

# NORMAL OPERATORS FOR MOMENTUM RAY TRANSFORMS, II: SAINT VENANT OPERATORS

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**ABSTRACT.** The momentum ray transform  $I_m^k$  integrates a rank  $m$  symmetric tensor field  $f$  on  $\mathbb{R}^n$  over lines with the weight  $t^k$ ,  $I_m^k f(x, \xi) = \int_{-\infty}^{\infty} t^k \langle f(x + t\xi), \xi^m \rangle dt$ . Let  $N_m^k = (I_m^k)^* I_m^k$  be the normal operator of  $I_m^k$ . To what extent is a symmetric  $m$ -tensor field  $f$  determined by the data  $(N_m^0 f, \dots, N_m^r f)$  given for some  $0 \leq r \leq m$ ? The Saint Venant operator  $W_m^r$  is a linear differential operator of order  $m - r$  with constant coefficients on the space of symmetric  $m$ -tensor fields. We derive an explicit formula expressing  $W_m^r f$  in terms of  $(N_m^0 f, \dots, N_m^r f)$ . The tensor field  $W_m^r f$  represents the full local information on  $f$  that can be extracted from the data  $(N_m^0 f, \dots, N_m^r f)$ .

**Keywords.** Ray transform, inverse problems, Saint Venant operator, tensor tomography, momentum ray transform.

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## 1. INTRODUCTION

This article is a follow-up to our prior work [JKKS24]. To ensure a self-contained presentation, we have chosen to provide only a condensed version in the introduction and Section 2. We refer the reader to [JKKS24] for more details.

Let  $f$  be a Schwartz class symmetric  $m$ -tensor field on  $\mathbb{R}^n$ . The  $k^{\text{th}}$  momentum ray transform  $I_m^k f$  of  $f$  is defined by

$$(1.1) \quad I_m^k f(x, \xi) = \int_{\mathbb{R}} t^k f_{i_1 \dots i_m}(x + t\xi) \xi^{i_1} \dots \xi^{i_m} dt \quad (x \in \mathbb{R}^n, \xi \in \mathbb{R}^n, |\xi| = 1, \langle x, \xi \rangle = 0).$$

As in (1.1), with repeating indices, the Einstein summation convention is used throughout the article.

Let  $(I_m^k)^*$  be the  $L^2$  adjoint of  $I_m^k$ . Instead of working directly with the momentum ray transforms, we work with the associated normal operators  $N_m^k = (I_m^k)^* I_m^k$ . Being an averaging operator,  $N_m^k$  represents a better measurement model than the momentum ray transforms themselves. An inversion formula was obtained in [JKKS24] which recovers a symmetric  $m$ -tensor  $f$  from the data  $(N_m^0 f, \dots, N_m^m f)$ . The formula is reproduced in Theorem 2.1 below.

In this work we investigate the problem of recovering a tensor field from partial data. To what extent is a symmetric  $m$ -tensor field  $f$  determined by the data  $(N_m^0 f, \dots, N_m^r f)$  given for some  $0 \leq r \leq m$ ?

In the next section, we will recall the definition of the Saint Venant operator

$$(1.2) \quad W_m^r : C^\infty(\mathbb{R}^n; S^m) \rightarrow C^\infty(\mathbb{R}^n; S^{m-r} \otimes S^m) \quad (0 \leq r \leq m).$$

It is a linear differential operator of order  $m - r$  with constant coefficients. This operator was briefly mentioned in [Sha94, Theorem 2.17.2], but the operator  $W = W_m^0$  was widely used throughout Chapter 2 of [Sha94]. It is closely related to the equation

$$(1.3) \quad dv = f.$$

where  $d = \sigma \nabla$  is the inner derivative defined in Section 2.3 below. Namely, the equation (1.3) is solvable in a simply connected domain  $U \subset \mathbb{R}^n$  if and only if the right-hand side satisfies  $W_m^0 f = 0$ , see [Sha94, Theorem 2.2.2]. In the case of  $m = 2$ , the condition  $W_2^0 f = 0$  is popular in linear elasticity and is called *the deformation consistency condition*, it was obtained by Saint Venant.

For  $f \in \mathcal{S}(\mathbb{R}^n; S^m)$ , the tensor field  $W_m^r f$  represents the full *local* information, on the field  $f$ , that can be extracted from the data  $(I_m^0 f, \dots, I_m^r f)$ , see [Sha94, Theorem 2.17.2]. In particular,  $W_m^r f$  is uniquely determined by  $(N_m^0 f, \dots, N_m^r f)$ . The paper [MS21] establishes that, for  $f \in \mathcal{S}(S^m)$  and for  $0 \leq r \leq m$ , the tensor field  $W_m^r f$  can be explicitly recovered from  $(I_m^0 f, \dots, I_m^r f)$ . In [MS23, Theorem 3.1], the kernel of the momentum ray transform is described using the Saint Venant operator. It is shown that for  $f \in \mathcal{S}(S^m)$ ,  $(I_m^0 f, \dots, I_m^r f) = 0$  if and only if  $W_m^r f = 0$ . We will derive an explicit formula expressing  $W_m^r f$  through  $(N_m^0 f, \dots, N_m^r f)$ ; see Theorem 2.2 below. The latter theorem is the main result of the current work.

## 2. BASIC DEFINITIONS AND MAIN RESULT

**2.1. Tensor algebra.** Let  $T\mathbb{R}^n = \bigoplus_{m=0}^{\infty} T^m \mathbb{R}^n$  be the complex tensor algebra over  $\mathbb{R}^n$ . Assuming  $n$  to be fixed, the notation  $T^m \mathbb{R}^n$  will be often abbreviated to  $T^m$ . For a fixed orthonormal basis  $(e_1, \dots, e_n)$  of  $\mathbb{R}^n$ , by  $u_{i_1 \dots i_m} = u^{i_1 \dots i_m} = u(e_{i_1}, \dots, e_{i_m})$  we denote *coordinates* (= *components*) of a tensor  $u \in T^m$  with respect to the basis. There is no distinction between covariant and contravariant tensors since we use orthonormal bases only. The standard dot product on  $\mathbb{R}^n$  extends to  $T^m$  by

$$\langle u, v \rangle = u^{i_1 \dots i_m} \overline{v_{i_1 \dots i_m}}.$$

Let  $S^m = S^m \mathbb{R}^n$  be the subspace of  $T^m$  consisting of symmetric tensors. *The partial symmetrization*  $\sigma(i_1 \dots i_m) : T^{m+k} \rightarrow T^{m+k}$  in the indices  $(i_1, \dots, i_m)$  is defined by

$$\sigma(i_1 \dots i_m) u_{i_1 \dots i_m j_1 \dots j_k} = \frac{1}{m!} \sum_{\pi \in \Pi_m} u_{i_{\pi(1)} \dots i_{\pi(m)} j_1 \dots j_k},$$

where the summation is performed over the group  $\Pi_m$  of all permutations of the set  $\{1, \dots, m\}$ . In particular,  $\sigma : T^m \rightarrow S^m$  is the symmetrization in all indices. Given  $u \in S^m$  and  $v \in S^k$ , *the symmetric product*  $uv \in S^{m+k}$  is defined by  $uv = \sigma(u \otimes v)$ . Being equipped with the symmetric product,  $S^* \mathbb{R}^n = \bigoplus_{m=0}^{\infty} S^m \mathbb{R}^n$  becomes a commutative graded algebra that is called *the algebra of symmetric tensors over  $\mathbb{R}^n$* .

Given  $u \in S^m$ , let  $i_u : S^k \rightarrow S^{m+k}$  be the operator of symmetric multiplication by  $u$  and let  $j_u : S^{m+k} \rightarrow S^k$  be the adjoint of  $i_u$ . These operators are written in coordinates as

$$\begin{aligned} (i_u v)_{i_1 \dots i_{m+k}} &= \sigma(i_1 \dots i_{m+k}) u_{i_1 \dots i_m} v_{i_{m+1} \dots i_{m+k}} \\ (j_u v)_{i_1 \dots i_k} &= v_{i_1 \dots i_{m+k}} u^{i_{k+1} \dots i_{m+k}}. \end{aligned}$$

For the Kronecker tensor  $\delta$ , the notations  $i_\delta$  and  $j_\delta$  will be abbreviated to  $i$  and  $j$  respectively.

**2.2. Tensor fields.** Recall that the Schwartz space  $\mathcal{S}(\mathbb{R}^n)$  is the topological vector space consisting of  $C^\infty$ -smooth complex-valued functions on  $\mathbb{R}^n$  that decay rapidly at infinity together with all derivatives, equipped with the standard topology. Let  $\mathcal{S}(\mathbb{R}^n; S^m) = \mathcal{S}(\mathbb{R}^n) \otimes S^m$  be the topological vector space of smooth fast decaying symmetric  $m$ -tensor fields, defined on  $\mathbb{R}^n$ . In Cartesian coordinates, such a tensor field is written as  $f =$

$(f_{i_1 \dots i_m})$  with coordinates (= components)  $f_{i_1 \dots i_m} = f^{i_1 \dots i_m} \in \mathcal{S}(\mathbb{R}^n)$  symmetric in all indices.

We use the Fourier transform  $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$ ,  $f \mapsto \widehat{f}$  in the form (hereafter  $i$  is the imaginary unit)

$$\mathcal{F}f(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\langle y, x \rangle} f(x) dx.$$

The Fourier transform  $\mathcal{F} : \mathcal{S}(\mathbb{R}^n; S^m) \rightarrow \mathcal{S}(\mathbb{R}^n; S^m)$ ,  $f \mapsto \widehat{f}$  of symmetric tensor fields is defined component-wise:

$$\widehat{f}_{i_1 \dots i_m} = \widehat{f_{i_1 \dots i_m}}.$$

The  $L^2$ -product on  $C_0^\infty(\mathbb{R}^n; T^m)$  is defined by

$$(2.1) \quad (f, g)_{L^2(\mathbb{R}^n; T^m)} = \int_{\mathbb{R}^n} \langle f(x), g(x) \rangle dx.$$

**2.3. Inner derivative and divergence.** The first-order differential operator

$$d : C^\infty(\mathbb{R}^n; S^m) \rightarrow C^\infty(\mathbb{R}^n; S^{m+1})$$

defined by

$$(df)_{i_1 \dots i_{m+1}} = \sigma(i_1 \dots i_{m+1}) \frac{\partial f_{i_1 \dots i_m}}{\partial x^{i_{m+1}}} = \frac{1}{m+1} \left( \frac{\partial f_{i_2 \dots i_{m+1}}}{\partial x^{i_1}} + \dots + \frac{\partial f_{i_1 \dots i_m}}{\partial x^{i_{m+1}}} \right)$$

is called the inner derivative.

The divergence

$$\operatorname{div} : C^\infty(\mathbb{R}^n; S^{m+1}) \rightarrow C^\infty(\mathbb{R}^n; S^m)$$

is defined by

$$(\operatorname{div} f)_{i_1 \dots i_m} = \delta^{jk} \frac{\partial f_{i_1 \dots i_m j}}{\partial x^k}.$$

The operators  $d$  and  $-\operatorname{div}$  are formally adjoint to each other with respect to the  $L^2$ -product (2.1).

**2.4. The space  $\mathcal{S}(T\mathbb{S}^{n-1})$ .** The Schwartz space  $\mathcal{S}(E)$  is well-defined for a smooth vector bundle  $E \rightarrow M$  over a compact manifold with the help of a finite atlas and partition of unity subordinate to the atlas.

In particular, the Schwartz space  $\mathcal{S}(T\mathbb{S}^{n-1})$  is well defined for the tangent bundle

$$T\mathbb{S}^{n-1} = \{(x, \xi) \in \mathbb{R}^n \times \mathbb{S}^{n-1} : \langle x, \xi \rangle = 0\} \rightarrow \mathbb{S}^{n-1}, \quad (x, \xi) \mapsto \xi$$

of the unit sphere  $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ .

The Fourier transform  $\mathcal{F} : \mathcal{S}(T\mathbb{S}^{n-1}) \rightarrow \mathcal{S}(T\mathbb{S}^{n-1})$ ,  $\varphi \mapsto \widehat{\varphi}$  is defined by

$$\mathcal{F}\varphi(y, \xi) = \frac{1}{(2\pi)^{(n-1)/2}} \int_{\xi^\perp} e^{-i\langle y, x \rangle} \varphi(x, \xi) dx,$$

where  $dx$  is the  $(n-1)$ -dimensional Lebesgue measure on the hyperplane  $\xi^\perp = \{x \in \mathbb{R}^n : \langle \xi, x \rangle = 0\}$ .

The  $L^2$ -product on  $\mathcal{S}(T\mathbb{S}^{n-1})$  is defined by

$$(2.2) \quad (\varphi, \psi)_{L^2(T\mathbb{S}^{n-1})} = \int_{\mathbb{S}^{n-1}} \int_{\xi^\perp} \varphi(x, \xi) \overline{\psi(x, \xi)} dx d\xi,$$

where  $d\xi$  is the  $(n-1)$ -dimensional Euclidean volume form on the unit sphere  $\mathbb{S}^{n-1}$ .

**2.5. Momentum ray transform.** It is convenient to parameterize the family of oriented lines in  $\mathbb{R}^n$  by points of the manifold  $T\mathbb{S}^{n-1}$ . Namely, a point  $(x, \xi) \in T\mathbb{S}^{n-1}$  determines the line  $\{x + t\xi : t \in \mathbb{R}\}$  through  $x$  in the direction  $\xi$ .

For an integer  $k \geq 0$ , the momentum ray transform

$$I_m^k : \mathcal{S}(\mathbb{R}^n; S^m) \rightarrow \mathcal{S}(T\mathbb{S}^{n-1})$$

is the linear continuous operator defined by (1.1).

**2.6. Normal operators.** The formal adjoint of the momentum ray transform  $I_m^k$  with respect to  $L^2$ -products (2.1) and (2.2)

$$(I_m^k)^* : \mathcal{S}(T\mathbb{S}^{n-1}) \rightarrow C^\infty(\mathbb{R}^n; S^m)$$

is expressed by

$$((I_m^k)^* \varphi)_{i_1 \dots i_m}(x) = \int_{\mathbb{S}^{n-1}} \langle x, \xi \rangle^k \xi_{i_1} \dots \xi_{i_m} \varphi(x - \langle x, \xi \rangle \xi, \xi) \, d\xi.$$

We emphasize that, for  $\varphi \in \mathcal{S}(T\mathbb{S}^{n-1})$ , the tensor field  $(I_m^k)^* \varphi$  does not need to fast decay at infinity.

Let

$$N_m^k = (I_m^k)^* I_m^k : \mathcal{S}(\mathbb{R}^n; S^m) \rightarrow C^\infty(\mathbb{R}^n; S^m)$$

be the normal operator for the momentum ray transform  $I_m^k$ . For  $f \in \mathcal{S}(\mathbb{R}^n; S^m)$ , the Fourier transform  $\widehat{N_m^k f} \in \mathcal{S}'(\mathbb{R}^n; S^m)$  is well defined at least in the distribution sense and the restriction of  $\widehat{N_m^k f}$  to  $\mathbb{R}^n \setminus \{0\}$  belongs to  $C^\infty(\mathbb{R}^n \setminus \{0\}; S^m)$ .

**2.7. The inversion formula.** Let  $\Gamma$  be Euler's Gamma function and let the operator  $(-\Delta)^{1/2}$  be defined with the help of the Fourier transform by  $|y|\mathcal{F} = \mathcal{F}(-\Delta)^{1/2}$ . We use the definition

$$(2l+1)!! = 1 \cdot 3 \cdots (2l+1), \quad (-1)!! = 1.$$

Let us reproduce [JKKS24, Theorem 3.1].

**Theorem 2.1.** *Given integers  $m \geq 0$  and  $n \geq 2$ , a tensor field  $f \in \mathcal{S}(\mathbb{R}^n; S^m)$  is recovered from the data  $(N_m^0 f, N_m^1 f, \dots, N_m^m f)$  by the inversion formula*

$$(2.3) \quad f(x) = (-\Delta)^{1/2} \sum_{k=0}^m D_{m,n}^k (N_m^k f)(x),$$

where the linear differential operator of order  $m+k$

$$D_{m,n}^k : C^\infty(\mathbb{R}^n; S^m) \rightarrow C^\infty(\mathbb{R}^n; S^m)$$

is defined by

$$(2.4) \quad D_{m,n}^k = c_{m,n}^k \sum_{p=k}^m (n+2m-2p-3)!! \times \sum_{q=0}^{\min(p, m-p, p-k)} \frac{(-1)^q}{2^q q! (m-p-q)! (p-k-q)!} d^{p-q} i^q j^q j_x^{p-k-q} \operatorname{div}^k$$

with the coefficient

$$(2.5) \quad c_{m,n}^k = \frac{(-1)^k}{(k!)^2} \frac{2^{m-2} \Gamma(\frac{2m+n-1}{2})}{\pi^{(n+1)/2} (n+2m-3)!!}$$

and the operators  $i$ ,  $j$ , and  $j_x$  are defined in Section 2.1.

**2.8. The Saint Venant operator.** For integers  $m$  and  $r$  satisfying  $0 \leq r \leq m$ , let  $S^{m-r} \otimes S^m$  be the space of  $(2m-r)$ -tensors on  $\mathbb{R}^n$  which are symmetric in first  $m-r$  and last  $m$  indices. The Saint Venant operator (1.2) is defined by

$$(2.6) \quad (W_m^r f)_{i_1 \dots i_{m-r} j_1 \dots j_m} = \sigma(i_1 \dots i_{m-r}) \sigma(j_1 \dots j_m) \sum_{l=0}^{m-r} (-1)^l \binom{m-r}{l} \times \frac{\partial^{m-r} f_{i_1 \dots i_{m-r-l} j_1 \dots j_{r+l}}}{\partial x_{i_{m-r-l+1}} \dots \partial x_{i_{m-r}} \partial x_{j_{r+l+1}} \dots \partial x_{j_m}}.$$

In particular  $W_m^m$  is the identity operator.

### 2.9. The main result.

**Theorem 2.2.** Let  $0 \leq r \leq m$  and  $n \geq 2$  be integers. For  $f \in \mathcal{S}(\mathbb{R}^n; S^m)$ , the tensor field  $W_m^r f$  is recovered from the data  $(N_m^0 f, \dots, N_m^r f)$  by the inversion formula

$$W_m^r f = (-\Delta)^{1/2} W_m^r \sum_{k=0}^r D_{m,n}^k (N_m^k f),$$

where the linear differential operator  $D_{m,n}^k$  is defined by (2.4).

Theorem 2.2 is a generalization of Theorem 2.1 since  $W_m^m$  is the identity operator. In the case of  $r=0$  Theorem 2.2 actually coincides with [Sha94, Theorem 2.12.3].

The first step in the proof of Theorem 2.2 is as follows. Since  $W_m^r$  is a differential operator with constant coefficients, it commutes with  $(-\Delta)^{1/2}$ . Applying the operator  $W_m^r$  to the equality (2.3), we write the result in the form

$$W_m^r f = (-\Delta)^{1/2} W_m^r \sum_{k=0}^r D_{m,n}^k (N_m^k f) + (-\Delta)^{1/2} W_m^r \sum_{k=r+1}^m D_{m,n}^k (N_m^k f).$$

Thus, to prove Theorem 2.2, it suffices to demonstrate that

$$(2.7) \quad W_m^r D_{m,n}^k = 0 \quad \text{for } 0 \leq r < k \leq m.$$

The proof of (2.7) is presented in the next section.

## 3. PROOF OF THEOREM 2.2

Applying the Fourier transform to (2.6), we obtain

$$\widehat{W_m^r f} = i^{m-r} \widehat{W_m^r f},$$

where  $i$  is the imaginary unit and the purely algebraic operator

$$\widehat{W_m^r} = \widehat{W_m^r}(y) : S^m \rightarrow S^{m-r} \otimes S^m \quad (y \in \mathbb{R}^n)$$

is defined by

$$\begin{aligned} (\widehat{W_m^r} h)_{i_1 \dots i_{m-r} j_1 \dots j_m} &= \sigma(i_1 \dots i_{m-r}) \sigma(j_1 \dots j_m) \sum_{l=0}^{m-r} (-1)^l \binom{m-r}{l} \times \\ &\times h_{i_1 \dots i_{m-r-l} j_1 \dots j_{r+l}} y_{i_{m-r-l+1}} \dots y_{i_{m-r}} y_{j_{r+l+1}} \dots y_{j_m}. \end{aligned}$$

This can be written in the coordinate-free form

$$(3.1) \quad \langle \widehat{W_m^r} h, u \otimes v \rangle = \sum_{l=0}^{m-r} (-1)^l \binom{m-r}{l} \langle h, (j_y^l u) (j_y^{m-r-l} v) \rangle \quad \text{for } u \in S^{m-r} \text{ and } v \in S^m.$$

On the other hand, applying the Fourier transform to (2.7), we see that (2.7) is equivalent to the statement

$$(3.2) \quad \widehat{W}_m^r \widehat{D}_{m,n}^k = 0 \quad \text{for } 0 \leq r < k \leq m,$$

where the operator  $\widehat{D}_{m,n}^k$  is defined by

$$(3.3) \quad \begin{aligned} \widehat{D}_{m,n}^k &= c_{m,n}^k \sum_{p=k}^m (-1)^p (n+2m-2p-3)!! \\ &\quad \times \sum_{q=0}^{\min(p, m-p, p-k)} \frac{1}{2^q q! (m-p-q)! (p-k-q)!} i_y^{p-q} i_y^q j_y^q \operatorname{div}^{p-k-q} j_y^k, \end{aligned}$$

see [JKKS24, formula (8.7)].

We will use only one property of the operator  $\widehat{D}_{m,n}^k$ : as is seen from (3.3),

$$(3.4) \quad \widehat{D}_{m,n}^k = i_y^{r+1} B_{m,n}^k, \quad \text{for } 0 \leq r < k,$$

with some linear operator  $B_{m,n}^k$ . Therefore, to prove (3.2), it suffices to demonstrate that

$$(3.5) \quad \widehat{W}_m^r i_y^{r+1} = 0 \quad \text{for } 0 \leq r \leq m-1.$$

By (3.1),

$$\begin{aligned} \langle \widehat{W}_m^r i_y^{r+1} h, u \otimes v \rangle &= \sum_{l=0}^{m-r} (-1)^l \binom{m-r}{l} \langle i_y^{r+1} h, (j_y^l u)(j_y^{m-r-l} v) \rangle \\ &= \left\langle h, \sum_{l=0}^{m-r} (-1)^l \binom{m-r}{l} j_y^{r+1} ((j_y^l u)(j_y^{m-r-l} v)) \right\rangle. \end{aligned}$$

This means that (3.5) holds for any  $h \in S^{m-1}$  if and only if

$$(3.6) \quad \sum_{l=0}^{m-r} (-1)^l \binom{m-r}{l} j_y^{r+1} ((j_y^l u)(j_y^{m-r-l} v)) = 0 \quad \text{for any } u \in S^{m-r} \text{ and } v \in S^m \text{ (} 0 \leq r < m \text{)}.$$

The left-hand side of (3.6) is homogeneous of degree  $m+1$  in  $y$ . It suffices to prove (3.6) for a unit vector  $y$ . In what follows,  $y \in \mathbb{R}^n$  is a fixed vector satisfying  $|y| = 1$ .

The complex vector space  $S^m = S^m \mathbb{R}^n$  is generated by powers  $x^m$  ( $x \in \mathbb{R}^n$ ). Therefore (3.6) is equivalent to the statement

$$\sum_{l=0}^{m-r} (-1)^l \binom{m-r}{l} j_y^{r+1} ((j_y^l x^{m-r})(j_y^{m-r-l} z^m)) = 0 \quad \text{for any } x, z \in \mathbb{R}^n \text{ (} 0 \leq r < m \text{)}.$$

Since  $j_y^l x^{m-r} = \langle x, y \rangle^l x^{m-r-l}$  and  $j_y^{m-r-l} z^m = \langle z, y \rangle^{m-r-l} z^{r+l}$ , the latter statement can be written as

$$(3.7) \quad \sum_{l=0}^{m-r} (-1)^l \binom{m-r}{l} \langle x, y \rangle^l \langle z, y \rangle^{m-r-l} j_y^{r+1} (x^{m-r-l} z^{r+l}) = 0$$

for any  $x, z \in \mathbb{R}^n$  and  $0 \leq r < m$ . The equality (3.7) holds in the case  $\langle x, y \rangle = \langle z, y \rangle = 0$  since all summands on the left-hand side are equal to zero.

Next, we prove (3.7) in the case  $\langle x, y \rangle = 0$  but  $\langle z, y \rangle \neq 0$ . In this case (3.7) looks as follows:

$$(3.8) \quad j_y^{r+1} (x^{m-r} z^r) = 0.$$

Let us write (3.8) in coordinates

$$y^{i_1} \dots y^{i_{r+1}} \sum_{\pi \in \Pi_m} x_{i_{\pi(1)}} \dots x_{i_{\pi(m-r)}} z_{i_{\pi(m-r+1)}} \dots z_{i_{\pi(m)}} = 0.$$

After pulling the factor  $y^{i_1} \dots y^{i_{r+1}}$  inside the sum, every summand contain at least one factor of the form  $y^k x_k = 0$ . This proves (3.8).

Quite similarly (3.7) is proved in the case  $\langle x, y \rangle \neq 0$  but  $\langle z, y \rangle = 0$ .

Now, we prove (3.7) in the general case when  $\alpha = \langle x, y \rangle \neq 0$  and  $\beta = \langle z, y \rangle \neq 0$ . We represent vectors  $x, z \in \mathbb{R}^n$  in the form

$$x = \alpha y + x', \quad \langle x', y \rangle = 0; \quad z = \beta y + z', \quad \langle z', y \rangle = 0.$$

From this

$$\begin{aligned} x^{m-r-l} z^{r+l} &= (\alpha y + x')^{m-r-l} (\beta y + z')^{r+l} \\ &= \sum_{p=0}^{m-r-l} \sum_{q=0}^{r+l} \binom{m-r-l}{p} \binom{r+l}{q} \alpha^{m-r-l-p} \beta^{r+l-q} y^{m-p-q} x'^p z'^q. \end{aligned}$$

Substituting this expression into (3.7), we obtain (up to a factor  $\alpha^{m-r} \beta^m$ )

$$\sum_{l=0}^{m-r} \sum_{p=0}^{m-r-l} \sum_{q=0}^{r+l} (-1)^l \binom{m-r}{l} \binom{m-r-l}{p} \binom{r+l}{q} \alpha^{-p} \beta^{-q} j_y^{r+1} (y^{m-p-q} x'^p z'^q) = 0.$$

Denoting  $\tilde{x} = \alpha^{-1} x'$  and  $\tilde{z} = \beta^{-1} z'$ , this can be written in the form

$$\sum_{l=0}^{m-r} \sum_{p=0}^{m-r-l} \sum_{q=0}^{r+l} (-1)^l \binom{m-r}{l} \binom{m-r-l}{p} \binom{r+l}{q} j_y^{r+1} (y^{m-p-q} \tilde{x}^p \tilde{z}^q) = 0.$$

To simplify notations, we denote  $\tilde{x}$  and  $\tilde{z}$  again by  $x$  and  $z$  respectively. Thus, we have to prove the statement

$$(3.9) \quad \sum_{l=0}^{m-r} \sum_{p=0}^{m-r-l} \sum_{q=0}^{r+l} (-1)^l \binom{m-r}{l} \binom{m-r-l}{p} \binom{r+l}{q} j_y^{r+1} (y^{m-p-q} x^p z^q) = 0$$

for  $x, z \in y^\perp$  and  $0 \leq r < m$ .

Since the last factor  $j_y^{r+1} (y^{m-p-q} x^p z^q)$  on the left-hand side of (3.9) is independent of  $l$ , it makes sense to change the order of summations. We first change the order of summations over  $l$  and  $p$

$$\sum_{p=0}^{m-r} \sum_{l=0}^{m-r-p} \sum_{q=0}^{r+l} (-1)^l \binom{m-r}{l} \binom{m-r-l}{p} \binom{r+l}{q} j_y^{r+1} (y^{m-p-q} x^p z^q) = 0$$

and then change the order of summations over  $l$  and  $q$

$$\sum_{p=0}^{m-r} \sum_{q=0}^{m-p} \sum_{l=\max(0, q-r)}^{m-r-p} (-1)^l \binom{m-r}{l} \binom{m-r-l}{p} \binom{r+l}{q} j_y^{r+1} (y^{m-p-q} x^p z^q) = 0$$

This can be written in the form

$$(3.10) \quad \sum_{p=0}^{m-r} \sum_{q=0}^{m-p} C(m, r, p, q) j_y^{r+1} (y^{m-p-q} x^p z^q) = 0 \quad (x, z \in y^\perp, 0 \leq r < m),$$

where

$$(3.11) \quad \begin{aligned} & C(m, r, p, q) \\ &= \sum_{l=\max(0, q-r)}^{m-r-p} (-1)^l \binom{m-r}{l} \binom{m-r-l}{p} \binom{r+l}{q} \quad (0 \leq p \leq m-r, 0 \leq q \leq m-p). \end{aligned}$$

From (3.10) and (3.11), for  $x, z \in y^\perp$ , we have

$$(3.12) \quad j_y^{r+1}(y^{m-p-q}x^p z^q) = 0 \quad \text{if } p \geq 0, q \geq 0, p+q \leq m, r+1 > m-p-q.$$

Indeed, writing in coordinates

$$\begin{aligned} & (y^{m-p-q}x^p z^q)_{i_1 \dots i_m} \\ &= \frac{1}{m!} \sum_{\pi \in \Pi_m} y_{i_{\pi(1)}} \cdots y_{i_{\pi(m-p-q)}} x_{i_{\pi(m-p-q+1)}} \cdots x_{i_{\pi(m-q)}} z_{i_{\pi(m-q+1)}} \cdots z_{i_{\pi(m)}}, \end{aligned}$$

we have

$$\begin{aligned} & (j_y^{r+1}(y^{m-p-q}x^p z^q))_{i_{m-r} \dots i_m} \\ &= \frac{1}{m!} \sum_{\pi \in \Pi_m} y^{i_1} \cdots y^{i_{r+1}} y_{i_{\pi(1)}} \cdots y_{i_{\pi(m-p-q)}} x_{i_{\pi(m-p-q+1)}} \cdots x_{i_{\pi(m-q)}} z_{i_{\pi(m-q+1)}} \cdots z_{i_{\pi(m)}}. \end{aligned}$$

In the case of  $r+1 > m-p-q$ , every summand of the sum contains either a factor of the form  $y^j x_j = 0$  or a factor of the form  $y^j z_j = 0$ .

In virtue of (3.12), the summation in (3.10) can be restricted to  $(p, q)$  satisfying

$$(3.13) \quad p \geq 0, \quad q \geq 0, \quad p+q \leq m-r-1.$$

In particular,  $r < m$  and  $p \leq m-r-1$ . In other words, (3.10) is equivalent to the statement

$$(3.14) \quad \sum_{p=0}^{m-r-1} \sum_{q=0}^{m-r-p-1} C(m, r, p, q) j_y^{r+1}(y^{m-p-q}x^p z^q) = 0 \quad (x, z \in y^\perp, 0 \leq r < m).$$

**Lemma 3.1.** *For integers  $m, r, p, q$  satisfying (3.13) and  $0 \leq r < m$ , the following equality holds:*

$$(3.15) \quad \sum_{l=\max(0, q-r)}^{m-r-p} (-1)^l \binom{m-r}{l} \binom{m-r-l}{p} \binom{r+l}{q} = 0.$$

With the help of Lemma 3.1, we immediately complete the proof of Theorem 2.2. Indeed, comparing (3.11) and (3.15), we see that all coefficients  $C(m, r, p, q)$  participating in (3.14) are equal to zero. This proves (3.10). As we have shown before, (3.10) implies the statement of Theorem 2.2.

*Proof of Lemma 3.1.* We assume binomial coefficients  $\binom{k}{p}$  to be defined for all integers  $k$  and  $p$  under the agreement

$$\binom{k}{p} = 0 \quad \text{if either } k < 0 \text{ or } p < 0 \text{ or } k < p.$$

Then

$$\begin{aligned}
(3.16) \quad C(m, r, p, q) &= \sum_{l=\max(0, q-r)}^{m-r-p} (-1)^l \binom{m-r}{l} \binom{m-r-l}{p} \binom{r+l}{q} \\
&= \sum_{l=-\infty}^{\infty} (-1)^l \binom{m-r}{l} \binom{r+l}{q} \binom{m-r-l}{p}.
\end{aligned}$$

From [Ego84, p. 10], we have for  $0 < \varepsilon \ll 1$ ,

$$\binom{n}{k} = \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{(1+z)^n}{z^{k+1}} dz.$$

In particular,

$$\binom{r+l}{q} = \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{(1+z)^{r+l}}{z^{q+1}} dz, \quad \binom{m-r-l}{p} = \frac{1}{2\pi i} \int_{|w|=\varepsilon} \frac{(1+w)^{m-r-l}}{w^{p+1}} dw.$$

With the help of these formulas, we transform (3.16) as follows:

$$\begin{aligned}
C(m, r, p, q) &= -\frac{1}{(2\pi)^2} \int_{|z|=\varepsilon} \int_{|w|=\varepsilon} \frac{(1+z)^r (1+w)^{m-r}}{z^{q+1} w^{p+1}} \sum_{l=-\infty}^{\infty} (-1)^l \binom{m-r}{l} \left(\frac{1+z}{1+w}\right)^l dw dz \\
&= -\frac{1}{(2\pi)^2} \int_{|z|=\varepsilon} \int_{|w|=\varepsilon} \frac{(1+z)^r (1+w)^{m-r}}{z^{q+1} w^{p+1}} \left(1 - \frac{1+z}{1+w}\right)^{m-r} dw dz \\
&= -\frac{1}{(2\pi)^2} \int_{|z|=\varepsilon} \int_{|w|=\varepsilon} \frac{(1+z)^r (w-z)^{m-r}}{z^{q+1} w^{p+1}} dw dz \\
&= -\frac{1}{(2\pi)^2} \int_{|z|=\varepsilon} \int_{|w|=\varepsilon} \frac{(1+z)^r}{z^{q+1} w^{p+1}} \sum_{l=-\infty}^{\infty} (-1)^l \binom{m-r}{l} z^l w^{m-r-l} dw dz.
\end{aligned}$$

We perform the integration with respect to  $w$ . By the Cauchy integral formula, the only summand that survives corresponds to  $l = m - r - p$ . Thus,

$$C(m, r, p, q) = \frac{(-1)^{m-r-p}}{2\pi i} \binom{m-r}{p} \int_{|z|=\varepsilon} (1+z)^r z^{m-r-p-q-1} dz.$$

The integrand is a holomorphic function if  $p + q \leq m - r - 1$ . Therefore  $C(m, r, p, q) = 0$  if  $p + q \leq m - r - 1$ .  $\square$

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