Report on the article

"Tropical Weil's reciprocity law and Weil's pairing"

by N. Kalinin and M. Magin

In the paper under review the authors study a tropical analog of classical Weil's reciprocity law.

Namely, they consider a (compact) tropical curve Γ and two (quasi-)meromorphic tropical functions f, g on Γ. For any point $p \in \Gamma$ the local symbol is defined as $[f,g]_p = \mathrm{ord}_p(g) \cdot f(p) - \mathrm{ord}_p(f) \cdot g(p)$. Clearly, this symbols is skew-symmetric and bilinear. It is shown that $\sum_{p\in\Gamma}[f,g] = 0$, which is the tropical Weil's reciprocity. By the definition of tropical ord, the proof is reduced to the case when Γ is a segment, in which case the statement is a straightforward exercise.

Besides, the authors claim to propose a new analytic proof of Weil's reciprocity law. The proof contains a gap and its correction consists in a well-known argument by Deligne. Namely, the expression (11) is not well-defined, because the cylinder C contains poles and zeroes of f, whence $\log f$ is not well-defined, also on the boundary of C. An attempt to make it rigourous leads directly to Deligne's construction in §§ 2.2–2.4 of [7] (which is cited in the present paper but not in an essential way). Indeed, a natural way to overcome the problem with $\log f$ is to cover C by open discs U_i, choose $\log(f)_i$ on each U_i, and to consider a line bundle on C with transition functions $g^{\frac{1}{2\pi\sqrt{-1}}(\log(f)_j - \log(f)_i)}$ on $U_i \cap U_j$ and with holomorphic connection given by differential forms $\frac{1}{2\pi\sqrt{-1}}\log(f)_i\frac{dg}{g}$ on each U_i. This is a well-defined holomorphic line bundle with holomorphic connection on the complement to poles and zeroes of f and g. The local monodromy around each pole or zero of f or g coincides with the Weil symbol at this point. A correct version of formula (11) is actually the monodromy around S_i. The product of all local monodromies vanish for a compact Riemann surface. The latter is true for any holomorphic line bundle with holomorphic connection and can be shown, for example, by considerations with cutting of a Riemann surface into cylinders as on Fig. 2 in the paper. This proof of Weil's reciprocity is offered in [7].

Of course, the definition of a tropical symbol and the statement of tropical Weil's reciprocity are very nice and deserve to be published somewhere, but, I believe, as a part of a broader research. For now, this is just a short observation. Thus, the paper is not recommended for publication.

Just few little additional remarks. First, an obvious observation that when $\max\{f, g\} = 0$, we have $[f,g]_p = 0$. This is a tropical analog of Steinberg relation for classical symbol. Does there exists a tropical K-theory based on tropical modules? Probably, the symbol is a boundary map for tropical K-groups. Secondly, it is not clear, whether Definition 6 should be called a tropical Weil pairing. The classical Weil pairing is between torsion points on Jacobian and one can directly propose a tropical analog of that taking into account tropical Weil's reciprocity. What is proposed in Definition 6, as well as its classical analogue, does depend on the divisors and is not well-defined for their classes in Jacobian. But, of course, the construction is nice and the analogy with real-normalized differentials is absolutely natural and perfectly makes sense.