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CLASSIFICATION OF 4-DIMENSIONAL CONFORMALLY FOLIATED LIE GROUPS WITH MINIMAL LEAVES AND NATURAL ALMOST HYPER-HERMITIAN STRUCTURE

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Abstract: Let (G, g) be a 4-dimensional Riemannian Lie group that admits a left-invariant conformal foliation with minimal leaves of codimension 2. Such a group possesses a natural almost hyper-Hermitian structure. The authors classify these Lie groups according to the types of this structure.

Keywords: foliations, almost hyper-Hermitian structures

1. Introduction

An *almost complex structure* on a real differentiable manifold M is a tensor field J such that at each point $x \in M$, the endomorphism J of the tangent space T_xM satisfies $J^2 = -1$, where 1 denotes the identity transformation. A manifold equipped with such a structure is called an *almost complex manifold*.

If an almost complex manifold M is endowed with a Riemannian metric g that is preserved by J, i.e. $g(J\cdot, J\cdot) = g(\cdot, \cdot)$, then the triple (M, g, J) is called an *almost Hermitian manifold*. In this case, the almost complex structure J is said to be *orthogonal* with respect to g. These structures jointly define a 2-form ω, known as the *Kähler form*, given by $\omega(\cdot, \cdot) = g(J\cdot, \cdot)$.

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An almost Hermitian manifold is a Riemannian $2n$-dimensional manifold whose orthogonal frame bundle reduces to the subgroup $U(n) \subset SO(2n)$. A well-known classification of such manifolds, due to Gray and Hervella [7], categorizes them based on the properties of the covariant derivative $\nabla\omega$ of the Kähler form.

The full Gray-Hervella classification comprises 16 classes for $\dim M > 4$. However, in the 4-dimensional case, it simplifies to only four classes: $\mathcal{K}$ (Kähler), $\mathcal{AK}$ (almost Kähler), $\mathcal{H}$ (Hermitian) and $\mathcal{W}$ where the class $\mathcal{W}$ imposes no conditions on $\nabla\omega$ and thus contains all almost Hermitian 4-manifolds. These classes are defined as follows:

- Class $\mathcal{K}$ of *Kähler manifolds*: $\nabla\omega = 0$.
- Class $\mathcal{AK}$ of *almost Kähler manifolds*: $d\omega = 0$.
- Class $\mathcal{H}$ of *Hermitian manifolds*: $\nabla_X(\omega)(Y,Z) = \nabla_{JX}(\omega)(JY,Z)$. This condition is just equivalent to the integrability of the almost complex structure J.

A Riemannian manifold M endowed with two almost Hermitian structures (g,I) and (g,J) satisfying $IJ = -JI$ is called an *almost hyper-Hermitian manifold*. Such a manifold has dimension $4n$, and its orthogonal frame bundle can be reduced to the subgroup $Sp(n)$ of $SO(4n)$. The structures I and J induce a third almost complex structure $K = IJ$. If I, J and K are all integrable, then (M,I,J,K) is a *hypercomplex manifold*, and (M,g,I,J,K) is called a *hyper-Hermitian manifold*. If, in addition, the structures (g,I), (g,J) and (g,K) are all Kähler, then M is a *hyper-Kähler manifold*. It is fair to say that hyper-Hermitian and, in particular, hyper-Kähler manifolds have been the most extensively studied.

We study the natural almost hyper-Hermitian structure on four-dimensional Lie groups that admit a conformal foliation with minimal leaves. Such Lie groups were classified in [8]. The main result of the paper is a classification of these kind of Lie groups whose natural structure (g,I,J,K) satisfies the condition that exactly one of the structures is Hermitian, while the other two are almost Kähler. These results are presented in Theorems 1 and 2. In Theorem 3, the authors describe the case of hyper-Hermitian structures.

2. About conformal foliation of 4-dimensional Lie group

Conformal foliation with minimal leaves arise in the theory of harmonic morphisms from Riemannian manifolds onto a surface. Recall that a *harmonic morphism* is a smooth map $\varphi : (M^m, g) \longrightarrow (N^n, h)$ between Riemannian manifolds, which preserves the Laplace equation, in the sense that, if $f : V \longrightarrow \mathbb{R}$ is a harmonic function on an open set $V \subset N$ (with $\varphi^{-1}(V)$ non-empty), then $f \circ \varphi$ is a harmonic function on $\varphi^{-1}(V)$.

It is known (see [6]) that

(1) if $m < n$, then harmonic morphisms are constant mappings.
(2) if $m = n = 2$, then a harmonic morphism is a conformal mapping.

(3) if $m = n \neq 2$, then a harmonic morphism is a conformal mapping with constant coefficient of conformality (i.e., a local isometry up to scaling).

In general, for $m \geq n \geq 1$ a mapping φ is a harmonic morphism if and only if it is both harmonic and weakly conformal. In particular, according to [3], a submersion $\varphi : M^m \longrightarrow N^2$ is a harmonic morphism if and only if it is horizontally conformal and has minimal leaves. Conversely, for a conformal foliation $\mathcal{F}$ on minimal submanifolds of codimension 2, its local projections $\pi : A \longrightarrow N^2$ are harmonic morphisms. Therefore, manifolds that admit conformal foliation with minimal leaves represent a distinct geometric interest.

Here we recall the basic construction used for 4-dimensional Riemannian Lie groups (G, g) equipped with a left-invariant, 2-dimensional conformal foliation $\mathcal{F}$. These groups have been studied in [8, 11] and other works by the same authors. We will adhere to the notation introduced in [8, 11].

Let G be a four-dimensional Lie group equipped with a left-invariant Riemannian metric g, and K be a 2-dimensional subgroup of G. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K respectively. Let $\mathfrak{m}$ be the 2-dimensional orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the metric g. Then $\mathfrak{k}$ generates a left-invariant, integrable distribution $\mathcal{V}$ and $\mathfrak{m}$ defines the orthogonal distribution $\mathcal{H}$. We denote by $\mathcal{F}$ the foliation of G tangent to $\mathcal{V}$.

Following to notation established in [8, 11], let $\{X, Y, W, Z\}$ be an orthonormal basis on $\mathfrak{g}$ such that $\{W, Z\}$ spans $\mathfrak{k}$ and $\{X, Y\}$ spans $\mathfrak{m}$.

Then the second fundamental form for $\mathcal{V}$ is given by

$$B^{\mathcal{V}}(U, V) = \frac{1}{2}\mathcal{H}(\nabla_U V + \nabla_V U),\ (U, V \in \mathcal{V})$$

and for $\mathcal{H}$

$$B^{\mathcal{H}}(U, V) = \frac{1}{2}\mathcal{V}(\nabla_U V + \nabla_V U),\ (U, V \in \mathcal{H})$$

In terms of second fundamental forms the foliation $\mathcal{F}$ is *conformal* if there exists a vector field W such that $B^{\mathcal{H}} = g \otimes W$. And foliation $\mathcal{F}$ has *minimal leaves* if trace $B^{\mathcal{V}} = 0$.

Without loss of generality, one may assume that $[W, Z] = \lambda W$ and $\mathcal{H}[X, Y] = \rho X$ for some $\lambda, \rho \in \mathbb{R}$. Taking into account the requirement that $\mathcal{F}$ be a conformal foliation with minimal leaves, one derives the structural equations for the Lie algebra:

$$\begin{aligned}
&[W, Z] = \lambda W, \\
&[Z, X] = \alpha X + \beta Y + z_1 Z + w_1 W, \\
&[Z, Y] = -\beta X + \alpha Y + z_2 Z + w_2 W, \\
&[W, X] = aX + bY + z_3 Z - z_1 W, \\
&[W, Y] = -bX + aY + z_4 Z - z_2 W, \\
&[Y, X] = rX + \theta_1 Z + \theta_2 W,
\end{aligned} \tag{1}$$

where the parameters are subject to the 14 Jacobi identity constraints derived in [8, 11]:

$$
\begin{aligned}
0 &= \lambda a,\\
0 &= \lambda b,\\
0 &= -w_2 z_3 + w_1 z_4 - 2\alpha\theta_1 + r z_1,\\
0 &= -2z_4 z_1 + 2z_3 z_2 - 2a\theta_1 + r z_3,\\
0 &= -\lambda z_3 - z_2 b + z_4\beta - z_1 a + z_3\alpha,\\
0 &= -\lambda z_4 - z_2 a + z_4\alpha + z_1 b - z_3\beta,\\
0 &= \lambda\theta_1 - w_1 z_4 + w_2 z_3 - 2a\theta_2 - r z_1,\\
0 &= -\lambda\theta_2 + 2z_1 w_2 - 2z_2 w_1 - 2\alpha\theta_2 + r w_1,\\
0 &= -w_2 a - w_1 b - z_2\alpha - z_1\beta - r\alpha,\\
0 &= -w_2 b + w_1 a - z_2\beta + z_1\alpha + r\beta,\\
0 &= \lambda z_1 - w_2 b - z_2\beta - w_1 a - z_1\alpha,\\
0 &= z_2 a + z_1 b - z_4\alpha - z_3\beta - r a,\\
0 &= z_2 b - z_1 a - z_4\beta + z_3\alpha + r b,\\
0 &= \lambda z_2 - w_2 a - z_2\alpha + w_1 b + z_1\beta.
\end{aligned}
\tag{2}
$$

In their work [8], the authors classified the left-invariant conformal foliations with minimal leaves of codimension 2 for 4-dimensional Lie groups.

3. Natural almost hyper-Hermitian structure on conformally foliated 4-dimensional Lie group

The geometry of these Lie groups permits the construction of a natural left-invariant almost hyper-Hermitian structure. This structure on (G, g), associated with the foliation, is defined by the following properties:

1) $I\mathcal{V} = \mathcal{V}$, $I\mathcal{H} = \mathcal{H}$;
2) $J\mathcal{V} = \mathcal{H}$;
3) Both almost complex structures are left-invariant and g-orthogonal.

Due to left-invariance, it is sufficient to consider the restriction of almost complex structures to the Lie algebra. Let X be an arbitrary unit vector in $\mathfrak{m}$, and define $Y = IX$, $W = JY$, $Z = IW$. From the properties of I and J stated above, the set $\{X, Y, W, Z\}$ forms an g-orthonormal basis for $\mathfrak{g}$ with $X, Y \in \mathfrak{m}$, $W, Z \in \mathfrak{k}$.

Now, since $g(JX, W) = g(JX, JY) = g(X, Y) = 0$, and given that $JX \in \mathfrak{k}$ while $W \perp Z$ it follows that $JX = Z$ or $JX = -Z$. If $JX = Z$, then $IJ = -JI$; if $JX = -Z$ then $IJ = JI$. Therefore:

$$IX = Y, IW = Z; JX = Z, JY = W; K = IJ : KX = -W, KY = Z \tag{3}$$

Next, we will define (I, J, K) using formulas (3) in an orthonormal basis of $(\mathfrak{g}, g)$ such that $\{W, Z\}$ is a basis for $\mathfrak{k}$ and $\{X, Y\}$ is a basis for $\mathfrak{m}$.

We say that an almost hyper-Hermitian structure (g, I, J, K) belongs to the class $(*_I, *_J, *_K)$ if $(G, g, T) \in *_T$ for each $T \in \{I, J, K\}$, where $*_T$

denotes one of the classes $\mathcal{AK}$, $\mathcal{H}$ or $\mathcal{K}$. For example, the structure is said to be in the class $(\mathcal{AK}, \mathcal{AK}, \mathcal{H})$ if $d\omega_I = d\omega_J = 0$, and the almost complex structure K is integrable.

Naturally, some structures may possess no additional symmetries, meaning they lie in the largest class $\mathcal{W}$ of all almost Hermitian manifolds. We will not consider this case.

Lemma 1. [1] *If any two of the almost complex structures I, J, K are integrable, then the third one is also integrable.*

Lemma 2. [2] *If any two of the almost Hermitian structures (g, I, ω_I), (g, J, ω_J), (g, K, ω_K) are almost Kähler, then the third structure is Hermitian.*

If each structure belongs to one of the three classes, then in principle there are $3^3 = 27$ possible combinations. However, Lemmas 1 and 2 show that only the following combinations can actually occur:
1) $(g, I, J, K) \in (\mathcal{AK}, \mathcal{AK}, \mathcal{H})$, $(\mathcal{AK}, \mathcal{H}, \mathcal{AK})$ or $(\mathcal{H}, \mathcal{AK}, \mathcal{AK})$,
2) $(g, I, J, K) \in (\mathcal{H}, \mathcal{H}, \mathcal{H})$,
3) $(g, I, J, K) \in (\mathcal{K}, \mathcal{H}, \mathcal{H})$, $(\mathcal{H}, \mathcal{K}, \mathcal{H})$ or $(\mathcal{H}, \mathcal{H}, \mathcal{K})$,
4) $(g, I, J, K) \in (\mathcal{K}, \mathcal{K}, \mathcal{H}), (\mathcal{K}, \mathcal{H}, \mathcal{K}), (\mathcal{H}, \mathcal{K}, \mathcal{K})$.

Moreover cases 3) and 4) are only possible for an abelian Lie group. Consequently, our further research will be restricted to cases 1) and 2) for non-abelian Lie groups.

Lemma 3. *The necessary and sufficient conditions for a structures (g, T) on G to be almost Kähler, Hermitian, and Kähler are given in the following table*

	I	J	K
$\mathcal{AK}$	$\theta_1 + 2a = 0$ $\theta_2 - 2\alpha = 0$	$r - z_2 + w_1 = 0$ $\lambda - b - \alpha = 0$ $a - \beta = 0$ $z_1 + z_4 = 0$	$r + z_2 - z_3 = 0$ $\lambda - b - \alpha = 0$ $a - \beta = 0$ $z_1 + w_2 = 0$
$\mathcal{H}$	$2z_1 + z_4 + w_2 = 0$ $2z_2 - z_3 - w_1 = 0$	$r + z_2 - z_3 = 0$ $\lambda - b + \alpha - \theta_2 = 0$ $a + \beta + \theta_1 = 0$ $z_1 + w_2 = 0$	$r - z_2 + w_1 = 0$ $\lambda - b + \alpha - \theta_2 = 0$ $a + \beta + \theta_1 = 0$ $z_1 + z_4 = 0$
$\mathcal{K}$	$\theta_1 + 2a = 0$ $\theta_2 - 2\alpha = 0$ $2z_1 + z_4 + w_2 = 0$ $2z_2 - z_3 - w_1 = 0$	$r + z_2 - z_3 = 0$ $r - z_2 + w_1 = 0$ $a - \beta = 0$ $\lambda - b - \alpha = 0$	$\theta_2 - 2\alpha = 0$ $\theta_1 + 2a = 0$ $z_1 + z_4 = 0$ $z_1 + w_2 = 0$

TABLE 1. Conditions for the almost Hermitian structures (g, I), (g, J), (g, K) to be almost Kähler, Hermitian, or Kähler

Proof. The structure (g, T) is almost Kähler if and only if the 2-form $\omega_T(\cdot,\cdot) = g(T\cdot,\cdot)$ is closed. We compute:

$$\begin{aligned} d\omega_I(X,Y,Z) =& - g(I[X,Y],Z) - g(I[Y,Z],X) - g(I[Z,X],Y) \\ =& g(I(rX+\theta_1 Z+\theta_2 W),Z) - g(I(\beta X-\alpha Y - z_2 Z - w_2 W),X) \\ & - g(I(\alpha X+\beta Y+z_1 Z+w_1 W),Y) = \theta_2 - 2\alpha \end{aligned}$$

Similarly, we have $d\omega_I(Y,Z,W) = -z_2+z_2 = 0$, $d\omega_I(Z,W,X) = z_1-z_1 = 0$, $d\omega_I(W,X,Y) = -\theta_1 - 2a$. Thus, we obtain the conditions $(\mathcal{AK}_I)$.

The computations for $d\omega_J$ and $d\omega_K$ proceed similarly. For the structure J, we find: $d\omega_J(X,Y,Z) = r-z_2+w_1$, $d\omega_J(Y,Z,W) = \alpha-\lambda+b$, $d\omega_J(Z,W,X) = -a+\beta$, $d\omega_J(W,X,Y) = -z_1+z_4$. Setting these expressions to zero yields the almost Kähler conditions for J.

For almost complex structure K: $d\omega_K(X,Y,Z) = w_2+z_1$, $d\omega_K(Y,Z,W) = \beta - a$, $d\omega_K(Z,W,X) = \lambda - b - \alpha$, $d\omega_K(W,X,Y) = z_3 - r - z_2$.

According to the Newlander-Nirenberg theorem, an almost complex structure is integrable if and only if its Nijenhuis tensor vanishes.

We compute the Nijenhuis tensor $N_I(U,V)$ for the almost complex structure I for each pair of basis vectors $U, V \in \mathfrak{g}$. It is straightforward to verify that two of the six component equalities hold identically. Actually,

$$\begin{aligned} N_I(X,Y) &= [X,Y] + I[IX,Y] + I[X,IY] - [IX,IY] \\ &= [X,Y] - [Y,-X] + I([Y,Y] + [X,-X]) = 0 \end{aligned}$$

A similar computation shows that $N_I(Z,W) = 0$. The remaining pairs yield non-trivial identities:

$$\begin{aligned} N_I(X,W) &= N_I(X,-IZ) = -\big([X,IZ] + I[IX,IZ] + I[X,I^2Z] - [IX,I^2Z]\big) \\ &= IN_I(X,Z), \\ N_I(Y,Z) &= N_I(X,Z) = -IN_I(X,Z), N_I(Y,W) = N_I(IX,-IZ) = N_I(X,Z) \\ N_I(X,Z) &= -\alpha X - \beta Y - z_1 Z - w_1 W + bX - aY - z_4 Z + z_2 W \\ &\quad + I(\beta X - \alpha Y - z_2 Z - w_2 W + aX + bY + z_3 Z - z_1 W) \\ &= (-\alpha + b + \alpha - b)X + (-\beta - a + \beta + a)Y \\ &\quad + (-z_1 - z_4 - w_2 - z_1)Z + (-w_1 + z_2 + z_2 - z_3)W \\ &= (-2z_1 - z_4 - w_2)Z + (2z_2 - z_3 - w_1)W \end{aligned}$$

This yields the necessary and sufficient conditions for the integrability of the almost complex structure I.

Similarly, for the almost complex structure J, we have the identities: $N_J(X,Z) = N_J(Y,W) = 0$ and $N_J(X,W) = N_J(X,JY) = -JN(X,Y)$, $N_J(Y,Z) = N_J(Y,JX) = JN(X,Y)$, $N_J(Z,W) = N_J(JX,JY) = -N_J(X,Y)$. The expression for $N_J(X,Y)$ is given by:

$$N_J(X,Y) = (-r-z_2+z_3)X + (-w_2-z_1)Y + (\theta_1-\beta-a)Z + (-\theta_2+\alpha-b+\lambda)W$$

From this we derive the necessary and sufficient conditions for the integrability of J.

For the almost complex structure K, we obtain the following identities: $N_K(X,W) = N_K(Y,Z) = 0$ and $N_K(X,Z) = N_K(X,KY) = -KN_K(X,Y)$, $N_K(Y,-KX) = KN_K(X,Y)$, $N_K(Z,W) = N_K(KY,-KX) = -N_K(X,Y)$. The explicit form of $N_K(X,Y)$ is given by:

$$N_K(X,Y) = (-r+z_2-w_1)X+(z_1+z_4)Y+(-\theta_1-a-\beta)Z+(-\theta_2+\lambda-b+\alpha)W$$

From this expression, we derive the integrability condition for K.

Finally the conditions for Kähler structures are obtained by combining the criteria for the classes $\mathcal{H}$ and $\mathcal{AK}$. □

4. The case of two almost Kähler and one Hermitian structure.

4.1. **Structures of the class** $(\mathcal{H}, \mathcal{AK}, \mathcal{AK})$**.** Lemma 3 provides the following conditions:

$$\begin{cases} \lambda - b - \alpha = 0 \\ a - \beta = 0 \\ z_1 + z_4 = 0 \\ z_1 + w_2 = 0 \\ r - z_2 + w_1 = 0 \\ r + z_2 - z_3 = 0 \end{cases} \tag{4}$$

4.1.1. *Case* $\lambda = 0$. Applying the Jacobi identities (2) to this case, we obtain:

$$\begin{cases} -2\alpha\theta_1 + 3z_1 r = 0 \\ 2z_1^2 - 2a\theta_1 + z_3(r + 2z_2) = 0 \\ -2z_1 a + \alpha(z_2 + z_3) = 0 \\ 2a\theta_2 + 3z_1 r = 0 \\ -2z_1^2 - 2\alpha\theta_2 + w_1(r - 2z_2) = 0 \\ \alpha r = 0 \\ ar = 0 \\ 2z_1 a - \alpha(z_2 + w_1) = 0 \end{cases} \tag{5}$$

If $a^2 + \alpha^2 \neq 0$ then systems (4) and (5) become:

$$\begin{cases} \lambda = r = 0 \\ \beta = a \\ b = -\alpha \\ w_1 = z_2 = z_3 \\ z_4 = w_2 = -z_1 \\ \alpha\theta_1 = 0 \\ a\theta_2 = 0 \\ z_1^2 + z_2^2 - a\theta_1 = 0 \\ z_1^2 + z_2^2 + \alpha\theta_2 = 0 \\ z_1 a - \alpha z_2 = 0 \end{cases} \iff \begin{cases} z_1 = z_2 = z_3 = z_4 = 0 \\ \theta_1 = \theta_2 = 0 \\ w_1 = w_2 = r = \lambda = 0 \\ \beta = a \\ b = -\alpha \end{cases}$$

which defines a 2-parameter family of Lie algebras characterized by the conditions $\beta = a$, $b = -\alpha$, and all other structure constants vanishing:

$$\begin{aligned} [Z, X] &= \alpha X + aY \\ [Z, Y] &= -aX + \alpha Y \\ [W, X] &= aX - \alpha Y \\ [W, Y] &= \alpha X + aY \end{aligned} \tag{6}$$

If $a^2 + \alpha^2 = 0$, then system (4) with restrictions (5) becomes

$$\begin{cases} b = \beta = \alpha = a = 0 \\ z_4 = w_2 = -z_1 \\ rz_1 = 0 \\ 2z_1^2 + z_3(r + 2z_2) = 0 \\ -2z_1^2 + w_1(r - 2z_2) = 0 \\ r = z_2 + w_1 \\ r = z_3 - z_2 \end{cases} \iff \begin{cases} r = w_1 = w_2 = 0 \\ z_1 = z_2 = z_3 = z_4 = 0 \\ b = \beta = \alpha = a = 0 \end{cases} \tag{7}$$

The solutions to this system yield a 2-parameter family of Lie algebras:

$$[Y, X] = \theta_1 Z + \theta_2 W \tag{8}$$

4.1.2. *Case* $\lambda \neq 0$. The combined system of equations (2) and (4) is given by:

$$\begin{cases} a = b = \beta = 0 \\ \alpha = \lambda \\ -2\lambda\theta_1 + 3z_1 r = 0 \\ 2z_1^2 + z_3(r + 2z_2) = 0 \\ \lambda\theta_1 - 3z_1 r = 0 \\ -3\lambda\theta_2 - 2z_1^2 + w_1(r - 2z_2) = 0 \\ z_1\lambda = z_3\lambda = 0 \\ r = -z_2 \\ r = z_2 - w_1 = z_3 - z_2 \end{cases} \iff \begin{cases} a = b = \beta = 0 \\ z_1 = z_3 = z_4 = w_2 = 0 \\ \theta_1 = 0 \\ \alpha = \lambda \\ w_1 = 2z_2 \\ r = -z_2 \\ \lambda\theta_2 + 2z_2^2 = 0 \end{cases}$$

Solving this system yields a 2-parameter family of Lie algebras, defined by the following non-zero brackets:

$$\begin{aligned} [W, Z] &= \lambda W \\ [Z, X] &= \lambda X + 2z_2 W \\ [Z, Y] &= \lambda Y + z_2 Z \\ [W, Y] &= -z_2 W \\ [Y, X] &= -z_2 X - \frac{2z_2^2}{\lambda} W \end{aligned} \tag{9}$$

Let us define the four-dimensional Lie algebras from Mubarakzyanov's classification [10] that are isomorphic to Lie algebras (6), (8) and (9). A description of all four-dimensional Lie algebras arising in our classification is provided in the Appendix.

Lemma 4. *Every almost hyper-Hermitian Lie algebra* $\mathfrak{g}, \mu g, I, J, K)$*, defined by (6) is isomorphic to Lie algebra* $(\mathfrak{g}_{4,10}$ *with structure* $(\mu g_0, I_0, J_0, K_0)$*, where* $I_0 e_1 = e_2, I_0 e_4 = e_3; J_0 e_1 = -e_3, J_0 e_2 = -e_4$ *and the basis* e_1, e_2, e_3, e_4 *is* g_0*-orthonormal.*

Proof. An isomorphism of family (6) to $\mathfrak{g}_{4,10}$, is defined by the formulas:

$$X = \frac{\alpha e_1 - a e_2}{\sqrt{a^2 + \alpha^2}}$$
$$Y = \frac{a e_1 + \alpha e_2}{\sqrt{a^2 + \alpha^2}}$$
$$Z = \frac{a e_4 - \alpha e_3}{\sqrt{a^2 + \alpha^2}}$$
$$W = \frac{-\alpha e_4 - a e_3}{\sqrt{a^2 + \alpha^2}}$$

Regardless of the parameters a and α the following hold: $I e_1 = e_2, I e_4 = e_3; J e_1 = -e_3, J e_2 = -e_4$. □

Lemma 5. *Every almost hyper-Hermitian Lie algebra* $(\mathfrak{g}, \mu g, I, J, K)$*, defined by (8) is isomorphic to Lie algebra* $\mathfrak{g}_{3,1} \oplus \mathfrak{g}_1$ *with structure* $(\mu_0 g_0, I_0, J_0, K_0)$*, where* $I_0 e_2 = e_3, I_0 e_1 = -e_4; J_0 e_2 = e_1, J_0 e_3 = e_4$ *and the basis* e_1, e_2, e_3, e_4 *is* g_0*-orthonormal.*

Proof. The isomorphism of family (8) to $\mathfrak{g}_{3,1} \oplus \mathfrak{g}_1$, is defined by formulas:

$$X = \theta_1 e_2 - \theta_2 e_3$$
$$Y = \theta_2 e_2 + \theta_1 e_3$$
$$W = \theta_2 e_1 + \theta_1 e_4$$
$$Z = \theta_1 e_1 - \theta_2 e_4$$

□

Put

$$\begin{cases} e_1 = \frac{\lambda X + z_2 W}{\lambda^2 + z_2^2} \\ e_2 = \frac{W}{\lambda} \\ e_3 = \frac{\lambda Y + z_2 Z}{\lambda^2 + z_2^2} \\ e_4 = \frac{Z}{\lambda} \end{cases} \tag{10}$$

Formulas (10) establish an isomorphism between the algebras of family (9) and the Lie algebra $\mathfrak{g}_{4,5}, [e_1, e_4] = p e_1, [e_2, e_4] = d e_2, [e_3, e_4] = q e_3$ for $p = q = -1, d = 1$.

Lemma 6. *Every almost hyper-Hermitian Lie algebra* $(\mathfrak{g}, \mu g, I, J, K)$*, defined by (9) for* $t = \frac{z_2}{\lambda} \in \mathbb{R}$ *is isomorphic to* $\mathfrak{g}_{4,5}$ *with structure* $(\mu_0 g_t, I_t, J_t, K_t)$*, where* $I_t e_1 = e_3, I_t e_2 = e_4; J_t e_1 = e_4 - t e_3, J_t e_3 = e_2 - t e_1$ *and basis* $\{(1+t^2) e_1 - t e_2, e_2, (1+t^2) e_3 - t e_4, e_4\}$ *is* g_t*- orthonormal.*

Combining the results of Lemmas 4, 5, and 6, we obtain the following theorem.

Theorem 1. *Any four-dimensional conformally foliated Lie group* (G, g) *with minimal leaves and a natural left-invariant almost hyper-Hermitian structure defined by (3) in the* g*-orthogonal basis* $\{X, Y, W, Z\}$ *belongs to class* $(\mathcal{H}, \mathcal{AK}, \mathcal{AK})$ *if its Lie algebra is isomorphic to one of the following:*

- $\mathfrak{g}_{4,10}$ *with* $X = e_1$, $Y = e_2$, $W = -e_4$, $Z = -e_3$.
- $\mathfrak{g}_{3,1} \oplus \mathfrak{g}_1$ *with* $X = e_2$, $Y = e_3$, $W = e_4$, $Z = e_1$.
- $\mathfrak{g}_{4,5}$ *with a one-parameter family of bases:* $X_t = (1 + t^2)e_1 - te_2$, $Y_t = (1 + t^2)e_3 - te_4$, $W_t = e_2$, $Z_t = e_4$, *where* $t \in \mathbb{R}$.

4.2. **Structures of class** $(\mathcal{AK}, \mathcal{AK}, \mathcal{H})$**.** Lemma 3 restricts the parameters for this class to

$$\begin{cases} \theta_1 + 2a = 0 \\ \theta_2 - 2\alpha = 0 \\ r = z_2 - w_1 \\ \lambda = \alpha + b \\ \beta = a \\ z_4 = -z_1 \end{cases} \tag{11}$$

4.2.1. *Case* $\lambda = 0$. For this class, system (2) becomes

$$\begin{cases} -w_2 z_3 + 4a\alpha + z_1(r - w_1) = 0 \\ 2z_1^2 + 4a^2 + z_3(r + 2z_2) = 0 \\ -2az_1 + \alpha(z_2 + z_3) = 0 \\ 2\alpha z_1 + a(z_2 + z_3) = 0 \\ 2z_1 w_2 - 4\alpha^2 + w_1(r - 2z_2) = 0 \\ a(z_1 + w_2) + 2\alpha r = 0 \\ \alpha(z_1 + w_2) = 0 \\ a(z_2 + w_1) + \alpha(z_1 - w_2) = 0 \\ a(r - z_2 + z_3) = 0 \\ \alpha(r + z_2 - z_3) = 0 \\ a(z_1 - w_2) - \alpha(z_2 + w_1) = 0 \end{cases} \tag{12}$$

Assuming $a^2 + \alpha^2 \neq 0$ lines 8 and 11 of (12) give $w_2 = z_1$ and $w_1 = -z_2$. From lines 3 and 4 we obtain $z_1 = 0$, $z_3 = -z_2$. Under these conditions, the 5th line forces $\alpha = 0$, yielding:

$$\begin{cases} \theta_1 = -2a \\ w_2 = z_4 = z_1 = \theta_2 = \alpha = b = 0 \\ w_1 = z_3 = -z_2 \\ r = 2z_2 \\ \beta = a \\ a^2 = z_2^2 \end{cases} \tag{13}$$

Thus we obtain two one-parameter families of Lie algebras:

$$\begin{aligned} [Z,X] &= \pm z_2 Y - z_2 W \\ [Z,Y] &= \mp z_2 X + z_2 Z \\ [W,X] &= \pm z_2 X - z_2 Z \\ [W,Y] &= \pm z_2 Y - z_2 W \\ [Y,X] &= 2z_2 X \mp 2z_2 Z \end{aligned} \tag{14}$$

If $a^2 + \alpha^2 = 0$ then

$$\begin{cases} -w_2 z_3 + z_1(z_2 - 2w_1) = 0 \\ 2z_1^2 + z_3(3z_2 - w_1) = 0 \\ 2z_1 w_2 - w_1(z_2 + w_1) = 0 \\ r = z_2 - w_1 \\ z_4 = -z_1 \end{cases} \tag{15}$$

If $w_2 \neq 0$, we obtain $2z_1 = \frac{w_1(z_2+w_1)}{w_2}$, hence the first and second equations then form the system

$$\begin{pmatrix} z_2 - 2w_1 & -w_2 \\ \frac{w_1(z_2+w_1)}{w_2} & w_2(3z_2 - w_1) \end{pmatrix} \begin{pmatrix} z_1 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with determinant equal to $3(z_2 - w_1)^2$. If $z_2 \neq w_1$ then the parameters satisfy $z_1 = z_3 = 0$ and $w_1(z_2 + w_1) = 0$. This leads to the following two-parameter families of Lie algebras:

$$[Z,X] = -z_2 W,\ [Z,Y] = z_2 Z + w_2 W,\ [W,Y] = -z_2 W,\ [Y,X] = 2z_2 X \tag{16}$$

and

$$[Z,Y] = z_2 Z + w_2 W, \quad [W,Y] = -z_2 W, \quad [Y,X] = z_2 X \tag{17}$$

If $w_1 = z_2$, system (15) simplifies to

$$\begin{cases} z_1 z_2 + w_2 z_3 = 0 \\ z_1^2 + z_2 z_3 = 0 \\ z_1 w_2 - z_2^2 = 0 \end{cases} \iff \left[\begin{array}{l} z_1 = z_2 = z_3 = 0 \\ \begin{cases} z_1 z_2 z_3 \neq 0 \\ w_2 = \frac{z_2^2}{z_1} \\ z_3 = -\frac{z_1^2}{z_2}. \end{cases} \end{array} \right.$$

The first line yields a one-parameter family of Lie algebras (the case (16) or (17) with $z_2 = 0$)

$$[Z,Y] = w_2 W \tag{18}$$

In the second case we obtain the following Lie algebra structure:

$$\begin{aligned} [Z,X] &= z_1 Z + z_2 W \\ [Z,Y] &= z_2 Z + \frac{z_2^2}{z_1} W \\ [W,X] &= -\frac{z_1^2}{z_2} Z - z_1 W \\ [W,Y] &= -z_1 Z - z_2 W \end{aligned} \tag{19}$$

If $w_2 = 0$ then the system (15) can be written as

$$\begin{cases} z_1(z_2 - 2w_1) = 0 \\ 2z_1^2 + z_3(3z_2 - w_1) = 0 \\ w_1(z_2 + w_1) = 0 \\ r = z_2 - w_1 \\ z_4 = -z_1 \end{cases}$$

The system admits the following nonzero solutions:

- $z_1 = z_3 = 0$, $w_1 = -z_2, r = 2z_2$ – this corresponds to structure equation (16) with $w_2 = 0$;
- $z_1 = z_3 = w_1 = 0$, $r = z_2$ – this corresponds to structure equation (17) with $w_2 = 0$;
- $z_1 = z_2 = w_1 = 0$ – this yields a one-parameter family of Lie algebras:

$$[W, X] = z_3 Z \tag{20}$$

4.2.2. *Case* $\lambda \neq 0$. Systems (2) and (11) reduce to

$$\begin{cases} \beta = a = b = \theta_1 = 0 \\ \theta_2 = 2\alpha \\ r = z_2 - w_1 \\ \lambda = \alpha \\ z_4 = -z_1 \\ w_2 z_3 + w_1 z_1 - r z_1 = 0 \\ 2z_1^2 + 2z_2 z_3 + r z_3 = 0 \\ 6\alpha^2 - 2z_1 w_2 + 2z_2 w_1 - r w_1 = 0 \\ (r + z_2)\alpha = 0 \\ z_1 \alpha = 0 \\ z_3 \alpha = 0 \end{cases}$$

Since $\alpha = \lambda \neq 0$, the last three lines imply $z_1 = z_3 = 0$, $r = -z_2$. Then $w_1 = 2z_2$, which leads to $\alpha^2 + z_2^2 = 0$. However, this contradicts the constraint $\lambda \neq 0$.

Lemma 7. *Every almost hyper-Hermitian Lie algebra* $(\mathfrak{g}, \mu g, I, J, K)$*, defined by (14) is isomorphic to* $(\mathfrak{g}_{4,10}, \mu g_0, I_0, J_0, K_0)$*, where* $I_0 e_1 = -e_2 \pm 2e_3, I_0 e_3 = e_4; J_0 e_1 = \pm e_1 + 2e_4, J_0 e_3 = e_2 \mp e_3$ *and the basis* $\mp e_1 - e_4, \pm e_2 - e_3, e_3, e_4$ *is* g_0*-orthonormal.*

Proof. The isomorphism of family (14) to $\mathfrak{g}_{4,10}$, is defined by formulas:

$$\begin{aligned} X &= z_2(\mp e_1 - e_4) \\ Y &= z_2(\pm e_2 - e_3) \\ Z &= \mp z_2 e_4 \\ W &= \mp z_2 e_3 \end{aligned}$$

□

Lemma 8. *Every almost hyper-Hermitian Lie algebra* $(\mathfrak{g}, \mu g, I, J, K)$*, defined by (16) for* $t = \frac{w_2}{2z_2} \in \mathbb{R}$ *is isomorphic to* $\mathfrak{g}_{4,8}$ *with almost hyper-Hermitian* *structure* $(\mu g_t, I_t, J_t, K_t)$, *where* $I_te_1 = e_3 - te_1, I_te_2 = -2e_4; J_te_1 = 2e_4, J_te_2 = e_3 - te_1$, *and the basis* $e_1, e_2, e_3 - \frac{1}{2}te_1, 2e_4$ *is* g_t*-orthonormal.*

Proof. The family (16) is isomorphic to Lie algebra $\mathfrak{g}_{4,8}$ with $\beta = -\frac{1}{2}$ (see Appendix). An explicit isomorphism is given by:

$$X = z_2e_2$$
$$Y = -2z_2e_4$$
$$Z = z_2e_3 - \tfrac{1}{2}w_2e_1$$
$$W = z_2e_1$$

The claimed expressions for I_t, J_t and g_t follow directly from this identification.

□

Lemma 9. *Every almost hyper-Hermitian Lie algebra* $(\mathfrak{g}, \mu g, I, J, K)$ *defined by (17) satisfies the following:*

- *If* $z_2 \neq 0$, *then for* $t = \frac{w_2}{z_2}$ *it is isomorphic to Lie algebra* $\mathfrak{g}_{4,5}$ *endowed with the almost hyper-Hermitian structure* $(\mu_0 g_t, I_t, J_t, K_t)$, *where* $I_te_1 = e_4$, $I_te_2 = \frac{t}{2}e_2 + \frac{(t^2+4)}{2}e_3; J_te_1 = e_2 + te_3, J_te_3 = \frac{1}{2}e_4$ *and the basis* $e_1, e_4, e_2 + te_3, 2e_3$ *is* g_t*-orthonormal.*

- *If* $z_2 = 0$ *and* $w_2 \neq 0$ *then it is isomorphic to* $\mathfrak{g}_{3,1} \oplus \mathfrak{g}_1$ *with almost hyper-Hermitian structure* $(\mu g_0, I_0, J_0, K_0)$, *where* $I_0e_1 = e_2, I_0e_3 = -e_4; J_0e_1 = -e_3, J_0e_2 = -e_4$ *and basis* e_1, e_2, e_3, e_4 *is* g_0*-orthonormal.*

Proof. The isomorphism of family (17) in case $z_2 \neq 0$ to $\mathfrak{g}_{4,5}$ ($p = q = -1$, $d = 1$) is defined by formulas:

$$X = z_2e_1$$
$$Y = z_2e_4$$
$$Z = z_2(e_2 + te_3)$$
$$W = -2z_2e_3$$

For the case $z_2 = 0$, $w_2 \neq 0$ an isomorphism from the family (17) to $\mathfrak{g}_{3,1} \oplus \mathfrak{g}_1$ is defined by:

$$X = w_2e_4$$
$$Y = w_2e_3$$
$$Z = w_2e_2$$
$$W = w_2e_1$$

□

Lemma 10. *Any almost hyper-Hermitian Lie algebra* $(\mathfrak{g}, \mu g, I, J, K)$ *defined by (19) with* $t = \frac{z_1}{z_2} \in \mathbb{R}^*$ *is isomorphic to* $\mathfrak{g}_{3,1} \oplus \mathfrak{g}_1$ *endowed with the almost hyper-Hermitian structure* $(\mu_0 g_t, I_t, J_t, K_t)$, *where* $I_te_1 = \frac{1-t^2}{2t}e_1 + \frac{1+t^2}{2t}e_2, I_te_3 = \frac{1-t^2}{2t}e_3 - \frac{1+t^2}{2t}e_4; J_te_1 = -\frac{1+t^2}{2t}e_3 + \frac{1-t^2}{2t}e_4, J_te_2 = \frac{1-t^2}{2t}e_3 - \frac{1+t^2}{2t}e_4$ *and the basis* $e_1 + e_2, t(e_1 - e_2), t(e_3 + e_4), e_3 - e_4$ *is* g_t*-orthonormal.*

Proof. An isomorphism from the family (19) to $\mathfrak{g}_{3,1} \oplus \mathfrak{g}_1$ is given by:

$$\begin{aligned} X &= 2z_1(e_3 + e_4) \\ Y &= 2z_2(e_3 - e_4) \\ Z &= 2z_2(e_1 + e_2) \\ W &= 2z_1(e_1 - e_2) \end{aligned}$$

□

Lemma 11. *Any almost hyper-Hermitian Lie algebra* $(\mathfrak{g}, \mu g, I, J, K)$ *defined by (20) is isomorphic to* $\mathfrak{g}_{3,1} \oplus \mathfrak{g}_1$ *endowed with the almost hyper-Hermitian structure* $(\mu g_0, I_0, J_0, K_0)$, *where* $I_0 e_1 = -e_2, I_0 e_3 = e_4; J_0 e_1 = -e_3, J_0 e_2 = -e_4$ *and the basis* e_1, e_2, e_3, e_4 *is* g_0*-orthonormal.*

Proof. An isomorphism from the family (20) to $\mathfrak{g}_{3,1} \oplus \mathfrak{g}_1$ is given by:

$$\begin{aligned} X &= z_3 e_3 \\ Y &= z_3 e_4 \\ Z &= -z_3 e_1 \\ W &= -z_3 e_2 \end{aligned}$$

□

Combining the results of Lemmas 7—11, we obtain the following theorem.

Theorem 2. *Any four-dimensional conformally foliated Lie group* (G, g) *with minimal leaves and a natural left-invariant almost hyper-Hermitian structure defined by (3) in the* g*-orthogonal basis* $\{X, Y, W, Z\}$ *belongs to class* $(\mathcal{AK}, \mathcal{AK}, \mathcal{H})$ *if its Lie algebra is isomorphic to one of the following:*

- $\mathfrak{g}_{4,10}$ *with either of the following bases:* $X_1 = e_1 + e_4$, $Y_1 = e_2 + e_3$, $W_1 = e_3$, $Z_1 = e_4$ *or* $X_2 = e_1 - e_4$, $Y_2 = e_2 - e_3$, $W_2 = e_3$, $Z_2 = e_4$.
- $\mathfrak{g}_{4,8}$ *with either of the following one-parameter families of bases:* $X_1^t = e_2$, $Y_1^t = -2e_4$, $W_1^t = e_1$, $Z_1^t = e_3 - \frac{1}{2}te_1$ *or* $X_2^t = e_1$, $Y_2^t = e_4$, $W_2^t = -2e_3$, $Z_2^t = e_2 + te_3$, *where* $t \in \mathbb{R}$.
- $\mathfrak{g}_{3,1} \oplus \mathfrak{g}_1$ *with a one-parameter family of bases:* $X_t = t(e_3 + e_4)$, $Y_t = e_3 - e_4$, $W_t = t(e_1 - e_2)$, $Z_t = e_1 + e_2$.

Proof. For $\mathfrak{g}_{3,1} \oplus \mathfrak{g}_1$, it is enough to observe that $t = 1$ and $t = -1$ correspond to the cases treated in Lemma 9 ($z_2 = 0$) and Lemma 11, respectively. □

We have not yet considered the case of structures of class $(\mathcal{AK}, \mathcal{H}, \mathcal{AK})$. It is obvious that if we passing from the basis $\{X, Y, W, Z\}$ to the basis $\{X, Y, -Z, W\}$, the structure (I, J, K) defined by formulas (3) transforms into $(I, K, -J)$. However, this transformation of the Lie algebra into itself will not always be an automorphism.

Lemma 12. *Let* $\mathfrak{g}$ *be the Lie algebra defined by structure equations (1) and* $\lambda = 0$. *Then the map*

$$\phi : \mathfrak{g} \longrightarrow \mathfrak{g}, \phi(X) = X, \phi(Y) = Y, \phi(Z) = W, \phi(W) = -Z$$

is an automorphism of $\mathfrak{g}$.

Proof. This follows by direct verification. □

Now since all possible cases of the structure $(\mathcal{AK}, \mathcal{AK}, \mathcal{H})$ arise when $\lambda = 0$ the corresponding structures of class $(\mathcal{AK}, \mathcal{H}, \mathcal{AK})$ are obtained via the automorphism ϕ from Theorem 2.

5. Hyper-Hermitian structures

In this section, we investigate hyper-Hermitian structures. The study of hyper-Hermitian manifolds is well-developed, particularly in dimension 4. Notably, in 1988, C.P. Boyer [5] classified compact hyper-Hermitian four-manifolds up to conformal equivalence. A complete classification of left-invariant hypercomplex structures on compact Lie groups was achieved by D. Joyce [9] in 1992, who showed that in dimension 4 only the groups T^4, $U(1) \times SU(2)$, and $U(1) \times SO(3)$ admit such structures. Subsequently, M.L. Barberis [4] in 1997 classified all invariant hypercomplex structures on four-dimensional real Lie groups G.

While our classification pertains to a more restricted class of structures compared to that of M.L. Barberis, we include an analysis of hyper-Hermitian structures in our paper for completeness.

Lemma 3 imposes restrictions on the parameters for this class:

$$\begin{cases} \lambda - b + \alpha - \theta_2 = 0 \\ a + \beta + \theta_1 = 0 \\ z_1 + z_4 = 0 \\ z_1 + w_2 = 0 \\ r + z_2 - z_3 = 0 \\ r - z_2 + w_1 = 0 \end{cases}$$

5.1. **Case $\lambda = 0$.** The system (2) for this class is as follows:

$$\begin{cases} -2\alpha\theta_1 + 3rz_1 = 0 \\ -2a\theta_2 - 3rz_1 = 0 \\ 2z_1^2 - 2a\theta_1 + z_3(r + 2z_2) = 0 \\ -2z_1^2 - 2\alpha\theta_2 + w_1(r - 2z_2) = 0 \\ z_1\theta_1 - z_2 b + z_3\alpha = 0 \\ -z_1\theta_1 - z_2\alpha + w_1 b = 0 \\ z_1\theta_2 + z_2 a + z_3\beta = 0 \\ -z_1\theta_2 - z_2\beta - w_1 a = 0 \\ z_1(\alpha + b) - z_3\beta + w_1 a = 0 \\ z_1(a - \beta) - z_3\alpha - w_1 b = 0 \\ -z_1(a - \beta) + z_3(\alpha + b) = 0 \\ z_1(\alpha + b) + w_1(a - \beta) = 0 \end{cases} \tag{21}$$

Adding the 3rd and 4th, the 5th and 6th, and the 7th and 8th lines, respectively, we obtain the following three relations:

$$\begin{cases} -2a\theta_1 - 2\alpha\theta_2 + 6rz_2 = 0 \\ r(\alpha - b) = 0 \\ r(a + \beta) = 0 \end{cases} \tag{22}$$

Let $r = 0$. Then $w_1 = z_3 = z_2$ and the first four equations of system (21) become

$$\begin{cases} \alpha\theta_1 = 0 \\ a\theta_2 = 0 \\ z_1^2 - a\theta_1 + z_2^2 = 0 \\ z_1^2 + \alpha\theta_2 + z_2^2 = 0 \end{cases} \tag{23}$$

Hence $z_1 = z_2 = 0$ and

$$\begin{cases} \alpha\theta_i = 0 \\ a\theta_i = 0, \end{cases}$$

There are two distinct cases.

- $a^2 + \alpha^2 \neq 0$, which implies $\theta_1 = \theta_2 = 0$. This yields a two-parameter family of Lie algebras defined by the bracket relations:

$$\begin{aligned} [Z, X] &= \alpha X - aY \\ [Z, Y] &= aX + \alpha Y \\ [W, X] &= aX + \alpha Y \\ [W, Y] &= -\alpha X + aY \end{aligned} \tag{24}$$

- $a^2 + \alpha^2 = 0$, which implies $\theta_1 = -\beta$ and $\theta_2 = -b$. This gives another two-parameter family of Lie algebras, with the following structure:

$$\begin{aligned} [Z, X] &= \beta Y \\ [Z, Y] &= -\beta X \\ [W, X] &= bY \\ [W, Y] &= -bX \\ [Y, X] &= -\beta Z - bW \end{aligned} \tag{25}$$

If $r \neq 0$ then we obtain $b = \alpha$ and $\beta = -a$. This implies $\theta_1 = \theta_2 = 0$ and $z_2 = z_3 = 0$ which contradicts the assumption $r \neq 0$.

5.2. **Case $\lambda \neq 0$.** The system (2) takes the following form:

$$\begin{cases} 2\alpha\beta + 3rz_1 = 0 \\ 2z_1^2 + z_3(r + 2z_2) = 0 \\ -z_3(\lambda - \alpha) - z_1\beta = 0 \\ z_1(\lambda - \alpha) - z_3\beta = 0 \\ -\lambda\beta - 3rz_1 = 0 \\ -(\lambda + \alpha)(\lambda + 2\alpha) - 2z_1^2 + w_1(r - 2z_2) = 0 \\ -\alpha z_3 - z_1\beta = 0 \\ z_1\alpha - w_1\beta = 0 \\ z_1(\lambda - \alpha) - z_2\beta = 0 \\ z_1\alpha - z_3\beta = 0 \\ z_1\beta + z_3\alpha = 0 \\ z_2(\lambda - \alpha) + z_1\beta = 0 \end{cases} \tag{26}$$

If $z_1^2 + z_3^2 \neq 0$, then the 10th and 11th equations imply $\alpha = \beta = 0$. Substituting this into the 3rd, 4th, and 12th equations yields $z_1 = z_2 = z_3 = 0$, which contradicts the assumption. Therefore, we must have $z_1^2 + z_3^2 = 0$. Consequently $r = -z_2$ and $w_1 = 2z_2$. The 5th equation gives $\beta = 0$, and the system simplifies to::

$$\begin{cases} (\lambda + \alpha)(\lambda + 2\alpha) + 6z_2^2 = 0 \\ z_2(\lambda - \alpha) = 0 \end{cases}$$

If $z_2 \neq 0$ then the equation $z_2(\lambda - \alpha) = 0$ implies $\alpha = \lambda$. Substituting into $(\lambda + \alpha)(\lambda + 2\alpha) + 6z_2^2 = 0$ yields $6\alpha^2 + 6z_2^2 = 0$, which forces $\alpha = \lambda = z_2 = 0$ – a contradiction. Therefore, we must have $z_2 = 0$, and the system produces two families:

- $\lambda = -\alpha$. This gives a one-parameter family of Lie algebras:

$$\begin{aligned} [W, Z] &= \lambda W \\ [Z, X] &= -\lambda X \\ [Z, Y] &= -\lambda Y \end{aligned} \tag{27}$$

- $\lambda = -2\alpha$. This implies $\theta_2 = -\alpha$ and $\beta = w_1 = w_2 = 0$, leading to another one-parameter family of Lie algebras:

$$\begin{aligned} [W, Z] &= -2\alpha W \\ [Z, X] &= \alpha X \\ [Z, Y] &= \alpha Y \\ [Y, X] &= -\alpha W \end{aligned} \tag{28}$$

Lemma 13. *Any hyper-Hermitian Lie algebra $(\mathfrak{g}, \mu g, I, J, K)$, defined by (24) is isomorphic to $(\mathfrak{g}_{4,10}, \mu g_0, I_0, J_0, K_0)$, where $I_0e_1 = -e_2, I_0e_3 = -e_4; J_0e_1 = -e_4, J_0e_2 = -e_3$ and the basis e_1, e_2, e_3, e_4 is g_0-orthonormal.*

Proof. The isomorphism from the family (24) to $\mathfrak{g}_{4,10}$, is given by formulas:

$$X = \frac{-ae_1 + \alpha e_2}{\sqrt{a^2+\alpha^2}}$$
$$Y = \frac{\alpha e_1 + ae_2}{\sqrt{a^2+\alpha^2}}$$
$$Z = \frac{ae_4 - \alpha e_3}{\sqrt{a^2+\alpha^2}}$$
$$W = \frac{-\alpha e_4 - ae_3}{\sqrt{a^2+\alpha^2}}$$

This transformation is orthogonal, and regardless of the parameters a and α: $Ie_1 = -e_2, Ie_3 = -e_4; Je_1 = -e_4, Je_2 = -e_3$. □

Lemma 14. *Any hyper-Hermitian Lie algebra* $(\mathfrak{g}, \mu g, I, J, K)$*, defined by (25) is isomorphic to* $(\mathfrak{g}_{3,7} \oplus \mathfrak{g}_1, \mu g_0, I_0, J_0, K_0)$*, where* $I_0e_1 = e_4, I_0e_2 = e_3; J_0e_1 = -e_3, J_0e_2 = e_4$ *and the basis* e_1, e_2, e_3, e_4 *is* g_0*-orthonormal.*

Proof. The isomorphism of family (25) to $\mathfrak{g}_{3,7} \oplus \mathfrak{g}_1$, is given by formulas:

$$X = \beta e_3 + be_2$$
$$Y = be_3 - \beta e_2$$
$$Z = \beta e_1 + be_4$$
$$W = be_1 - \beta e_4$$

□

Lemma 15. *Any hyper-Hermitian Lie algebra* $(\mathfrak{g}, \mu g, I, J, K)$*, defined by (27) is isomorphic to* $(\mathfrak{g}_{4,5}, \mu g_0, I_0, J_0, K_0)$*, where* $I_0e_1 = e_2, I_0e_3 = e_4; J_0e_1 = e_4, J_0e_2 = e_3$ *and the basis* e_1, e_2, e_3, e_4 *is* g_0*-orthonormal.*

Proof. The isomorphism of family (27) to $\mathfrak{g}_{4,5}$, is given by formulas:

$$X = \lambda e_1$$
$$Y = \lambda e_2$$
$$Z = \lambda e_4$$
$$W = \lambda e_3$$

□

Lemma 16. *Any hyper-Hermitian Lie algebra* $(\mathfrak{g}, \mu g, I, J, K)$*, defined by (28) is isomorphic to* $(\mathfrak{g}_{4,8}, \mu g_0, I_0, J_0, K_0)$*, where* $I_0e_1 = -e_4, I_0e_2 = e_3; J_0e_1 = -e_3, J_0e_2 = -e_4$ *and basis* e_1, e_2, e_3, e_4 *is* g_0*-orthonormal.*

Proof. Isomorphism of family (28) to $\mathfrak{g}_{4,8}$ $(\beta = 1)$, is defined by formulas:

$$X = \alpha e_2$$
$$Y = \alpha e_3$$
$$Z = -\alpha e_4$$
$$W = \alpha e_1$$

□

Combining the results of Lemmas 13–16, we obtain the following theorem.

Theorem 3. *Any four-dimensional conformally foliated Lie group (G, g) with minimal leaves and a natural left-invariant hyper-Hermitian structure defined by (3) in the g-orthogonal basis $\{X, Y, W, Z\}$ has Lie algebra isomorphic to one of the following:*

- $\mathfrak{g}_{4,10}$ *with* $X = -e_1$, $Y = e_2$, $W = -e_3$, $Z = e_4$.
- $\mathfrak{g}_{4,8}$ *with* $X = e_2$, $Y = e_3$, $W = e_1$, $Z = -e_4$.
- $\mathfrak{g}_{4,5}$ *with* $X = e_1$, $Y = e_2$, $W = e_3$, $Z = e_4$.
- $\mathfrak{g}_{3,7} \oplus \mathfrak{g}_1$ *with* $X = e_2$, $Y = e_3$, $W = e_1$, $Z = e_4$.

According to the results of [4], the list of Lie algebras obtained in Theorem 3 exhausts all 4-dimensional Lie algebras admitting a hypercomplex structure. In [4], the full set of such hyper-Hermitian structures on these algebras is described. Theorem 3 selects from each of these families the unique structure compatible with the foliation considered in the present work.

6. Appendix

Here we list all 4-dimensional Lie algebras which isomorphic to Lie algebras from the paper. We save denotations of G.M. Mubarakzyanov [10].

6.1. $\mathfrak{g}_{3,1} \oplus \mathfrak{g}_1$. This nilpotent Lie algebra is defined by the structure equation:

$$[e_2, e_3] = e_1 \tag{29}$$

It is isomorphic to the direct sum $\mathfrak{h}_3(\mathbb{R}) \oplus \mathbb{R}$, where $\mathfrak{h}_3(\mathbb{R})$ is Lie algebra of the 3-dimensional Heisenberg group

$$H_3(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & b & d \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid b, c, d \in \mathbb{R} \right\}$$

A basis for this Lie algebra is given by the matrices:

$$e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} e_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix};$$

With this basis, the (29) is satisfied, while all other brackets among the basis elements are zero. The vector e_4 is basis for abelian Lie algebra $\mathfrak{g}_1 \cong \mathbb{R}$.

6.2. $\mathfrak{g}_{3,7} \oplus \mathfrak{g}_1$. This is 4-dimensional unsolvable Lie algebra, defined by the following structure equations:

$$[e_1, e_2] = e_3, [e_2, e_3] = e_1, [e_3, e_1] = e_2 \tag{30}$$

It is isomorphic to Lie algebra of Lie group $SU(2) \times U(1)$.

6.3. $\mathfrak{g}_{4,5}$. This is 4-dimensional solvable non-nilpotent Lie algebra, defined by the following structure equations:

$$[e_1, e_4] = pe_1, [e_2, e_4] = de_2, [e_3, e_4] = qe_3 \tag{31}$$

This Lie algebra has 3-dimensional abelian $\mathfrak{g}'$.

For example in case $p = q = -1$, $d = 1$ this is Lie algebra of Lie group $G = \mathbb{R}^* \ltimes \mathbb{R}^3$, where $\mathbb{R}^*$ is multiplicative group of real numbers, $\mathbb{R}^3$ is additive group. Group operation on G is

$$(a, b, c, d) * (a', b', c', d') = (aa', ab' + b, a^{-1}c' + c, ad' + d)$$

6.4. $\mathfrak{g}_{4,8}$. This is 4-dimensional solvable, non-nilpotent Lie algebra, defined by the structure equations:

$$[e_2, e_3] = e_1, [e_1, e_4] = (1 + \beta)e_1, [e_2, e_4] = e_2, [e_3, e_4] = \beta e_3, \tag{32}$$

where $-1 \leq \beta \leq 1$.

For instance, when $\beta = -\frac{1}{2}$ this algebra is isomorphic to the Lie algebra of the Lie group $G = H_3(\mathbb{R}) \rtimes_\alpha \mathbb{R}$, where $\alpha : \mathbb{R} \longrightarrow \mathrm{Aut}(H_3)$ is given by

$$\alpha(s)(b, c, d) = (e^{-s}b, e^{s/2}c, e^{-s/2}d)$$

i.e. the group multiplication is defined by

$$((b, c, d), s) \cdot ((b', c', d'), s') = (b + e^{-s}b', c + e^{s/2}c', d + e^{-s/2}d' + e^{s/2}bc', s + s')$$

6.5. $\mathfrak{g}_{4,10}$. This is 4-dimensional solvable non-nilpotent Lie algebra, defined by the following structure equations:

$$[e_1, e_3] = e_1, [e_2, e_3] = e_2, [e_1, e_4] = -e_2, [e_3, e_4] = e_1 \tag{33}$$

It is isomorphic to the real Lie algebra $\mathfrak{aff}(\mathbb{C})$ which corresponds to the Lie group $\mathrm{Aff}(\mathbb{C})$ of affine transformations of the complex line. These transformations are of the form $z \longrightarrow az + b$, where $a \in \mathbb{C}^*$, $b \in \mathbb{C}$.

Interpreting $\mathbb{C}$ as $\mathbb{R}^2$ these transformations can be represented in matrix form as:

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} tA & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}, t \in \mathbb{R}^*, A \in O(2), b, x \in \mathbb{R}^2$$

A basis for this Lie algebra is given by the matrices:

$$e_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$e_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

With this basis, the commutator relations (33) are satisfied.

References

[1] D.V. Alekseevsky, S. Marchiafava, M. Pontecorvo, *Compatible Complex Structures on Almost Quaternionic Manifolds*, Trans.Amer.Math.Soc.,**351**:3 (1999), 997–1014.

[2] M.F. Atiyah, N. Hitchin,*The Geometry and Dynamics of Magnetic Monopoles*, Princeton University Press, Princeton, NJ, 1988.

[3] P. Baird and J. Eells, *A conservation law for harmonic maps*, in: *Geometry Symposium, Utrecht 1980* (E. Looijenga, D. Siersma and F. Takens, eds.), Lecture Notes in Math., Springer-Verlag, Berlin, Heidelberg, New York, **894** (1981), 1-25.

[4] M.L. Barberis, *Hypercomplex structures on four-dimensional Lie groups*, Proceeding of the AMS, **125**:4 (1997), 1043-1054.

[5] C.P. Boyer, *A note on hyperhermitian four-manifolds*, Proc. Amer. Math. Soc.**102**:1 (1988), 157-164.

[6] B. Fuglede, *Harmonic morphisms between Riemannian manifolds*, Ann. Inst. Fourier Grenoble, **28**:2 (1978), 107–144.

[7] A. Gray, L.M. Hervella, *The sixteen classes of almost Hermitian manifolds and their linear invariants*, Ann. Mat. Pura Appl., **123** (1980), 35–58.

[8] S. Gudmundsson, M. Svensson, *Harmonic morphisms from four-dimensional Lie groups*, J. Geom. Phys., **83** (2014), 1-11.

[9] D. Joyce, *Compact hypercomplex and quaternionic manifolds*, J. Differential Geom., **35**:3 (1992), 743-761.

[10] G.M. Mubarakzyanov, *On solvable Lie algebras*, Izv. Vys. Ucheb. Zaved. Matematika. **1**:32 (1963), 114–123.

[11] E.A. Svensson, S. Gudmundsson, *Natural almost Hermitian structures on conformally foliated 4-dimensional Lie groups with minimal leaves*, Rend. Circ. Mat. Palermo, **72**:3 (2023), 2265–2286.

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