ON THE DEGREE POLYNOMIAL OF VERTICES IN SOME CLASSES OF GRAPHS

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Abstract. The degree polynomial of vertices is a recently introduced descriptive parameter on the degrees of a simple graph. This notion leads to a concept named the degree polynomial sequence which is stronger than the concept of degree sequence. In this note, we characterize all graphs whose degree polynomial sequences are formed by polynomials with only one term. We show that it is not possible to provide a necessary and sufficient condition for a graph to be connected, also to be a tree, in terms of its degree polynomial sequence. Also we calculate the degree polynomial for the vertices of strong product and corona product of two simple graphs. Several open problems concerning these subjects is given as well.

1. Introduction

The degree sequence of a graph is one of the important invariants of a graph. In recent decades, many mathematicians have studied some various aspects and applications of this invariant. For instance, see [1], [2], [4], [5], [6], [8], [9], [10].

Recently, the author has introduced another descriptive parameter on the degrees of a simple graph, named the degree polynomial sequence of the graph (see [7]). This parameter is derived from a concept called the degree polynomial for the vertices of the graph. The degree polynomial sequence gives more information about a graph than a degree sequence does.

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In [7] the degree polynomial sequence for some well-known graphs have been obtained. Also the behavior of the degree polynomial under the join, Cartesian product, tensor product, and lexicographic product of two simple graphs have been studied and the degree polynomials of the vertices of the complement of a simple graph has been obtained in terms of the degree polynomials of the vertices of the graph itself.

In this note, first we answer the open problems raised in [7]. More precisely, we characterize all graphs whose degree polynomial sequences are formed by polynomials with only one term and we show that it is not possible to provide a necessary and sufficient condition for a graph to be connected, also to be a tree, in terms of its degree polynomial sequence. Then we calculate the degree polynomial for the vertices of strong product and corona product of two graphs. Several new open problems concerning these subjects is given as well.

2. **Preliminaries**

In the following, we use [3] for the basic terminologies and notations in graph theory. Also all graphs are finite and simple.

In a graph G, for two vertices $u, v \in V(G)$, if u is adjacent to v, we write $u \sim v$.

Let G be a graph of order n. A non-increasing sequence of nonnegative integers $q = (d_1, \ldots, d_n)$ is said the degree sequence of G, whenever there exists an ordering $v_1, \ldots, v_n$ of the vertices of G such that d_i be the degree of v_i, for $1 \leq i \leq n$. A sequence $q = (d_1, \ldots, d_n)$ of integers is realizable, if there exists a graph G such that q is the degree sequence of G. Since adding a finite number of isolated vertices to a graph and deleting a finite number of such vertices from a nonempty graph makes no change in the degree of the other vertices, we can consider only the case in which each $d_i, 1 \leq i \leq n$, is positive.

For a graph G, the degree polynomial of G, denoted by dp(G), is the polynomial $\sum_i t_i x^i$ in $\mathbb{R}[x]$ in which t_i is the number of vertices of G with degree i (specially, t_0 is the number of isolated vertices of G). If Δ be the maximum degree of G, then dp(G) is of degree Δ.

For a polynomial $f(x) = \sum_{i=1}^{n} a_i x^i \in \mathbb{R}[x]$ with $a_n \neq 0$, we show the sum of a_i's for $1 \leq i \leq n$, by sc(f). Also sec(f) and soc(f) are used for the sum of a_i's for even i, and the sum of a_i's for odd i, respectively. We define $\mathrm{sc}(0) = 0$ as well.

The total order $<_{\rm pol}$ on the set of all nonzero polynomials with coefficients in nonnegative integers is defined such that $<_{\rm pol}$ compares two distinct polynomials $f = \sum_{i=0}^{n} a_i x^i$ and $g = \sum_{i=0}^{m} b_i x^i$ with nonnegative integer coefficients and with $a_n, b_m \neq 0$, as follows:

If sc(f) $\neq$ sc(g), then which one of f and g whose sum of coefficients is greater (as an integer), will be greater;

If sc(f) = sc(g), then supposing that $i_1 = \max\{i|\ a_i \neq 0\ \mathit{or}\ b_i \neq 0\}$, if $a_{i_1} \neq b_{i_1}$, then whichever of f and g has greater coefficient in x^{i_1}, will be greater;

If sc(f) = sc(g) and $a_{i_1} = b_{i_1}$, then supposing that $i_2 = \max\{i|\ i < i_1,\ a_i \neq 0\ \mathit{or}\ b_i \neq 0\}$, if $a_{i_2} \neq b_{i_2}$, then whichever of f and g has greater coefficient in x^{i_2}, will be greater;

Continue on.

Let G be a graph. For a vertex v of G, the degree polynomial of v denoted by dp(v), is a polynomial with nonnegative integer coefficients, in which the coefficient of x^i is the number of neighbors of v each of degree i; Especially, for an isolated vertex v, dp(v) = 0.

Since adding a finite number of isolated vertices to a graph and deleting a finite number of such vertices from a nonempty simple graph makes no change in the degree polynomials of the other vertices, we will consider only the graphs which has no isolated vertices.

For a graph G of order n without any isolated vertex, a sequence $q = (f_1, f_2, \ldots, f_n)$ of polynomials is said to be the degree polynomial sequence of G, if

(a) $f_1 \geq_{\rm pol} \cdots \geq_{\rm pol} f_n$,

(b) there exists an ordering $V_1, \ldots, V_n$ of the vertices of G, such that f_i is the degree polynomial of v_i, for $1 \leq i \leq n$.

For the definitions and notations above, see [7].

Let G and H be simple graphs with disjoint vertex sets. The strong product of G and H, denoted by $G \boxtimes H$, is a simple graph with vertex set $V(G) \times V(H)$, in which for two vertices (u_1, v_1) and (u_2, v_2), $(u_1, v_1) \sim (u_2, v_2)$, if and only if

(1) $u_1 = u_2$ and $v_1 \sim v_2$ (in H), or

(2) $v_1 = v_2$ and $u_1 \sim u_2$ (in G), or

(3) $u_1 \sim u_2$ (in G) and $v_1 \sim v_2$ (in H).

The corona product of two graphs G and H, denoted by $G \odot H$, is defined as the graph obtained by taking one copy of G and $|V(G)|$ copies of H and joining the i-th vertex of G to every vertex in the i-th copy of H.

3. **The main results**

The following theorem characterizes all graphs whose degree polynomial sequences are formed by polynomials with only one term.

Theorem 3.1. *The degree polynomial sequences of a graph G is formed by polynomials with only one term, if and only if every connected component of G is a nonempty regular graph or a bipartite graph $(V_1, V_2; E)$ in which the degrees of all elements of V_i are equal, for $i = 1, 2$.*

Proof. Let the degree polynomial sequences of the graph G be formed by polynomials with only one term. Suppose that H is a connected component of G. Consider a vertex v in H. By hypothesis, the degree polunomial of v is cx^r where c and r are positive integer. We consider the following two cases:

(1) The integers r and c are equal. The degree of v is r. Consider any vertex $u(\neq v)$ in H. Upon the connectedness of H, there exists a path $(v =)v_1 \cdots v_d(= u)$ (of vetices of H) which connects v to u. Since $\mathrm{dp(v) = rx^r}$ and $v_2 \sim v$, the degree of v_2 is r. Sines the degree polynomial of v_2 has only one term, $\mathrm{dp(v_2) = rx^r}$. Repeating the same method implies that the degree of u is r. Hence the component H is r-regular.

(2) The integers r and c are distinct. The vertex v has c neighbors of degree r and has not any other neighbor. Let u is one of the neighbors of v. The degree of u is r and since u has a neighbor of degree c, $\mathrm{dp(u) = rx^c}$. Put

$$V_1 = \{x|\ x \in H,\ \mathrm{dp(x) = cx^r}\}, \quad \mathrm{V_2 = \{x|\ x \in H,\ dp(x) = rx^c\}}.$$

We show that H is bipartite with parts V_1 and V_2. First, note that no two elements of V_i are adjacent, for $i = 1, 2$. Secondary, $V(H)$ is the union of V_1 and V_2 because for any vertex x of H, by connectedness, there exist a path $(v =)v_1 \cdots v_d(= x)$ in H and therefore the degree polynomial of x is cx^r or rx^c.

Conversely, suppose that every connected component of the graph G is a nonempty regular graph or a bipartite graph $(V_1, V_2; E)$ in which the degrees of all elements of V_i are equal, for $i = 1, 2$. It is clear that the degree polynomial of any vertex in G is of one term. □

The following example shows that it is not possible to provide a necessary and sufficient condition for a graph to be connected, also to be a tree, in terms of its degree polynomial sequence.

Example 3.2. *Consider two following graphs:*

G_1 G_2

The graph G_1 is a tree also connected and G_2 is neither a tree nor connected but G_1 and G_2 have the same degree polynomial sequence.

Now we calculate the degree polynomial for the vertices of strong product and corona product of two graphs.

For convenience, first we introduce a special product of polynomials with nonnegative integer coefficients, namely strong product.

Definition 3.3. *Let $f = \sum_{a_i \neq 0} a_i x^i$ and $g = \sum_{b_j \neq 0} b_j x^j$ be two nonzero polynomials with nonnegative integer coefficients where $a_i x^i$'s and $b_j x^j$'s are the nonzero terms of f and g, respectively. The strong product of f and g, denoted by $f \boxtimes g$, is the polynomial $\sum c_t x^t$ in which t's are the distinct $i + j + ij$'s and*

$$c_t = \sum_{i+j+ij=t} a_i b_j.$$

Also we set $0 \boxtimes 0 = 0, f \boxtimes 0 = 0, 0 \boxtimes f = 0$, where 0 is the zero polynomial.

Under the conditions of Definition 3.3, it is observed simply that first, $f \boxtimes g$ can be achieved by strong-multiplying the nonzero terms of f by the nonzero terms of g, one by one, and secondly, for each two polynomials f and g with variable x and nonnegative integer coefficients, $f \boxtimes g = g \boxtimes f$.

Example 3.4. *Consider two polynomials*

$$f = 3 + 2x^4, \quad g = 2 + x + 5x^2.$$

We have

$$f \boxtimes g = 6 + 3x + 15x^2 + 4x^4 + 2x^9 + 10x^{14}.$$

Theorem 3.5. *If G and H be two simple graphs and u and v be vertices of G and H, respectively, then*

$$\mathrm{dp}_{G \boxtimes H}((u, v)) = (x^{\deg u} \boxtimes \mathrm{dp}(v)) + (x^{\deg v} \boxtimes \mathrm{dp}(u)) + (\mathrm{dp}(u) \boxtimes \mathrm{dp}(v)).$$

Proof. If u in G and v in H are isolated, then by definition of $G\boxtimes H$, the vertex (u,v) is isolated vertex in $G\boxtimes H$ and therefore $\mathrm{dp_{G\boxtimes H}((u,v))}=0$. On the other hand, $\mathrm{dp(u)}=0$ and $\mathrm{dp(v)}=0$. Thus the conclusion holds.

If u is isolated in G but v be non-isolated in H, supposing that $\mathrm{dp(v)}=\sum_{\mathrm{r_j}\neq 0}\mathrm{r_j x^j}$ where r_j's are positive integers and j's are the disjoint degrees of the neighbors of v in H, by definition of $G\boxtimes H$, since u has not any adjacent vertex in G, each neighbor of (u,v) in $G\boxtimes H$ is in the form (u,v') where $v'\sim v$. Meanwhile, the degree of such (u,v') in $G\boxtimes H$ is $\deg u+\deg v'+\deg u\deg v'$. Since for each j, the vertex v has r_j neighbors of degree j, for each j, the number of neighbors of (u,v) of degree j, will be r_j. Thus $\mathrm{dp_{G\boxtimes H}((u,v))}=\sum_{\mathrm{r_j}}\mathrm{r_j x^j}=\mathrm{dp(v)}$. But in this case, $\mathrm{dp(u)}=0$ and therefore the conclusion holds.

The argument in the case that v is isolated but u is not, is similar to the argument in the previous case.

Now do not let any of the vertices u and v be isolated. Suppose that $\mathrm{dp(u)}=\sum_{\mathrm{s=1}}^{\mathrm{k}}\mathrm{c_{i_s}x^{i_s}}$ and $\mathrm{dp(v)}=\sum_{\mathrm{t=1}}^{\mathrm{k'}}\mathrm{r_{j_t}x^{j_t}}$ where c_{i_s}'s and r_{j_t}'s are positive integers and i_s's and j_t's are the disjoint degrees of the neighbors of u and v, respectively. This means that the neighbors of u in G are restricted to

$$\begin{gathered} c_{i_1} \text{ vertices of degree } i_1,\\ \vdots\\ c_{i_k} \text{ vertices of degree } i_k, \end{gathered}$$

and the neighbors of v in H are restricted to

$$\begin{gathered} r_{j_1} \text{ vertices of degree } j_1,\\ \vdots\\ r_{j_{k'}} \text{ vertices of degree } j_{k'}. \end{gathered}$$

By definition of $G\boxtimes H$, the adjacent vertices of (u,v) in $G\boxtimes H$, are of three kinds below

(i) the vertices in the form (u,b) where b is adjacent to v in H,

(ii) the vertices in the form (a,v) where a is adjacent to u in G,

(iii) the vertices in the form (a,b) where a is adjacent to u in G, and b is adjacent to v in H.

Since in all vertices of kind (i), u is fixed, such vertices are restricted to

$$r_{j_1} \text{ vertices of degree } \deg u + j_1 + j_1 \deg u,$$
$$\vdots$$
$$r_{j_{k'}} \text{ vertices of degree } \deg u + j_{k'} + j_{k'} \deg u.$$

Also, since in the vertices of kind (ii), v is fixed, such vertices are restricted to

$$c_{i_1} \text{ vertices of degree } i_1 + \deg v + i_1 \deg v,$$
$$\vdots$$
$$c_{i_k} \text{ vertices of degree } i_k + \deg v + i_k \deg v.$$

Finally, we know that if a is a neighbor of u of degree i, and b is a neighbor of v of degree j, then (a, b) is a neighbor of (u, v) of kind (iii) and of degree $i + j + ij$. On the other hand, any neighbor of (u, v) of kind (iii) is in the form (a, b) of degree $\deg a + \deg b + \deg a \deg b$ where a is adjacent to u, and b is adjacent to v. Therefore since for each i, u has exactly c_i neighbors of degree i, and for each j, v has exactly r_j neighbors of degree j, the number $c_i r_j$ is calculated in the coefficient of x^{i+j+ij} in the degree polynomial of (u, v) and the sum of all $c_i r_j$'s which $i + j + ij = s$ is the number of the neighbors of (u, v) of kind (iii) and of degree s. Thus

$$\begin{aligned}\mathrm{dp}_{G\boxtimes H}((u,v)) = (\mathrm{r_{j_1} x^{\deg u + j_1 + j_1 \deg u}} + \cdots + \mathrm{r_{j_{k'}} x^{\deg u + j_{k'} + j_{k'} \deg u}}) + \\ (c_{i_1} x^{i_1 + \deg v + i_1 \deg v} + \cdots + c_{i_k} x^{i_k + \deg v + i_k \deg v}) + (\mathrm{dp(u)} \boxtimes \mathrm{dp(v)}) \\ = (x^{\deg u} \boxtimes \mathrm{dp(v)}) + (\mathrm{x^{\deg v}} \boxtimes \mathrm{dp(u)}) + (\mathrm{dp(u)} \boxtimes \mathrm{dp(v)}).\end{aligned}$$

□

Theorem 3.6. *If G and H are two graphs and $V(G) = \{u_1, \cdots, u_n\}$,*
(1) $\mathrm{dp_{G\odot H}(u_i) = x^{|V(H)|} dp_G(u_i) + x dp(H), 1 \leq i \leq n}$,
(2) $\mathrm{dp_{G\odot H}(v) = x dp_H(v) + x^{\deg_G u_i + |V(H)|}}$, *for every vertex v in i-th copy of H in the graph $G \odot H$.*

Proof. (1) Consider an arbitrary integer $1 \leq i \leq n$. If u_i is an isolated vertex in G, supposing that H has t_0 vertices of degree 0, t_1 vertices of degree 1, … , t_Δ vertices of degree Δ (Δ is the maximum degree of H), by definition of $G \odot H$, since u_i is not adjacent with any u_j in $G \odot H$, $1 \leq i \leq n$, the neighbors of u in $G \odot H$ are restricted to

$$t_0 \text{ vertices of degree } 1+0,$$

$$t_1 \text{ vertices of degree } 1+1,$$

$$\vdots$$

$$t_\Delta \text{ vertices of degree } 1+\Delta,$$

and therefore the degree polynomial of u_i in $G \odot H$ is

$$t_0x^1 + t_1x^2 + \cdots + t_\Delta x^{1+\Delta} = x\mathrm{dp(H)},$$

and therefore the conclusion holds. Now let u_i is non-isolated in G. Suppose that $\mathrm{dp_G(u_i)} = \sum_{\mathrm{s=1}}^{\mathrm{k}} \mathrm{c_{i_s} x^{i_s}}$ where c_{i_s}'s are positive integers and i_s's are the distinct degrees of neighbors of u is G. It means that the neighbors of u_i in G are restricted to

$$c_{i_1} \text{ vertices of degree } i_1,$$

$$\vdots$$

$$c_{i_k} \text{ vertices of degree } i_k.$$

Now by definition of $G \odot H$, u_i will be adjacent in $G \odot H$ to all the vertices above and also to any vertex in i-th copy of H. Thus the neighbors of u_i in $G \odot H$ are restricted to

$$c_{i_1} \text{ vertices of degree } |V(H)| + i_1,$$

$$\vdots$$

$$c_{i_k} \text{ vertices of degree } |V(H)| + i_k,$$

$$t_0 \text{ vertices of degree } 1+0,$$

$$t_1 \text{ vertices of degree } 1+1,$$

$$\vdots$$

$$t_\Delta \text{ vertices of degree } 1+\Delta,$$

where $\mathrm{dp(H)} = \sum_{\mathrm{i=0}}^{\Delta} \mathrm{t_i x^i}$. Therefore

$$\begin{aligned}\mathrm{dp_{G\odot H}(u_i)} &= \mathrm{c_{i_1} x^{|V(H)|+i_1} + \cdots + c_{i_k} x^{|V(H)|+i_k} + t_0 x^1 + t_1 x^2 + \cdots + t_\Delta x^{1+\Delta}} \\ &= x^{|V(H|}\mathrm{dp_G(u_i) + xdp(H)}.\end{aligned}$$

(2) Let v is in i-th copy of H in the graph $G \odot H$. If v is isolated in H, by definition of $G \odot H$, the only neighbor of v in $G \odot H$ is the vertex u_i (of degree $\deg_G u_i + |V(H)|$). Therefore $\mathrm{dp_{G\odot H}(v) = x^{\deg_G u_i + |V(H)|}}$ and therefore the conclusion holds. Now let v is non-isolated in H. Let $\mathrm{dp_H(v)} = \sum_{\mathrm{t=1}}^{\mathrm{k'}} \mathrm{r_{j_t} x^{j_t}}$ where r_{j_t}'s are positive integers and j_t's are the disjoint degrees of the neighbors of v in H. Therefore the neighbors of v in H are restricted to

$$\begin{gathered} r_{j_1} \text{ vertices of degree } j_1, \\ \vdots \\ r_{j_{k'}} \text{ vertices of degree } j_{k'}. \end{gathered}$$

By definition of $G \odot H$, the vertex v will be adjacent (in $G \otimes H$) to all the vertices above and also with the vertex u_i. Thus the neighbors of v in $G \odot H$ are restricted to

$$\begin{gathered} r_{j_1} \text{ vertices of degree } 1 + j_1, \\ \vdots \\ r_{j_{k'}} \text{ vertices of degree } 1 + j_{k'} \\ \\ \text{the vertex } u_i \text{ of degree } \deg_G u_i + |V(H)|. \end{gathered}$$

Therefore

$$\begin{aligned} \mathrm{dp_{G\odot H}(v)} &= \mathrm{r_{j_1} x^{1+j_1} + \cdots + r_{j_{k'}} x^{1+j_{k'}} + x^{\deg_G u_i + |V(H)|}} \\ &= x\mathrm{dp_H(v) + x^{\deg_G u_i + |V(H)|}}. \end{aligned}$$

□

4. **Some open problems**

The following open problems can be raised.

(1) The characterization of all sequences of polynomials which have at least one connected realization.

(2) The characterization of all realizable sequences of polynomials whose all realizations are connected.

(3) The characterization of all sequences of polynomials which have at least one realization in the form of a tree.

(4) The characterization of all realizable sequences of polynomials whose all realizations are a tree.

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