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CLASSIFICATION OF 4-DIMENSIONAL CONFORMALLY FOLIATED LIE GROUPS WITH MINIMAL LEAVES AND NATURAL ALMOST HYPER-HERMITIAN STRUCTURE

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Abstract: Let (G, g) is 4-dimensional Riemannian Lie group admitting a left-invariant conformal foliation with minimal leaves of codimension 2. It has natural almost hyper-Hermitian structure. Authors classify such Lie groups according to the types of this structure.

Keywords: foliations, almost hyper-Hermitian structures

1 Introduction

Conformal foliation with minimal leaves arise in the theory of harmonic morphisms of Riemannian manifolds onto a surface. Recall that a *harmonic morphism* is a smooth mapping of Riemannian manifolds $\varphi : (M^m, g) \longrightarrow (N^n, h)$, which preserves the Laplace equation, in the sense that $f \circ \varphi$ is a harmonic function $(\Delta^M f \circ \varphi = 0)$ on $\varphi^{-1}(V)$ for every function f which is harmonic in an open set $V \subset N$ (such that $\varphi^{-1}(V)$).

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It is known (see [1]) that

(1) if $m < n$, then harmonic morphisms are constant mappings.
(2) if $m = n = 2$, then a harmonic morphism is a conformal mapping.
(3) if $m = n \neq 2$, then a harmonic morphism is a conformal mapping with constant coefficient of conformality (i.e., a local isometry up to scaling).

In general, for $m \geq n \geq 1$ a mapping f is a harmonic morphism if and only if f is both harmonic and weakly conformal. In particular, according to [2], a submersion $\varphi : M^m \longrightarrow N^2$ is a harmonic morphism if and only if it is horizontally conformal and has minimal leaves. Conversely, for a conformal foliation $\mathcal{F}$ on minimal submanifolds of codimension 2, its local projections $\pi : A \longrightarrow N^2$ are harmonic morphisms. Therefore, manifolds that admit conformal foliation with minimal leaves represent a distinct geometric interest.

Moreover, if M is a 4-dimensional Lie group admitting such a foliation, then a naturally arising left-invariant almost hyper-Hermitian structure is aligned with the foliation structure on it.

In the paper, the authors have obtained a classification of Lie groups for which these structures have additional properties, specifically being almost Kähler, Hermitian or Kähler structures. The main result of the paper is classification theorem.

Theorem 1. *All possible classes of natural almost hyper-Hermitian structures on conformally foliated 4-dimensional non-abelian Lie groups with minimal leaves are listed in the Table 2.*

In section 6 the authors give examples of such Lie groups.

2 About conformal foliation of 4-dimensional Lie group

Here we remind basic construction used for 4-dimensional Riemannain Lie groups (G, g) eqquiped with a 2-dimensional, conformal and left-invariant foliation $\mathcal{F}$. The above groups have been studied in [3, 4] and other works by these authors. We will use the same notation that was introduced by the authors in [3, 4].

Let G be 4-dimensional Lie group with left-invariant Riemannian metric g, and K is its 2-dimensional subgroup. Let $\mathfrak{k}$ and $\mathfrak{g}$ are Lie algebras of the groups K and G respectively. Let $\mathfrak{m}$ be 2-dimensional orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Riemannian metric g on G. Then $\mathfrak{k}$ generates left-invariant integrable distribution $\mathcal{V}$. $\mathcal{H}$ is its orthogonal distribution given by $\mathfrak{m}$. Let's denote by $\mathcal{F}$ the foliation of G tangent to $\mathcal{V}$. Following to denotations of [3, 4], let $\{X, Y, Z, W\}$ is orthonormal basis on $\mathfrak{g}$ such that Z, W generates $\mathfrak{k}$ and X, Y generates $\mathfrak{m}$ as well.

Then the second fundamental form for $\mathcal{V}$ is given by

$$B^{\mathcal{V}}(U, V) = \frac{1}{2}\mathcal{H}(\nabla_U V + \nabla_V U),\ (U, V \in \mathcal{V})$$

and for $\mathcal{H}$

$$B^{\mathcal{H}}(U,V) = \frac{1}{2}\mathcal{V}(\nabla_U V + \nabla_V U),\ (U,V \in \mathcal{H})$$

In terms of second fundamental forms the foliation $\mathcal{F}$ is *conformal* if there exists a vector field W such that $B^{\mathcal{H}} = g \otimes W$. And foliation $\mathcal{F}$ has *minimal leaves* if trace $B^{\mathcal{V}} = 0$.

Without loss of generality, one can assume that $[W,Z] = \lambda W$ and $\mathcal{H}[X,Y] = \rho X$ for some $\lambda, \rho \in \mathbb{R}$. Taking into account the requirement for $\mathcal{F}$ to be conformal with minimal leaves, we can derive the structural equations for the Lie algebras satisfying

$$\begin{aligned}
&[W,Z] = \lambda W,\\
&[Z,X] = \alpha X + \beta Y + z_1 Z + w_1 W,\\
&[Z,Y] = -\beta X + \alpha Y + z_2 Z + w_2 W,\\
&[W,X] = aX + bY + z_3 Z - z_1 W,\\
&[W,Y] = -bX + aY + z_4 Z - z_2 W,\\
&[Y,X] = rX + \theta_1 Z + \theta_2 W,
\end{aligned}$$

where the parameters are related by 14 relationships generated by the Jacobi identity

$$\begin{aligned}
&0 = \lambda a,\\
&0 = \lambda b,\\
&0 = -2z_4z_1 + 2z_3z_2 - 2a\theta_1 + rz_3,\\
&0 = -\lambda z_3 - z_2 b + z_4\beta - z_1 a + z_3\alpha,\\
&0 = -\lambda z_4 - z_2 a + z_4\alpha + z_1 b - z_3\beta,\\
&0 = \lambda\theta_1 - w_1z_4 + w_2z_3 - 2a\theta_2 + rw_1,\\
&0 = -\lambda\theta_2 + 2z_1w_2 - 2z_2w_1 - 2\alpha\theta_2 + rw_1,\\
&0 = -w_2a - w_1b - z_2\alpha - z_1\beta - r\alpha,\\
&0 = -w_2b + w_1a - z_2\beta + z_1\alpha + r\beta,\\
&0 = \lambda z_1 - w_2b - z_2\beta - w_1a - z_1\alpha,\\
&0 = z_2a + z_1b - z_4\alpha - z_3\beta - ra,\\
&0 = z_2b - z_1a - z_4\beta + z_3\alpha + rb,\\
&0 = \lambda z_2 - w_2a - z_2\alpha + w_1b + z_1\beta.
\end{aligned} \tag{Y}$$

In the work [3] authors classified all 4-dimensional Riemannain Lie groups (G,g) eqquiped with a 2-dimensional, conformal and left-invariant foliation $\mathcal{F}$ with minimal leaves. They have got 20 multi-dimensional families of Lie algebras.

3 Natural almost hyper-Hermitian structure on conformally foliated 4-dimensional Lie group

The geometry of these Lie groups allows for the construction of a natural left-invariant almost hyper-Hermitian structure. Namely, the almost hyper-Hermitian structure on (G, g), corresponding to the foliation is defined by the following formulas:

$$IX = Y, IZ = -W; \qquad JX = Z, JY = W; \qquad K = IJ$$

It is obvious that all these almost complex structures preserve the metric g.

Four classes of almost Hermitian 4-dimensional manifolds (M, g, J, ω) are possible [5]. It can be almost Kähler ($\mathcal{A}K$) if $d\omega = 0$, Hermitian ($\mathcal{I}$) if J is integrable. If $d\omega = 0$ and Nijenhuis tensor of J is equal to zero then it is Kähler manifolds $\mathcal{K}$. The fourth class is not of interest as it includes any almost Hermitian 4-manifold without additional properties.

Lemma 1. [6] *If any two of the almost complex structures I, J, K are integrable, then the third one is integrable as well.*

Lemma 2. [7] *If any two of the almost Hermitian structures (g, I, ω_I), (g, J, ω_J), (g, K, ω_K) are almost Kähler, then the third structure is Hermitian.*

We will say that an almost hyper-Hermitian structure (g, I, J, K) belongs to the class $(*_I, *_J, *_K)$ if $(G, g, T) \in *_T$, where $T = I, J$ or K, and $*$ represents one of the classes $\mathcal{AK}$, $\mathcal{I}$, $\mathcal{K}$. For example, the belonging of the structure to the class $(\mathcal{AK}_I, \mathcal{AK}_J, \mathcal{I}_K)$ indicates that $d\omega_I = d\omega_J = 0$, and the almost complex structure K is integrable.

Because of Lemmas 1 and 2, it makes sense to single out from all the 4-dimensional Lie groups described above, those for which

1) $(g, I, J, K) \in (\mathcal{AK}_I, \mathcal{AK}_J, \mathcal{I}_K)$, $(\mathcal{AK}_I, \mathcal{I}_J, \mathcal{AK}_K)$ or $(\mathcal{I}_I, \mathcal{AK}_J, \mathcal{AK}_K)$,
2) $(g, I, J, K) \in (\mathcal{I}_I, \mathcal{I}_J, \mathcal{I}_K)$,
3) $(g, I, J, K) \in (\mathcal{K}_I, \mathcal{I}_J, \mathcal{I}_K)$, $(\mathcal{I}_I, \mathcal{K}_J, \mathcal{I}_K)$ or $(\mathcal{I}_I, \mathcal{I}_J, \mathcal{K}_K)$,
4) $(g, I, J, K) \in (\mathcal{K}_I, \mathcal{K}_J, \mathcal{I}_K)$.

Lemma 3. *The necessary and sufficient conditions for almost Kähler, Hermitian, Kähler structures (g, T) on G are given in the following table*

Proof. The structure (g, T) is almost Kähler iff the 2-form $\omega_T(\cdot, \cdot) = g(T\cdot, \cdot)$ is closed.

$$\begin{aligned} d\omega_I(X, Y, Z) &= -g(I[X, Y], Z) - g(I[Y, Z], X) - g(I[Z, X], Y) \\ &= -g(I(-rX - \theta_1 Z - \theta_2 W), Z) - g(I(\beta X - \alpha Y - z_2 Z - w_2 W), X) \\ &\quad - g(I(\alpha X + \beta Y + z_1 Z + w_1 W), Y) = \theta_2 - 2\alpha \end{aligned}$$

Similarly $d\omega_I(Y, Z, W) = -z_2 + z_2 = 0$, $d\omega_I(Z, W, X) = z_1 - z_1 = 0$, $d\omega_I(W, X, Y) = -\theta_1 - 2a$. Thus, we get conditions $(\mathcal{AK}_I)$.

Likewise, for $d\omega_J$: $d\omega_J(X, Y, Z) = r - z_2 + w_1$, $d\omega_J(Y, Z, W) = \alpha - \lambda + b$, $d\omega_J(Z, W, X) = -a + \beta$, $d\omega_J(W, X, Y) = -z_1 + z_4$. So we get conditions $(\mathcal{AK}_J)$. For $d\omega_K$: $d\omega_K(X, Y, Z) = w_2 + z_1$, $d\omega_K(Y, Z, W) = \beta - a$,

	I	J	K
$\mathcal{AK}$	$\theta_1 + 2a = 0$ $\theta_2 - 2\alpha = 0$	$r - z_2 + w_1 = 0$ $\lambda - b - \alpha = 0$ $a - \beta = 0$ $z_1 + z_4 = 0$	$r + z_2 - z_3 = 0$ $\lambda - b - \alpha = 0$ $a - \beta = 0$ $z_1 + w_2 = 0$
$\mathcal{I}$	$2z_1 + z_4 + w_2 = 0$ $2z_2 - z_3 - w_1 = 0$	$r + z_2 - z_3 = 0$ $\lambda - b + \alpha - \theta_2 = 0$ $a + \beta + \theta_1 = 0$ $z_1 + w_2 = 0$	$r - z_2 + w_1 = 0$ $\lambda - b + \alpha - \theta_2 = 0$ $a + \beta + \theta_1 = 0$ $z_1 + z_4 = 0$
$\mathcal{K}$	$\theta_1 + 2a = 0$ $\theta_2 - 2\alpha = 0$ $2z_1 + z_4 + w_2 = 0$ $2z_2 - z_3 - w_1 = 0$	$r + z_2 - z_3 = 0$ $r - z_2 + w_1 = 0$ $a - \beta = 0$ $\lambda - b - \alpha = 0$	$\theta_2 - 2\alpha = 0$ $\theta_1 + 2a = 0$ $z_1 + z_4 = 0$ $z_1 + w_2 = 0$

Таблица 1. The conditions for the almost Hermitian structures (g, I), (g, J), (g, K) to belong to the classes $\mathcal{AK}$, $\mathcal{I}$, $\mathcal{K}$

$d\omega_K(Z, W, X) = \lambda - b - \alpha$, $d\omega_K(W, X, Y) = z_3 - r - z_2$. By $d\omega_K = 0$ we get $(\mathcal{AK}_K)$.

According to the Newlander-Nirenberg theorem, for the integrability of an almost complex structure, it is necessary and sufficient for its Nijenhuis tensor to be zero.

Let's calculate the Nijenhuis tensor $N_I(U, V)$ for the almost complex structure I for each pair of basis vectors $U, V \in \mathfrak{g}$. It is easy to verify that two out of the six possible equalities hold identically.

Actually,

$$\begin{aligned} N_I(X, Y) &= [X, Y] + I[IX, Y] + I[X, IY] - [IX, IY] \\ &= [X, Y] - [Y, -X] + I([Y, Y] + [X, -X]) = 0 \end{aligned}$$

Similarly $N(Z, W) = 0$. Other couples of vectors give equalities:

$$\begin{aligned} N_I(X, W) &= N_I(X, -IZ) = -\big([X, IZ] + I[IX, IZ] + I[X, I^2Z] - [IX, I^2Z]\big) \\ &= IN_I(X, Z), \\ N_I(Y, Z) &= N_I(X, Z) = -IN_I(X, Z), N_I(Y, W) = N_I(IX, -IZ) = N_I(X, Z) \\ N_I(X, Z) &= -\alpha X - \beta Y - z_1 Z - w_1 W + bX - aY - z_4 Z + z_2 W \\ &\quad + I(\beta X - \alpha Y - z_2 Z - w_2 W + aX + bY + z_3 Z - z_1 W) \\ &= (-\alpha + b + \alpha - b)X + (-\beta - a + \beta + a)Y \\ &\quad + (-z_1 - z_4 - w_2 - z_1)Z + (-w_1 + z_2 + z_2 - z_3)W \\ &= (-2z_1 - z_4 - w_2)Z + (2z_2 - z_3 - w_1)W \end{aligned}$$

We obtain the necessary and sufficient conditions $(\mathcal{I}_I)$ for the integrability of the almost complex structure I.

Likewise for J: $N_J(X,Z) = N_J(Y,W) = 0$ and $N_J(X,W) = N_J(X,JY) = -JN(X,Y)$, $N_J(Y,Z) = N_J(Y,JX) = JN(X,Y)$, $N_J(Z,W) = N(JX,JY) = -N(X,Y)$.

$$N_J(X,Y) = (-r-z_2+z_3)X+(-w_2-z_1)Y+(\theta_1-\beta-a)Z+(-\theta_2+\alpha-b+\lambda)W$$

we receive the conditions $(\mathcal{I}_J)$.

In case of almost complex structure K we have two trivial equalities $N_K(X,W) = N_K(Y,Z) = 0$ and $N_K(X,Z) = N_K(X,KY) = -KN_K(X,Y)$, $N_K(Y,-KX) = KN_K(X,Y)$, $N_K(Z,W) = N_K(KY,-KX) = -N_K(X,Y)$.

$$N_K(X,Y) = (-r+z_2-w_1)X+(z_1+z_4)Y+(-\theta_1-a-\beta)Z+(-\theta_2+\lambda-b+\alpha)W$$

So we get conditions $(\mathcal{I}_K)$. We obtain the conditions for Kähler structures by combining the criteria for belonging to the classes $\mathcal{I}$ and $\mathcal{AK}$. □

4 The case of two almost Kähler and one Hermitian structure.

4.1. Structures of class $(\mathcal{AK}_I, \mathcal{AK}_J, \mathcal{I}_K)$.. Lemma 3 gives restrictions on parameters for this class

$$\begin{cases} \theta_1 + 2a = 0 \\ \theta_2 - 2\alpha = 0 \\ r = z_2 - w_1 \\ \lambda = \alpha + b \\ \beta = a \\ z_4 = -z_1 \end{cases}$$

4.1.1. Case $\lambda = 0$. The system (Y) for this class is given by

$$\begin{cases} -w_2z_3 + 4a\alpha + z_1(r - w_1) = 0 \\ 2z_1^2 + 4a^2 + z_3(r + 2z_2) = 0 \\ -2az_1 + \alpha(z_2 + z_3) = 0 \\ 2\alpha z_1 + a(z_2 + z_3) = 0 \\ 2z_1w_2 - 4\alpha^2 + w_1(r - 2z_2) = 0 \\ a(z_1 + w_2) + 2\alpha r = 0 \\ \alpha(z_1 + w_2) = 0 \\ a(z_2 + w_1) + \alpha(z_1 - w_2) = 0 \quad (1.8) \\ a(r - z_2 + z_3) = 0 \\ \alpha(r + z_2 - z_3) = 0 \\ a(z_1 - w_2) - \alpha(z_2 + w_1) = 0 \quad (1.11) \end{cases} \tag{1}$$

Let consider the subsystem (1.8), (1.11). If $a^2+\alpha^2\neq 0$ then $w_2=z_1$ and $w_1=-z_2$. System (1) is equivalent to

$$\begin{cases} z_1(3z_2-z_3)+4a\alpha=0 \\ z_1^2+2a^2+2z_2z_3=0 \\ -2az_1+\alpha(z_2+z_3)=0 \\ a(z_2+z_3)=0 \\ z_1^2-2\alpha^2=0 \quad (2.5) \\ az_1+2\alpha z_2=0 \\ \alpha z_1=0 \quad (2.7) \\ \alpha(3z_2-z_3)=0 \end{cases} \tag{2}$$

Equations (2.5), (2.7) gives us

$$\begin{cases} \alpha=z_1=0 \\ z_3=-z_2=\pm a\neq 0 \end{cases}$$

In this case we get two 1-dimensional subfamilies of nilpotent Lie algebras $\mathfrak{a}_5(z_2)\subset\mathfrak{g}_5(\alpha,a,\beta,b,r)$, $\alpha=b=0$, $\beta=-a=\pm z_2$, $r=2z_2$ (using the notation specified in [3])

$$\begin{aligned} [Z,X]&=\pm z_2Y-z_2W \\ [Z,Y]&=\mp z_2X+z_2Z \\ [W,X]&=\pm z_2X-z_2Z \\ [W,Y]&=\pm z_2Y-z_2W \\ [Y,X]&=2z_2X\mp 2z_2Z \end{aligned} \tag{A5}$$

If $a^2+\alpha^2=0$ then

$$\begin{cases} -w_2z_3+z_1(z_2-2w_1)=0 \\ 2z_1^2+z_3(3z_2-w_1)=0 \\ 2z_1w_2-w_1(z_2+w_1)=0 \\ r=z_2-w_1 \\ z_4=-z_1 \end{cases} \tag{3}$$

If $w_2\neq 0$, then $2z_1=\frac{w_1(z_2+w_1)}{w_2}$, hence the first and second equations define the system

$$\begin{pmatrix} z_2-2w_1 & -w_2 \\ \frac{w_1(z_2+w_1)}{w_2} & w_2(3z_2-w_1) \end{pmatrix}\begin{pmatrix} z_1 \\ z_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with determinant equals to $3(z_2-w_1)^2$. In case of $z_2\neq w_1$ parameters $z_1=z_3=0$. We get a 2-dimensional subfamily of nilpotent Lie algebras $\mathfrak{a}_7(z_2,w_2)\subset\mathfrak{g}_7(z_2,w_1,w_2,\theta_1,\theta_2)$, $\theta_1=\theta_2=0$, $w_1=-z_2$ satisfying

$$[Z,X]=-z_2W,\quad [Z,Y]=z_2Z+w_2W,\quad [W,Y]=-z_2W,\quad [Y,X]=2z_2X \tag{A7}$$

and 2-dimensional subfamily of nilpotent Lie algebras $\mathfrak{a}_8(z_2,w_2)\subset\mathfrak{g}_8(z_2,z_4,w_2,r,\theta_1,\theta_2)$, $z_4=\theta_1=\theta_2=0$, $r=z_2$ is given by

$$[Z,Y]=z_2Z+w_2W,\quad [W,Y]=-z_2W,\quad [Y,X]=z_2X \tag{A8}$$

In case $w_1 = z_2$, system (3) is equal to

$$\begin{cases} z_1z_2 + w_2z_3 = 0 \\ z_1^2 + z_2z_3 = 0 \\ z_1w_2 - z_2^2 = 0 \end{cases} \iff \left[\begin{array}{l} z_1 = z_2 = z_3 = 0 \\ \begin{cases} z_1z_2z_3 \neq 0 \\ w_2 = \frac{z_2^2}{z_1} \\ z_3 = -\frac{z_1^2}{z_2}. \end{cases} \end{array} \right.$$

The first line gives 1-dimensional subfamily of nilpotent Lie algebras $\mathfrak{a}_{14}(w_2) \subset \mathfrak{g}_{14}(z_2, z_4, w_2, \theta_1, \theta_2)$, $z_2 = z_4 = \theta_1 = \theta_2 = 0$ with

$$[Z, Y] = w_2W \tag{A14}$$

The second condition gives subfamily of nilpotent Lie algebras $\mathfrak{a}_{11}(z_1, z_2) \subset \mathfrak{g}_{11}(z_1, z_2, z_3, w_1, \theta_1, \theta_2)$, $\theta_1 = \theta_2 = 0$, $w_1 = z_2$, $z_3 = -\frac{z_1^2}{z_2}$ written as

$$\begin{aligned} [Z, X] &= z_1Z + z_2W \\ [Z, Y] &= z_2Z + \frac{z_2^2}{z_1}W \\ [W, X] &= -\frac{z_1^2}{z_2}Z - z_1W \\ [W, Y] &= -z_1Z - z_2W \end{aligned} \tag{A11}$$

If $w_2 = 0$ then the system (3) can be represented as

$$\begin{cases} z_1(z_2 - 2w_1) = 0 \\ 2z_1^2 + z_3(3z_2 - w_1) = 0 \\ w_1(z_2 + w_1) = 0 \quad r = z_2 - w_1 \\ z_4 = -z_1 \end{cases}$$

It has following non-zero solutions:

- $z_1 = z_3 = w_1 = 0$, $r = z_2$ corresponds to structure equation (A8) with $w_2 = 0$;
- $z_1 = z_3 = 0$, $w_1 = -z_2, r = 2z_2$ corresponds to structure equation (A7) with $w_2 = 0$;
- $z_1 = z_2 = w_1 = 0$ corresponds to 1-dimensional subfamily of nilpotent Lie algebras $\mathfrak{a}_{13}(z_3) \subset \mathfrak{g}_{13}(z_3, z_4, \theta_1, \theta_2)$, $z_4 = \theta_1 = \theta_2 = 0$. In such a way we obtain

$$[W, X] = z_3Z \tag{A13}$$

4.1.2. Case $\lambda \neq 0$. System (Y) is

$$\begin{cases} -w_2z_3 - w_1z_1 + rz_1 = 0 \\ 2z_1^2 + 2z_2z_3 + rz_3 = 0 \\ -6\alpha^2 + 2z_1w_2 - 2z_2w_1 + rw_1 = 0 \\ \alpha(r + z_2) = 0 \\ z_1\alpha = 0 \\ z_3\alpha = 0 \end{cases}$$

As $\alpha = \lambda \neq 0$, then $z_1 = z_3 = 0$, $r = -z_2$, and $w_1 = 2z_2$ from $(\mathcal{AK}_J)$. The third equation of system $-6\alpha^2 - 6z_2^2 = 0$ contradicts to constraint $\lambda \neq 0$.

Lie algebras from family $\mathfrak{a}_{13}$ are isomorphic to Lie algebras from $\mathfrak{a}_{14}$ when parameter w_2 of family $\mathfrak{a}_{14}$ is equal to parameter z_2 on $\mathfrak{a}_{13}$. There isomorphism keep almost hyper-Hermitian structures. The isomorphism f is defined by formulas

$$f(X) = Y, \quad f(Y) = -X, \quad f(Z) = W, \quad f(W) = -Z$$

Obviously, that $f \circ I = I \circ f$, $f \circ J = J \circ f$ and $f \circ K = K \circ f$.

4.2. Structures of the class $(\mathcal{AK}_I, \mathcal{I}_J, \mathcal{AK}_K)$**..** If $(G, g, I, J, K) \in (\mathcal{AK}_I, \mathcal{I}_J, \mathcal{AK}_K)$, then

$$\begin{cases} \theta_1 + 2a = 0, \\ \theta_2 - 2\alpha = 0, \\ \lambda - b - \alpha = 0, \\ a - \beta = 0, \\ z_1 + w_2 = 0, \\ r + z_2 - z_3 = 0 \end{cases}$$

4.2.1. Case $\lambda = 0$**.** System (Y) is

$$\begin{cases} z_1z_3 + z_4w_1 + 4a\alpha + rz_1 = 0 & \\ -2z_1z_4 + 2z_2z_3 + 4a^2 + rz_3 = 0 & (4.2) \\ \alpha(z_2 + z_3) + a(z_4 - z_1) = 0 & (4.3) \\ a(z_2 + z_3) - \alpha(z_4 - z_1) = 0 & (4.4) \\ -2z_1^2 - 2z_2w_1 - 4\alpha^2 + rw_1 = 0 & \\ \alpha(r + z_2 - w_1) = 0 & \\ a(r - z_2 + w_1) = 0 & \\ 2z_1\alpha + a(z_2 + w_1) = 0 & (4.8) \\ \alpha(z_1 + z_4) + a(r - z_2 + z_3) = 0 & \\ a(z_1 + z_4) = 0 & \\ 2z_1a - \alpha(z_2 + w_1) = 0 & (4.11) \end{cases} \qquad (4)$$

Let's consider subsystems (4.3), (4.4) and (4.8), (4.11). If $a^2 + \alpha^2 \neq 0$, then $z_1 = z_4 = 0$, $w_1 = z_3 = -z_2$, $r = 2z_3$, and $a = 0$ from (4.2). Therefore, system (4) is equivalent to

$$4z_3^2 - 4\alpha^2 = 0$$

and gives two 1-dimensional subfamilies $\mathfrak{b}_5(z_3) \subset \mathfrak{g}_5(\alpha, a, \beta, b, r)$, $\alpha = \pm z_3$, $b = \mp z_3$, $r = 2z_3$, $a = \beta = 0$ with

$$\begin{aligned}
[Z, X] &= \pm z_3 X + z_3 W \\
[Z, Y] &= \pm z_3 Y - z_3 Z \\
[W, X] &= \mp z_3 Y + z_3 Z \\
[W, Y] &= \pm z_3 X + z_3 W \\
[Y, X] &= 2z_3 X \pm 2z_3 W
\end{aligned} \tag{B5}$$

If $a^2 + \alpha^2 = 0$, then (4) is equivalent to

$$\begin{cases} w_1 z_4 + z_1(r + z_3) = 0 \\ -2z_1^2 + w_1(r - 2z_2) = 0 \\ -2z_1 z_4 + z_3(r + 2z_2) = 0, \end{cases} \tag{5}$$

One can repeat reasoning similar to that for system (3) and obtain the following Lie algebras:

- 2-dimensional subfamily of nilpotent Lie algebras $\mathfrak{b}_8(z_2, z_4) \subset \mathfrak{g}_8(z_2, z_4, w_2, r, \theta_1, \theta_2)$, $\theta_1 = \theta_2 = w_2 = 0$, $r = -z_2$

$$\begin{aligned}
[Z, Y] &= z_2 Z \\
[W, Y] &= z_4 Z - z_2 W \\
[Y, X] &= -z_2 X
\end{aligned} \tag{B8}$$

- 2-dimensional subfamily of nilpotent Lie algebras $\mathfrak{b}_9(z_2, z_4) \subset \mathfrak{g}_9(z_2, z_3, z_4, \theta_1, \theta_2)$, $\theta_1 = \theta_2 = 0$, $z_3 = -z_2$

$$\begin{aligned}
[Z, Y] &= z_2 Z \\
[W, X] &= -z_2 Z \\
[W, Y] &= z_4 Z - z_2 W \\
[Y, X] &= -2z_2 X
\end{aligned} \tag{B9}$$

- 1-dimensional subfamily of nilpotent Lie algebras $\mathfrak{b}_{14}(z_4) \subset \mathfrak{g}_{14}(z_2, z_4, w_2, \theta_1, \theta_2)$, $z_2 = w_2 = \theta_1 = \theta_2 = 0$

$$[W, Y] = z_4 Z \tag{B14}$$

- 2-dimensional subfamily of nilpotent Lie algebras $\mathfrak{b}_{11}(z_1, z_2) \subset \mathfrak{g}_{11}(z_1, z_2, z_3.w_1, \theta_1, \theta_2)$, $\theta_1 = \theta_2 = 0$, $z_3 = z_2$, $w_1 = -\frac{z_1^2}{z_2}$

$$\begin{aligned}
[Z, X] &= z_1 Z - \frac{z_1^2}{z_2} W \\
[Z, Y] &= z_2 Z - z_1 W \\
[W, X] &= z_2 Z - z_1 W \\
[W, Y] &= \frac{z_2^2}{z_1} Z - z_2 W
\end{aligned} \tag{B11}$$

- 1-dimensional subfamily of nilpotent Lie algebras $\mathfrak{b}_{12}(w_1) \subset \mathfrak{g}_{12}(z_3, w_1, w_2, \theta_1, \theta_2)$, $w_1 = w_2 = \theta_1 = \theta_2 = 0$

$$[Z, X] = w_1 W \tag{B12}$$

Lie algebras of family $\mathfrak{b}_{12}(w_1)$ are isomorphic to Lie algebras of $\mathfrak{b}_{14}(z_4)$ in case the parameter w_1 in family $\mathfrak{b}_{12}$ is equal to parameter z_4 on $\mathfrak{b}_{14}$. Given isomorphism preserves almost hyper-Hermitian structures. Isomorphism f is defined by formulas

$$f(X) = -Y, \quad f(Y) = X, \quad f(Z) = -W, \quad f(W) = Z,$$

and $f \circ I = I \circ f$, $f \circ J = J \circ f$, $f \circ K = K \circ f$. There is isomorphism between Lie algebras $\mathfrak{g}_1 \in \mathfrak{a}_{14}(w_2)$ and $\mathfrak{g}_2 \in \mathfrak{b}_{12}(w_1)$, $w_1 = w_2$ given by

$$f : (\mathfrak{g}_1, g, I, J, K) \longrightarrow (\mathfrak{g}_2, g, I, -K, J), \quad f(X) = -Y, f(Y) = X, f|_{\mathcal{V}} = id|_{\mathcal{V}}$$

There is isomorphism between Lie algebras $\mathfrak{g}_1 \in \mathfrak{a}_8(z_2, w_2)$ and $\mathfrak{g}_2 \in \mathfrak{b}_8(-z_2, -w_2)$given by

$$f : (\mathfrak{g}_1, g, I, J, K) \longrightarrow (\mathfrak{g}_2, g, I, -K, J), \quad f(Z) = W, f(W) = -Z, f|_{\mathcal{H}} = id|_{\mathcal{H}}$$

There is isomorphism between Lie algebras $\mathfrak{g}_1 \in \mathfrak{a}_7(z_2, w_2)$ and $\mathfrak{g}_2 \in \mathfrak{b}_9(z_2, w_2)$ given by

$$f : (\mathfrak{g}_1, g, I, J, K) \longrightarrow (\mathfrak{g}_2, g, I, -K, J), \quad f(Z) = W, f(W) = -Z, f|_{\mathcal{H}} = -id|_{\mathcal{H}}$$

4.2.2. Case $\lambda \neq 0$. System (Y) can be written in following form

$$\begin{cases} w_1 z_4 + z_1(r + z_3) = 0 \\ -2z_1 z_4 + z_3(r + 2z_2) = 0 \\ -6\alpha^2 - 2z_1^2 + w_1(r - 2z_2) = 0 \\ \alpha(r + z_2) = 0 \\ z_1\alpha = z_3\alpha = z_4\alpha = 0 \end{cases}$$

As $\alpha = \lambda \neq 0$, then $z_1 = z_3 = z_4 = 0$, $r = -z_2$, $z_2 \neq 0$, $w_1 \neq 0$. Therefore (Y) consists in only one condition

$$-6\alpha^2 - 3z_2 w_1 = 0$$

which gives us 2-dimensional subfamily of nilpotent Lie algebras $\mathfrak{b}_4(\lambda, z_2) \subset \mathfrak{g}_4(\lambda, z_2, w_1, w_2)$, $w_2 = 0$, $w_1 = -\frac{2\lambda^2}{z_2}$:

$$\begin{aligned} &[W, Z] = \lambda W \\ &[Z, X] = \lambda X - \frac{2\lambda^2}{z_2} W \\ &[Z, Y] = \lambda Y + z_2 Z \\ &[W, Y] = -z_2 W \\ &[Y, X] = -z_2 X + 2\lambda W \end{aligned} \tag{B4}$$

4.3. Structures of the class $(\mathcal{I}_I, \mathcal{AK}_J, \mathcal{AK}_K)$.. For those structures Lemma 3 gives conditions

$$\begin{cases} \lambda - b - \alpha = 0 \\ a - \beta = 0 \\ z_1 + z_4 = 0 \\ z_1 + w_2 = 0 \\ r - z_2 + w_1 = 0 \\ r + z_2 - z_3 = 0 \end{cases} \tag{6}$$

4.3.1. Case $\lambda = 0$. System (Y) for this class is

$$\begin{cases} -2\alpha\theta_1 + 3z_1 r = 0 \\ 2z_1^2 - 2a\theta_1 + z_3(r + 2z_2) = 0 \\ -2z_1 a + \alpha(z_2 + z_3) = 0 \\ 2a\theta_2 + 3z_1 r = 0 \\ -2z_1^2 - 2\alpha\theta_2 + w_1(r - 2z_2) = 0 \\ \alpha r = 0 \\ a r = 0 \\ 2z_1 a - \alpha(z_2 + w_1) = 0 \end{cases} \tag{7}$$

Let $a^2 + \alpha^2 \neq 0$ then $r = 0$ and $w_1 = z_3 = z_2$. System (7) is equivalent to

$$\begin{cases} \alpha\theta_1 = 0 \\ a\theta_2 = 0 \\ z_1^2 + z_2^2 - a\theta_1 = 0 \\ z_1^2 + z_2^2 + \alpha\theta_2 = 0 \\ z_1 a - \alpha z_2 = 0 \end{cases} \iff z_1 = z_2 = \theta_1 = \theta_2 = 0$$

and gives 2-dimensional subfamily of nilpotent Lie algebras $\mathfrak{c}_{10}(\alpha, a) \subset \mathfrak{g}_{10}(\alpha, a, \beta, b)$, $a = \beta$, $b = -\alpha$:

$$\begin{aligned} [Z, X] &= \alpha X + aY \\ [Z, Y] &= -aX + \alpha Y \\ [W, X] &= aX - \alpha Y \\ [W, Y] &= \alpha X + aY \end{aligned} \tag{C10}$$

If $a^2 + \alpha^2 = 0$, then system (7) with restrictions (6) is equivalent to

$$\begin{cases} r z_1 = 0 \\ 2z_1^2 + z_3(r + 2z_2) = 0 \\ -2z_1^2 + w_1(r - 2z_2) = 0 \\ b = \beta = 0 \\ z_4 = w_2 = -z_1 \\ r = z_2 + w_1 \\ r = z_3 - z_2 \end{cases} \iff r = z_1 = z_2 = z_3 = z_4 = w_1 = w_2 = 0 \tag{8}$$

and gives 2-dimensional subfamily of nilpotent Lie algebras $\mathfrak{c}_{14}(\theta_1, \theta_2) \subset \mathfrak{g}_{14}(z_2, z_4, w_2, \theta_1, \theta_2)$, $z_2 = z_4 = w_2 = 0$:

$$[Y, X] = \theta_1 Z + \theta_2 W \tag{C14}$$

4.3.2. Case $\lambda \neq 0$. System (Y) together with (8):

$$\begin{cases} -2\lambda\theta_1 + 3z_1 r = 0 \\ 2z_1^2 + z_3(r + 2z_2) = 0 \\ \lambda\theta_1 - 3z_1 r = 0 \\ -3\lambda\theta_2 - 2z_1^2 + w_1(r - 2z_2) = 0 \\ z_1\lambda = z_3\lambda = 0 \\ \lambda = b + \alpha \\ a = \beta \\ r = z_2 - w_1 = z_3 - z_2 \end{cases}$$

Therefore $z_1 = z_3 = 0$, $r = -z_2$, $\theta_1 = 0$ and

$$3\lambda\theta_2 + 6z_2^2 = 0$$

So we have 2-dimensional subfamily of nilpotent Lie algebras $\mathfrak{c}_4(\lambda, z_2) \subset \mathfrak{g}_4(\lambda, z_2, w_1, w_2)$, $w_2 = 0$, $w_1 = 2z_2$ given by

$$\begin{aligned} &[W, Z] = \lambda W \\ &[Z, X] = \lambda X + 2z_2 W \\ &[Z, Y] = \lambda Y + z_2 Z \\ &[W, Y] = -z_2 W \\ &[Y, X] = -z_2 X - \frac{2z_2^2}{\lambda} W \end{aligned} \tag{C4}$$

5 Hypercomplex structures

Structures of the class $(\mathcal{I}_I, \mathcal{I}_J, \mathcal{I}_K)$ are called *hypercomplex*. A complete classification of left-invariant hypercomplex structures on compact Lie groups was obtained by D. Joyce [8]. In dimension 4 only the groups T^4, $U(1) \times SU(2)$, $U(1) \times SO(3)$ are compact and hypercomplex. Our groups are not necessary compact.

Lemma 3 gives restrictions on parameters for this class:

$$\begin{cases} \lambda - b + \alpha - \theta_2 = 0 \\ a + \beta + \theta_1 = 0 \\ z_1 + z_4 = 0 \\ z_1 + w_2 = 0 \\ r + z_2 - z_3 = 0 \\ r - z_2 + w_1 = 0 \end{cases}$$

5.1. Case $\lambda = 0$. The system (Y) for this class:

$$\begin{cases} -2\alpha\theta_1 + 3rz_1 = 0 \\ -2a\theta_2 - 3rz_1 = 0 \\ 2z_1^2 - 2a\theta_1 + z_3(r + 2z_2) = 0 \\ -2z_1^2 - 2\alpha\theta_2 + w_1(r - 2z_2) = 0 \\ z_1\theta_1 - z_2 b + z_3\alpha = 0 \\ -z_1\theta_1 - z_2\alpha + w_1 b = 0 \\ z_1\theta_2 + z_2 a + z_3\beta = 0 \\ -z_1\theta_2 - z_2\beta - w_1 a = 0 \\ z_1(\alpha + b) - z_3\beta + w_1 a = 0 \\ z_1(a - \beta) - z_3\alpha - w_1 b = 0 \\ -z_1(a - \beta) + z_3(\alpha + b) = 0 \\ z_1(\alpha + b) + w_1(a - \beta) = 0 \end{cases} \tag{9}$$

Adding lines 3rd and 4th, 5th and 6th, 7th and 8th, we obtain, respectively, three relationships:

$$\begin{cases} -2a\theta_1 - 2\alpha\theta_2 + 6rz_2 = 0 \\ r(\alpha - b) = 0 \\ r(a + \beta) = 0 \end{cases} \tag{10}$$

Let $r = 0$, then $w_1 = z_3 = z_2$ and the part of system (9) is

$$\begin{cases} z_1^2 - a\theta_1 + z_2^2 = 0 \\ z_1^2 + \alpha\theta_2 + z_2^2 = 0 \\ z_1\theta_1 + z_2(\alpha - b) = 0 \\ z_1\theta_2 + z_2(a + \beta) = 0 \\ z_1(a - \beta) - z_2(\alpha + b) = 0 \\ z_1(\alpha + b) + z_2(a - \beta) = 0 \end{cases} \tag{11}$$

If $z_1^2 + z_2^2 \neq 0$, then $\beta = a$, $b = -\alpha$. As $a\theta_1 - \alpha\theta_2$ equals to $2(z_1^2 + z_2^2) \neq 0$ it provides a pair of equalities $z_1 = z_2 = 0$ which leads to contradiction.

If $z_1^2 + z_2^2 = 0$, then for $i = 1, 2$

$$\begin{cases} \alpha\theta_i = 0 \\ a\theta_i = 0, \end{cases}$$

This system gives two cases.

- $a^2 + \alpha^2 \neq 0 \implies \theta_1 = \theta_2 = 0$. We get 2-dimensional subfamily of nilpotent Lie algebras $\mathfrak{d}_{10}(\alpha, a) \subset \mathfrak{g}_{10}(\alpha, a, \beta, b)$ with

$$\begin{aligned} [Z, X] &= \alpha X - aY \\ [Z, Y] &= aX + \alpha Y \\ [W, X] &= aX + \alpha Y \\ [W, Y] &= -\alpha X + aY \end{aligned} \tag{D10}$$

- $a^2 + \alpha^2 = 0 \implies \theta_1 = -\beta$ and $\theta_2 = -b$. We get 2-dimensional subfamily of nilpotent Lie algebras $\mathfrak{d}_{18}(\beta, b) \subset \mathfrak{g}_{18}(\beta, b, z_3, z_4, \theta_1, \theta_2)$

given by

$$\begin{aligned}
&[Z, X] = \beta Y \\
&[Z, Y] = -\beta X \\
&[W, X] = bY \\
&[W, Y] = -bX \\
&[Y, X] = -\beta Z - bW
\end{aligned} \tag{D18}$$

If $r \neq 0$ then $b = \alpha$ and $\beta = -a$. This implies $\theta_1 = \theta_2 = 0$ and $z_2 = z_3 = 0$ which contradicts the condition $r \neq 0$.

5.2. Case $\lambda \neq 0$. Let $\lambda \neq 0$, then system (Y) is:

$$\begin{cases}
2\alpha\beta + 3rz_1 = 0 \\
2z_1^2 + z_3(r + 2z_2) = 0 \\
-z_3(\lambda - \alpha) - z_1\beta = 0 \\
z_1(\lambda - \alpha) - z_3\beta = 0 \\
-\lambda\beta - 3rz_1 = 0 \quad (12.5) \\
-(\lambda + \alpha)(\lambda + 2\alpha) - 2z_1^2 + w_1(r - 2z_2) = 0 \quad (12.6) \\
-\alpha z_3 - z_1\beta = 0 \\
z_1\alpha - w_1\beta = 0 \\
z_1(\lambda - \alpha) - z_2\beta = 0 \\
z_1\alpha - z_3\beta = 0 \quad (12.10) \\
z_1\beta + z_3\alpha = 0 \quad (12.11) \\
z_2(\lambda - \alpha) + z_1\beta = 0
\end{cases} \tag{12}$$

If $z_1^2 + z_3^2 \neq 0$, then from subsystem (12.10), (12.11) we have $\alpha = \beta = 0$. This implies $z_1 = z_2 = z_3 = 0$. Those conditions together with (12.6) contradict to $\lambda \neq 0$. Therefore $z_1^2 + z_3^2 = 0$ and $r = -z_2$, $w_1 = 2z_2$. Equation (12.5) implies $\beta = 0$, and system (Y) is simplified to:

$$\begin{cases}
(\lambda + \alpha)(\lambda + 2\alpha) + 6z_2^2 = 0 \\
z_2(\lambda - \alpha) = 0
\end{cases}$$

If $z_2 \neq 0$ then $\alpha = \lambda$ and $6\alpha^2 + 6z_2^2 = 0$, and we come to a contradiction with $\alpha = \lambda = 0$. If $z_2 = 0$ then we get two families:

- $\lambda = -\alpha$ gives 1-dimensional subfamily of nilpotent Lie algebras $\mathfrak{d}_2(\lambda) \subset \mathfrak{g}_2(\lambda, \alpha, \beta, w_1, w_2)$:

$$\begin{aligned}
&[W, Z] = \lambda W \\
&[Z, X] = -\lambda X \\
&[Z, Y] = -\lambda Y
\end{aligned} \tag{D2}$$

- $\lambda = -2\alpha$ gives 1-dimensional subfamily of nilpotent Lie algebras $\mathfrak{d}_3(\alpha) \subset \mathfrak{g}_3(\alpha, \beta, w_1, w_2, \theta_2)$, $\theta_2 = -\alpha$, $\beta = w_1 = w_2 = 0$:

$$[W, Z] = -2\alpha W$$
$$[Z, X] = \alpha X$$
$$[Z, Y] = \alpha Y$$
$$[Y, X] = -\alpha W$$

6 Examples

1. $G = \mathbb{R} \times H_3(\mathbb{R})$, where $\mathbb{R}$ is additive group of real numbers and

$$H_3(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & b & d \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid b, c, d \in \mathbb{R} \right\}$$

is Heisenberg group. Operation $*$ on this group is

$$(a, b, c, d) * (a', b', c', d') = (a + a', b + b', c + c', d + d' + bc')$$

Let

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}; \quad R = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is basis of Lie algebra $\mathfrak{h}_3(\mathbb{R})$, with structure equations $[P, Q] = R$, $[P, R] = [Q, R] = 0$.

If foliation $\mathcal{F}$ corresponds to some commutative subalgebra $\mathfrak{k} \subset \mathfrak{h}_3(\mathbb{R})$, $\mathfrak{k} = \langle P, R \rangle$ or $\mathfrak{k} = \langle Q, R \rangle$ then natural almost hyper-Hermitian structure is $(\mathcal{AK}_I, \mathcal{AK}_J, \mathcal{I}_K)$ or $(\mathcal{AK}_I, \mathcal{I}_J, \mathcal{AK}_K)$. If foliation $\mathcal{F}$ corresponds to commutative subalgebra $\mathfrak{k} = \langle U, R \rangle \subset \mathfrak{g}$, where U is left invariant vector field tangent to first multiplier, then the structure is $(\mathcal{I}_I, \mathcal{AK}_J, \mathcal{AK}_K)$.

2. $G = \mathbb{R}^* \ltimes \mathbb{R}^3$, where $\mathbb{R}^*$ is multiplicative group of real numbers, $\mathbb{R}^3$ is additive group. Group operation on G is

$$(a, b, c, d) * (a', b', c', d') = (aa', ab' + b, a^{-1}c' + c, ad' + d)$$

Let $P = (1, 0, 0, 0)$, $Q = (0, 1, 0, 0)$, $R = (0, 0, 1, 0)$, $S = (0, 0, 0, 1)$ is basis of corresponding Lie algebra. Then $[Q, P] = -Q$, $[R, P] = R$, $[S, P] = -S$.

If foliation $\mathcal{F}$ corresponds to commutative subalgebra $\mathfrak{k} \subset \mathbb{R}^3$, $\mathfrak{k} = \langle S, R \rangle$ or $\mathfrak{k} = \langle Q, R \rangle$ then the natural almost hyper-Hermitian structure on this group is $(\mathcal{AK}_I, \mathcal{AK}_J, \mathcal{I}_K)$ or $(\mathcal{AK}_I, \mathcal{I}_J, \mathcal{AK}_K)$ (example for $\mathfrak{a}_8$ and $\mathfrak{b}_8$). If foliation $\mathcal{F}$ corresponds to commutative subalgebra $\mathfrak{k} = \langle P, S \rangle \subset \mathfrak{g}$, then the structure is $(\mathcal{I}_I, \mathcal{AK}_J, \mathcal{AK}_K)$ (example for $\mathfrak{c}_4$).

3. Lie algebra from family $\mathfrak{d}_2$ with $\lambda = 1$ correspond to Lie group $G = \mathbb{R}^* \ltimes \mathbb{R}^3$, with group operation

$$(a, b, c, d) * (a', b', c', d') = (aa', ab' + b, ac' + c, ad' + d)$$

This group is named a *group of dilations*. It consists of all homotheties, translations and their compositions on $\mathbb{R}^3$. In matrix form

$$[a,b,c,d] = \begin{pmatrix} a & 0 & 0 & b \\ 0 & a & 0 & c \\ 0 & 0 & a & d \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Foliation $\mathcal{F}$ is defined by subgroup $K = \left\{ \begin{pmatrix} a & 0 & 0 & b \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$. Natural almost hyper-Hermitian structure, corresponding to this foliation is hypercomplex.

4. $G = (\mathbb{R}^* \ltimes \mathbb{R}) \ltimes \mathbb{R}^2$. Group operation on G is

$$(a,b,c,d) * (a',b',c',d') = (aa', a^2b' + b, a^{-1}c' + c, ad' + d - bc')$$

Let $Y = (1,0,0,0)$, $X = (0,1,0,0)$, $Z = (0,0,1,0)$, $W = (0,0,0,1)$ is basis of corresponding Lie algebra. Then $[Y,X] = 2X$, $[Z,Y] = Z$, $[W,Y] = -W$, $[Z,X] = -W$.

The natural almost hyper-Hermitian structure on this group is $(\mathcal{AK}_I, \mathcal{AK}_J, \mathcal{I}_K)$ or $(\mathcal{AK}_I, \mathcal{I}_J, \mathcal{AK}_K)$ (example for $\mathfrak{a}_7$ and $\mathfrak{b}_9$).

5. Let us construct an example for a Lie algebra from the family $\mathfrak{c}_{10}$. Mapping f:

$$X' = \frac{\alpha X + aY}{\sqrt{\alpha^2 + a^2}}, Y' = \frac{-aX + \alpha Y}{\sqrt{\alpha^2 + a^2}}$$

$$Z' = \frac{\alpha Z + aW}{\sqrt{\alpha^2 + a^2}}, W' = \frac{-aZ + \alpha W}{\sqrt{\alpha^2 + a^2}}$$

is automorphism of Lie algebra with almost hyper-Hermitian structure. Now structural equations in case $\alpha^2 + a^2 = 1$ are:

$$[Z', X'] = X', [Z', Y'] = Y', [W', X'] = -Y', [W', Y'] = X'$$

The corresponding Lie group is $G = \mathbb{R}^* \ltimes E(2)$, where $E(2)$ is the group of affine transformations of $\mathbb{R}^2$ which preserve the Euclidean metric

$$E(2) = \left\{ \begin{pmatrix} W & T \\ 0 & 1 \end{pmatrix} : W \in O(2), T = \begin{pmatrix} x \\ y \end{pmatrix} \right\}$$

Group operation on G is defined by formula:

$$\left[z, \begin{pmatrix} W & T \\ 0 & 1 \end{pmatrix} \right] * \left[z', \begin{pmatrix} W' & T' \\ 0 & 1 \end{pmatrix} \right] = \left[zz', \begin{pmatrix} W & T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} W' & zT' \\ 0 & 1 \end{pmatrix} \right]$$

Basis of corresponding Lie algebra is

$$Z' = [1,0], W' = \left[0, \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right],$$

$$X' = \left[0, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right], Y' = \left[0, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\right]$$

For different values of parameters a, α in this example X', Y' is orthonormal basis of $\mathfrak{m}$ oriented as X, Y and the natural almost hyper-Hermitian structure on this group is $(\mathcal{I}_I, \mathcal{AK}_J, \mathcal{AK}_K)$. If orthonormal basis X', Y' is oriented in the opposite way then natural structure on this group is hypercomplex and this example corresponds to case $\mathfrak{d}_{10}$.

6. If $\beta = 1$, $b = 0$ for Lie algebra from family $\mathfrak{d}_{18}(\beta, b)$, then corresponding Lie group is $U(1) \times SO(3)$ or $U(1) \times SU(2)$. Foliation $\mathcal{F}$ here is fiber bundle $\pi : S^1 \times S^3 \longrightarrow S^2$, $\pi(u, v) = \pi_1(v)$, where $\pi_1 : S^3 \longrightarrow S^2$ is projection of Hopf bundle.

7 Classification

To finish classification we remark, that any class with some Kähler structure is just realized on abelian Lie algebra. All classes of natural almost hyper-Hermitian structures on conformally foliated 4-dimensional non-abelian Lie groups with minimal leaves are represented in the Table 2.

Class	List of families of the Lie algebras
$(\mathcal{AK}_I, \mathcal{AK}_J, \mathcal{I}_K)$	$\mathfrak{a}_5(z_2) \subset \mathfrak{g}_5(\alpha, a, \beta, b, r)$, $\alpha = b = 0$, $\beta = -a = \pm z_2$, $r = 2z_2$ $\mathfrak{a}_7(z_2, w_2) \subset \mathfrak{g}_7(z_2, w_1, w_2, \theta_1, \theta_2)$, $\theta_1 = \theta_2 = 0$, $w_1 = -z_2$, $z_2 \neq 0$ $\mathfrak{a}_8(z_2, w_2) \subset \mathfrak{g}_8(z_2, z_4, w_2, r, \theta_1, \theta_2)$, $z_4 = \theta_1 = \theta_2 = 0$, $r = z_2$, $z_2 \neq 0$ $\mathfrak{a}_{11}(z_1, z_2) \subset \mathfrak{g}_{11}(z_1, z_2, z_3, w_1, \theta_1, \theta_2)$, $\theta_1 = \theta_2 = 0$, $w_1 = z_2$, $z_3 = -\frac{z_1^2}{z_2}$ $\mathfrak{a}_{14}(w_2) \subset \mathfrak{g}_{14}(z_2, z_4, w_2, \theta_1, \theta_2)$, $z_2 = z_4 = \theta_1 = \theta_2 = 0$
$(\mathcal{AK}_I, \mathcal{I}_J, \mathcal{AK}_K)$	$\mathfrak{b}_4(\lambda, z_2) \subset \mathfrak{g}_4(\lambda, z_2, w_1, w_2)$, $w_2 = 0$, $w_1 = -\frac{2\lambda^2}{z_2}$ $\mathfrak{b}_5(z_3) \subset \mathfrak{g}_5(\alpha, a, \beta, b, r)$, $\alpha = \pm z_3$, $b = \mp z_3$, $r = 2z_3$, $a = \beta = 0$ $\mathfrak{b}_8(z_2, z_4) \subset \mathfrak{g}_8(z_2, z_4, w_2, r, \theta_1, \theta_2)$, $\theta_1 = \theta_2 = w_2 = 0$, $r = -z_2$ $\mathfrak{b}_9(z_2, z_4) \subset \mathfrak{g}_9(z_2, z_3, z_4, \theta_1, \theta_2)$, $\theta_1 = \theta_2 = 0$, $z_3 = -z_2$ $\mathfrak{b}_{11}(z_1, z_2) \subset \mathfrak{g}_{11}(z_1, z_2, z_3.w_1, \theta_1, \theta_2)$, $\theta_1 = \theta_2 = 0$, $z_3 = z_2$, $w_1 = -\frac{z_1^2}{z_2}$ $\mathfrak{b}_{12}(w_1) \subset \mathfrak{g}_{12}(z_3, w_1, w_2, \theta_1, \theta_2)$, $w_1 = w_2 = \theta_1 = \theta_2 = 0$
$(\mathcal{I}_I, \mathcal{AK}_J, \mathcal{AK}_K)$	$\mathfrak{c}_4(\lambda, z_2) \subset \mathfrak{g}_4(\lambda, z_2, w_1, w_2)$, $w_2 = 0$, $w_1 = 2z_2$ $\mathfrak{c}_{10}(\alpha, a) \subset \mathfrak{g}_{10}(\alpha, a, \beta, b)$, $a = \beta$, $b = -\alpha$ $\mathfrak{c}_{14}(\theta_1, \theta_2) \subset \mathfrak{g}_{14}(z_2, z_4, w_2, \theta_1, \theta_2)$, $z_2 = z_4 = w_2 = 0$
$(\mathcal{I}_I, \mathcal{I}_J, \mathcal{I}_K)$	$\mathfrak{d}_2(\lambda) \subset \mathfrak{g}_2(\lambda, \alpha, \beta, w_1, w_2)$, $\alpha = -\lambda$, $\beta = w_1 = w_2 = 0$ $\mathfrak{d}_3(\alpha) \subset \mathfrak{g}_3(\alpha, \beta, w_1, w_2, \theta_2)$, $\theta_2 = -\alpha$, $\beta = w_1 = w_2 = 0$ $\mathfrak{d}_{10}(\alpha, a) \subset \mathfrak{g}_{10}(\alpha, a, \beta, b)$, $\beta = b = 0$ $\mathfrak{d}_{18}(\beta, b) \subset \mathfrak{g}_{10}(\beta, b, z_3, z_4, \theta_1, \theta_2)$, $\theta_1 = -\beta$, $\theta_2 = -b$, $z_3 = z_4 = 0$

Таблица 2. Classes of natural almost hyper-Hermitian structures on conformally foliated 4-dimensional non-abelian Lie groups with minimal leaves

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