{c-r}} 0$ and $\text{sol}(\mathcal{B}) \xrightarrow{\text{c-r}} 0$.

Take $u, w \in E_+$ and two increasing sequences (α'_n) and (α''_n) of indices such that $|x^b_\alpha| \le \frac{1}{n}u$ for $\alpha \ge \alpha'_n$ and $b \in B$, and $|x^c_\alpha| \le \frac{1}{n}w$ for $\alpha \ge \alpha''_n$ and $c \in C$. Let us pick, for each n, any $\alpha_n \geq \alpha'_n, \alpha''_n$. Since $|x^d_\alpha| \leq \frac{1}{n}(u+w)$ for $\alpha \geq \alpha_n$ and $d \in B \cup C$ then $\mathcal{B} \cup \mathcal{C} \xrightarrow{\text{c-r}} 0$. Since $|x^b_\alpha + x^c_\alpha| \leq \frac{1}{n}(u+w)$ for all $\alpha \geq \alpha_n$, it follows $\mathcal{B} + \mathcal{C} \xrightarrow{\text{c-r}} 0$.

In order to show $\text{co}(\mathcal{B}) \xrightarrow{\text{c-r}} 0$, let $b_1, ..., b_n \in B$ and $r_1, ..., r_n \in \mathbb{R}_+$ with $\sum_{k=1}^n r_k = 1$. Since, for $\alpha \geq \alpha'_n$,

$$\Big|\sum_{k=1}^n r_k x^{b_k}_\alpha\Big| \leq \sum_{k=1}^n r_k |x^{b_k}_\alpha| \leq \sum_{k=1}^n r_k \frac{1}{n} u = \frac{1}{n} u,$$

and $b_1, ..., b_n \in B$, $r_1, ..., r_n \in \mathbb{R}_+$ are arbitrary, we conclude $\text{co}(\mathcal{B}) \xrightarrow{\text{c-r}} 0$.

The remaining part of the proof is obvious. □

2.1 Collectively order continuous sets.

Definition 1.2 gives rise to the following notions.

Definition 2.1. Let $\mathcal{T} \subseteq \mathcal{L}(E,F)$, where E and F are VLs. We say that $\mathcal{T}$ is

a) *collectively order bounded* (briefly, $\mathcal{T} \in \mathbf{L}_{ob}(E,F)$) if the set $\mathcal{T}[0,b] = \{Tx : T \in \mathcal{T}; x \in [0,b]\}$ is order bounded for every $b \in E_+$.

b) *collectively order continuous* (briefly, $\mathcal{T} \in \mathbf{L}_{oc}(E,F)$) if $\mathcal{T}(x_\alpha) = \{(Tx_\alpha)\}_{T\in\mathcal{T}} \xrightarrow{\text{c-o}} 0$ whenever $x_\alpha \xrightarrow{\text{o}} 0$.

c) *collectively* uo-*continuous* (briefly, $\mathcal{T} \in \mathbf{L}_{uoc}(E,F)$) if $\mathcal{T}(x_\alpha) \xrightarrow{\text{c-uo}} 0$ whenever $x_\alpha \xrightarrow{\text{uo}} 0$.

d) *collectively* r-*continuous* (briefly, $\mathcal{T} \in \mathbf{L}_{rc}(E,F)$) if $\mathcal{T}(x_\alpha) \xrightarrow{\text{c-r}} 0$ whenever $x_\alpha \xrightarrow{\text{r}} 0$.

Clearly, an operator T is order bounded (resp., o-, uo-, r-continuous) iff the set $\{T\}$ is collectively order bounded (resp., collectively o-, uo-, r-continuous).

Via considering of a sequence $f_n \in (\ell^2)'$, $f_n(x) = x_n$, we see that infinite uniformly bounded sets of o-continuous (resp., uo-continuous, r-continuous) functionals need not to satisfy the property collectively.

It is easy to see that, for nonempty $\mathcal{T}_1, \mathcal{T}_2 \subseteq \mathbf{L}_{ob}(E,F)$ and all $r_1, r_2 \in \mathbb{R}$, $r_1\mathcal{T}_1 + r_2\mathcal{T}_2 = \{r_1T_1 + r_2T_2 : T_1 \in \mathcal{T}_1, T_2 \in \mathcal{T}_2\} \in \mathbf{L}_{ob}(E,F)$. By Proposition 2.1, if $\mathcal{T}_1$ and $\mathcal{T}_2$ are nonempty subsets of $\mathbf{L}_{oc}(E,F)$ (resp., of $\mathbf{L}_{uoc}(E,F)$, $\mathbf{L}_{rc}(E,F)$) then $\mathcal{T}_1 \cup \mathcal{T}_2$ and $r_1\mathcal{T}_1 + r_2\mathcal{T}_2$ are subsets of $\mathbf{L}_{oc}(E,F)$ (resp., of $\mathbf{L}_{uoc}(E,F)$, $\mathbf{L}_{rc}(E,F)$) for $r_1, r_2 \in \mathbb{R}$.

2.2 Conditions for collectively order boundedness of sets.

Y. Abramovich, and G. Sirotkin proved that every order continuous operator in an Archimedean VL is order bounded [1, Thm. 2.1]. We extend their result as follows with the key idea of proof coming from [1].

Theorem 2.1. *Let E and F be VLs with E Archimedean. The following statements hold.*

i) Let $\mathcal{T} \subseteq \mathcal{L}(E,F)$. If $\mathcal{T}x_\alpha \xrightarrow{\text{c-o}} 0$ whenever $x_\alpha \downarrow 0$, then $\mathcal{T} \in \mathbf{L}_{ob}(E,F)$. In particular, $\mathbf{L}_{oc}(E,F) \subseteq \mathbf{L}_{ob}(E,F)$.

ii) If $\dim(F) < \infty$ then $\mathbf{L}_{uoc}(E,F) \subseteq \mathbf{L}_{ob}(E,F)$.

Moreover, $\mathbf{L}_{ob}(E,F) = \mathbf{L}_{rc}(E,F)$ for all VLs E and F.

Proof. $i)$ Let $[0,b]$ be an order interval in E. Like in the proof of [1, Thm. 2.1], let $\mathcal{I} = \mathbb{N} \times [0,b]$ be a set directed with the lexicographical order: $(m,z) \geq (n,y)$ iff either $m > n$ or $m = n$ and $z \geq y$. Let $x_{(k,y)} = \frac{1}{k}y \in [0,b]$. Since $0 \leq x_{(m,y)} = \frac{1}{m}y \leq \frac{1}{n}b = x_{(n,b)}$ for $(m,y) \geq (n,b)$ then $x_{(m,y)} \xrightarrow{\text{r}} 0$, and hence $x_{(m,y)} \downarrow 0$ because E is Archimedean.

By the assumption, there exists a net $g_\beta \downarrow 0$ such that, for every β there exists (m_β, y_β) satisfying $|Tx_{(m,y)}| \leq g_\beta$ for all $(m, y) \geq (m_\beta, y_\beta)$ and $T \in \mathcal{T}$. Pick any g_{β_0}. Since $(m_{\beta_0} + 1, y) \geq (m_{\beta_0}, y_{\beta_0})$, it follows

$$\Big|T\Big(\frac{y}{m_{\beta_0}+1}\Big)\Big| = \big|Tx_{(m_{\beta_0}+1,y)}\big| \leq g_{\beta_0} \quad (y \in [0,b], T \in \mathcal{T}).$$

Then $|Ty| \leq (m_{\beta_0} + 1)g_{\beta_0}$ for $y \in [0, b]$ and $T \in \mathcal{T}$. Since $[0, b]$ is arbitrary, the set $\mathcal{T}$ is collective order bounded.

ii) Let $[0, b] \subset E$. Take a directed set $\mathcal{I}$ as in the proof of *i)*, and consider a net $x_{(k,y)} = \frac{1}{k}y$ in $[0, b]$. Then $x_{(m,y)} \xrightarrow{\text{r}} 0$, and hence $x_{(m,y)} \xrightarrow{\text{uo}} 0$ as E is Archimedean. Because of $\dim(F) < \infty$, F has a strong unit, say w. Since $\mathcal{T} \in \mathbf{L}_{uoc}(E, F)$, there exists a net $g_\beta \downarrow 0$ in F such that, for every β, there exists (m_β, y_β) with $|Tx_{(m,y)}| \wedge w \leq g_\beta$ for all $(m, y) \geq (m_\beta, y_\beta)$ and $T \in \mathcal{T}$. As $\dim(F) < \infty$, by passing to a tail, we may assume $g_\beta \leq \frac{w}{2}$ for all β. So, $|Tx_{(m,y)}| \leq g_\beta$ for $(m, y) \geq (m_\beta, y_\beta)$ and $T \in \mathcal{T}$. Pick any β_0. Since $(m_{\beta_0} + 1, y) \geq (m_{\beta_0}, y_{\beta_0})$,

$$\Big|T\Big(\frac{y}{m_{\beta_0}+1}\Big)\Big| = \big|Tx_{(m_{\beta_0}+1,y)}\big| \leq g_{\beta_0} \quad (y \in [0,b], T \in \mathcal{T}).$$

Thus, $|T(y)| \leq (m_{\beta_0} + 1)g_{\beta_0}$ for all $y \in [0, b]$ and $T \in \mathcal{T}$, and hence $\mathcal{T} \in \mathbf{L}_{ob}(E, F)$.

For the rest of the proof, firstly let $\mathcal{T} \in \mathbf{L}_{ob}(E, F)$ and $x_\alpha \xrightarrow{\text{r}} 0$. Then, for some $u \in E_+$, there exists an increasing sequence (α_n) of indices such that $n|x_\alpha| \leq u$, and hence $0 \leq u + nx_\alpha \leq 2u$ for all $\alpha \geq \alpha_n$. Since $\mathcal{T} \in \mathbf{L}_{ob}(E, F)$, there exist $w_1, w_2 \in F_+$ satisfying $|Tu| \leq w_1$ for all $T \in \mathcal{T}$, and $|T(u + nx_\alpha)| \leq w_2$ for all $\alpha \geq \alpha_n$ and $T \in \mathcal{T}$. Then $n|Tx_\alpha| \leq |Tu| + w_2 \leq w_1 + w_2$, and hence $|Tx_\alpha| \leq \frac{w_1+w_2}{n}$ all $\alpha \geq \alpha_n$ and $T \in \mathcal{T}$. Consequently, $\mathcal{T} \in \mathbf{L}_{or}(E, F)$.

Finally, let $\mathcal{T} \in \mathbf{L}_{rc}(E, F)$ and $[0, b] \subset E$. Take a directed set $\mathcal{I}$ and a net $x_{(k,y)} \xrightarrow{\text{r}} 0$ as in the proof of *i)*. Then, for some $u \in E_+$ there exists an increasing sequence (k_n, y_n) in $\mathcal{I}$ with $\big|T\big(\frac{y}{k}\big)\big| = |Tx_{(k,y)}| \leq \frac{1}{n}u$ for all $(k, y) \geq (k_n, y_n)$ and $T \in \mathcal{T}$. In particular, $\big|T\big(\frac{y}{k_1+1}\big)\big| = \big|Tx_{(k_1+1,y)}\big| \leq u$ for all $y \in [0, b]$ and all $T \in \mathcal{T}$. This implies that $|Ty| \leq (k_1 + 1)u$ for for all $y \in [0, b]$ and all $T \in \mathcal{T}$. Since $[0, b]$ is arbitrary, $\mathcal{T} \in \mathbf{L}_{ob}(E, F)$. $\square$

It is an open question whether $\dim(F) < \infty$ can be dropped or weakened in Theorem 2.1 *ii)*. In favor of dropping this assumption, we have a relatively easy observation that each of non order bounded operators in the textbooks [2, 3, 7] ([2, Exer. 30; p. 48],[3, Ex. 2.38, Ex. 4.73],[7, Exer. 98.7]) is not uo-continuous. From the other hand, it is well known that a positive functional need not to be uo-continuous; e.g., $f \in (\ell^1)'$, $f(\mathbf{x}) = \sum_{k=1}^\infty x_k$, for $\mathbf{x} = \sum_{k=1}^\infty x_k\mathbf{e}_k \in \ell^1$, where $\mathbf{e}_k$ is the k-th unit vector of ℓ^1, is not uo-continuous as $f(\mathbf{e}_n) \equiv 1$ despite $\mathbf{e}_n \xrightarrow{\text{uo}} 0$ in ℓ^1.

2.3 Characterizations of collective order continuity. Every element x in a VL E gives rise to a set $\mathcal{T}_x = \{y^\sim : |y| \leq |x|\} \in \mathbf{L}_{oc}(E^\sim, \mathbb{R})$, where $y^\sim(f) = f(y)$ for each $f \in E^\sim = \mathcal{L}_{ob}(E, \mathbb{R})$. To see that $\mathcal{T}_x$ is collectively o-continuous let $f_\alpha \xrightarrow{\text{o}} 0$ in $E^\sim$ and pick $g_\beta \downarrow 0$ in $E^\sim$ so that, for each β there is an α_β with $|f_\alpha| \leq g_\beta$ for $\alpha \geq \alpha_\beta$. Thus, for all $|y| \leq |x|$ and $\alpha \geq \alpha_\beta$,

$$|y^\sim(f_\alpha)| = |f_\alpha(y)| \leq |f_\alpha|(|y|) \leq |f_\alpha|(|x|) \leq g_\beta(|x|).$$

By the Riesz–Kantorovich theorem (cf., [2, Thm. 1.67]), $g_\beta(|x|) \downarrow 0$. Then, $\mathcal{T}_x(f_\alpha) = \{(y^\sim(f_\alpha)) : |y| \leq |x|\} \xrightarrow{\text{c-o}} 0$. As $f_\alpha \xrightarrow{\text{o}} 0$ is arbitrary, $\mathcal{T}_x \in \mathbf{L}_{oc}(E^\sim, \mathbb{R})$.

Let E be a VL with the Dedekind completion E^δ and F be a Dedekind complete VL. The mapping $(\mathcal{L}_{oc})_+(E,F) \ni T \to T^\delta$, $T^\delta y = \sup\limits_{E^\delta_+ \ni y \geq x \in E} Tx$ has a unique extension to a Riesz isomorphism of $\mathcal{L}_{oc}(E,F)$ onto $\mathcal{L}_{oc}(E^\delta,F)$ (see, for example, [2, Thm. 1.84] [5, Thm. 3.2.3]). Therefore,

$$\mathcal{T}^\delta = \{T^\delta : T \in \mathcal{T}\} \in \mathbf{L}_{oc}(E^\sim, F) \iff \mathcal{T} \in \mathbf{L}_{oc}(E,F).$$

The following theorem provides a collective extension of [3, Thm. 1.56].

Theorem 2.2. *For a subset $\mathcal{T}$ of $\mathcal{L}(E,F)$, where E is an Archimedean VL and F is a Dedekind complete VL, the following statements are equivalent.*

i) $\mathcal{T} \in \mathbf{L}_{oc}(E,F)$.

ii) *If* $x_\alpha \downarrow 0$ *in* E *then* $\mathcal{T}x_\alpha \xrightarrow{\text{c-o}} 0$.

iii) *If* $x_\alpha \downarrow 0$ *in* E *then* $\inf\limits_\alpha \sup\limits_{T\in\mathcal{T};\alpha'\geq\alpha} |Tx_{\alpha'}| = 0$.

iv) $\{T^+\}_{T\in\mathcal{T}}, \{T^-\}_{T\in\mathcal{T}} \in \mathbf{L}_{oc}(E,F)$.

v) $|\mathcal{T}| = \{|T|\}_{T\in\mathcal{T}} \in \mathbf{L}_{oc}(E,F)$.

Proof. Implications *i)* $\Longrightarrow$ *ii)* and *ii)* $\Longrightarrow$ *iii)* are obvious, *iv)* $\Longrightarrow$ *v)* follows from $\{|T|\}_{T\in\mathcal{T}} \subseteq \{T^+\}_{T\in\mathcal{T}} + \{T^-\}_{T\in\mathcal{T}}$, and *v)* $\Longrightarrow$ *i)* is a consequence of $|Tx| \leq |T|(|x|)$.

iii) $\Longrightarrow$ *iv)* The Dedekind completeness of F ensures existence of $T^\pm$ for each $T \in \mathcal{T}$. It suffices to show $\{T^+\}_{T\in\mathcal{T}} \in \mathbf{L}_{oc}(E,F)$. In view of Theorem 2.1 *i)*, we have to show that $\{(T^+x_\alpha)\}_{T\in\mathcal{T}} \xrightarrow{\text{c-o}} 0$ for each $x_\alpha \downarrow 0$.

Suppose $x_\alpha \downarrow 0$ in E. Then $\sup\limits_{T\in\mathcal{T};\alpha'\geq\alpha_1} |Tx_{\alpha'}|$ exists in F for some α_1. Letting $u_\alpha = \sup\limits_{T\in\mathcal{T};\alpha'\geq\alpha} |Tx_{\alpha'}|$ for $\alpha \geq \alpha_1$, we obtain $u_\alpha \downarrow_{\alpha\geq\alpha_1} 0$ and $|Tx_\alpha| \leq u_\alpha$ for all $\alpha \geq \alpha_1$ and $T \in \mathcal{T}$. Therefore, $\mathcal{T}x_\alpha \xrightarrow{\text{c-o}} 0$. Theorem 2.1 *i)* implies $\mathcal{T} \in \mathbf{L}_{ob}(E,F)$.

Fix some $\alpha_0 \geq \alpha_1$. By Birkhoff's Inequality (cf., [2, Thm. 1.7]), for all $y \in [0, x_{\alpha_0}]$ and $\alpha \geq \alpha_0$,

$$0 \leq y - y \wedge x_\alpha = y \wedge x_{\alpha_0} - y \wedge x_\alpha \leq x_{\alpha_0} - x_\alpha.$$

Then, for all $y \in [0, x_{\alpha_0}]$, $T \in \mathcal{T}$, and $\alpha \geq \alpha_0$, we have

$$0 \leq Ty - T(y \wedge x_\alpha) \leq T^+(x_{\alpha_0} - x_\alpha) = T^+x_{\alpha_0} - T^+x_\alpha,$$

and hence $0 \leq T^+x_\alpha \leq T^+x_{\alpha_0} + |T(y \wedge x_\alpha)| - Ty$. Since $(y \wedge x_\alpha)\downarrow_{\alpha\geq\alpha_0} 0$ for each fixed $y \in [0, x_{\alpha_0}]$, it follows from *iii)* that $|T(y \wedge x_\alpha)| \leq g_\alpha = \sup\limits_{T\in\mathcal{T};\alpha'\geq\alpha} |T(y \wedge x'_\alpha)|$ for all $\alpha \geq \alpha_0$ and $T \in \mathcal{T}$. So, for each fixed $y \in [0, x_{\alpha_0}]$,

$$0 \leq T^+x_\alpha \leq g_\alpha + T^+x_{\alpha_0} - Ty \quad (\alpha \geq \alpha_0, T \in \mathcal{T}).$$

Then, for all $\alpha \geq \alpha_0$ and $T \in \mathcal{T}$,

$$0 \leq T^+x_\alpha \leq g_\alpha + T^+x_{\alpha_0} - \sup_{y\in[0,x_{\alpha_0}]} Ty = g_\alpha.$$

Since $g_\alpha \downarrow_{\alpha\geq\alpha_0} 0$, then $\{(T^+x_\alpha)\}_{T\in\mathcal{T}} \xrightarrow{\text{c-o}} 0$. □

2.4 The collective r-case of Theorem 2.2 (namely, $\mathcal{T} \in \mathbf{L}_{rc}(E,F) \iff |\mathcal{T}| \in \mathbf{L}_{rc}(E,F)$ if F is Dedekind complete) is trivial because $\mathbf{L}_{rc}(E,F) = \mathbf{L}_{ob}(E,F)$ by Theorem 2.1.

As an application of Theorem 2.2, we have the following.

Corollary 2.1. *Let E and F be VLs with E Archimedean and F Dedekind complete. Then* $\mathrm{sol}(\mathcal{T}) \in \mathbf{L}_{oc}(E,F)$ *whenever* $\mathcal{T} \in \mathbf{L}_{oc}(E,F)$.

Proof. Let $\mathcal{T} \in \mathbf{L}_{oc}(E,F)$. Then $|\mathcal{T}| \in \mathbf{L}_{oc}(E,F)$ by Theorem 2.2 v). Let $x_\alpha \downarrow 0$ in E. Theorem 2.2 iii) implies

$$0 \le \inf_{\alpha} \sup_{S \in \mathrm{sol}(\mathcal{T}); \alpha' \ge \alpha} |Sx_{\alpha'}| \le \inf_{\alpha} \sup_{T \in \mathcal{T}; \alpha' \ge \alpha} |T|x_{\alpha'} = 0.$$

It follows from Theorem 2.2 iii), $\mathrm{sol}(\mathcal{T}) \in \mathbf{L}_{oc}(E,F)$. □

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