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СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ
Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru
ISSN 1813-3304

Том 20, № 2, стр. 144–145 (2023) УДК 517.44
https://doi.org/10.33048/semi.2023.20.??? MSC ??X??

PROPAGATION OF THE STANDARD METHOD OF INVERTING THE RADON TRANSFORMATION TO THE CLASS OF DISCONTINUOUS FUNCTIONS

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INTRODUCED BY D.S. ANIKONOV

Abstract: The problem of inverting the Radon integral transformation in finite-dimensional Euclidean space is considered. Examples of application of this transformation in pure and applied mathematics are given. The relevance of this topic for probing issues is indicated. It is noted that for the latter direction, it is natural to consider the sub-integral function as discontinuous. However, the available inversion formulas are only proven for differentiable functions. Therefore, the question of obtaining formulas for discontinuous functions arises. It is set that the required results can be obtained by some modification of available proofs for smooth functions.

Keywords: Radon transformation, discontinuous functions, probing, tomography, integral geometry.

1 Introduction

The Radon transformation is defined as integrals of a function over hyperplanes in n-dimensional Euclidean space. This transformation has been used in many areas of mathematics. For example, in the theory of hyperbolic differential equations, it was possible to reduce the dimension of space to two. In this simple case, it was possible to use explicit representations for solutions of Cauchy problems. However, such representations were obtained for the images of the Radon transformation. Therefore, the next stage of the work was the question of inverting the transformation. These mathematical aspects have been investigated, for example, in [1,2]. It is important to keep in mind that the obtained inversion formulas were proven only for smooth functions. Another application of the Radon transformation relates to the theory of probing unknown media with various physical signals. In particular, the mathematical part of X-ray tomography theory is reduced to the inverse Radon transformation for $n = 2$ and $n = 3$.

Furthermore, the Radon transformation is widely used in seismic exploration, geophysical mapping, defectoscopy, and environmental monitoring. Some similar applications are represented in [3 – 14].

It is emphasized that for probing, it is natural to consider the sought characteristics as discontinuous functions. This is the way to describe inhomogeneous media consisting of different materials. However, as already mentioned, the available formulas are proven only for smooth functions.

In the present work a slight modification of available proofs allows to obtain explicit inversion formulas for discontinuous sub-integral functions. There are grounds to believe that the application of our formulas will noticeably expand the possibilities of using corresponding algorithms.

2 Notations, Definitions, and Problem Statement

We will use the following notations. $\mathbb{E}_n$ means an n-dimensional Euclidean space; $e_1, \ldots, e_n$ is the main orthonormal basis in $\mathbb{E}_n$; points x, y, w in the space $\mathbb{E}_n$ in a general basis are represented as $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, $\omega = (\omega_1, \ldots, \omega_n)$; Ω denotes the unit sphere in $\mathbb{E}_n$; ω is an element of the sphere in $\mathbb{E}_n$, $\Omega = \{\omega : \omega \in \mathbb{E}_n, |\omega|\} = 1\}; Y(\omega, p) = \{y : y \cdot \omega = p\}$, $y \in \mathbb{E}_n$, $\omega \in \Omega$, $-\infty < p < \infty$ represents a hyperplane in $\mathbb{E}_n$; ∂T denotes the boundary of the set T, $T \subset \mathbb{E}_n$.

Let G be a bounded open set containing pairwise disjoint open sets G_i, $i = 1, \ldots, N$. Denoting the union of these subsets by G_0, we require that $\overline{G}_0 = \overline{G}$. We assume that the boundary of the set G_0 has zero measure in the space $\mathbb{E}_n$. Define the class of functions F. Function $f(y)$, $y \in \mathbb{E}_n$, belongs to F, if the following conditions are satisfied: $|f(u) - f(v)| \leq \mathit{const}|u - v|^\alpha$, $u, v \in G_i$, $0 < \alpha \leq 1$ and $f(y) = 0$ for $y \in \mathbb{E}_n \setminus G$, where *const* is a positive number, $i = 1, \ldots, N$. Apart from the main basis, we will use spherical angles $\varphi = (\varphi_1, \ldots, \varphi_{n-1})$ for representing ω:

$$
\begin{aligned}
&\omega(\varphi) = \Big(\omega_1(\varphi), \ldots, \omega_n(\varphi)\Big), \quad \varphi_1 \in [0, 2\pi), \quad \varphi_2, \ldots, \varphi_{n-1} \in [0, \pi], \\
&\omega_1 = \sin\varphi_1 \sin\varphi_2 \ldots \sin\varphi_{n-1} \\
&\omega_2 = \cos\varphi_1 \sin\varphi_2 \ldots \sin\varphi_{n-1} \\
&\omega_3 = \cos\varphi_2 \sin\varphi_3 \ldots \sin\varphi_{n-1} \\
&\ldots \\
&\omega_{n-1} = \cos\varphi_{n-2} \cdot \sin\varphi_{n-1} \\
&\omega_n = \cos\varphi_{n-1}
\end{aligned}
$$

In this article integration over Ω is used. Therefore we give the following definitions.

The integral of the function $g(w)$ over Ω is defined by the equality:

$$\int_{\Omega} g(\omega)d\omega = \int_0^{\pi} \dots \int_0^{\pi} \int_0^{2\pi} g(\omega(\varphi)) \sin\varphi_2 \,(\sin\varphi_3)^2 \dots (\sin\varphi_{n-1})^{n-2}\, d\varphi_1 \dots d\varphi_{n-1}$$

We say that a subset Ω_0 of the sphere Ω is a set of measure zero if the projection of Ω_0 onto the hyperplane $y_n = 0$ is a set of zero measure in $\mathbb{E}_{n-1}$.

Let $f(y)$ be a function defined for $y \in \mathbb{E}_n$, which is integrable in $\mathbb{E}_n$. The Radon transform is defined by the equality:

$$[Rf](\omega, p) = \int\limits_{y\cdot\omega=p} f(y)d_y\sigma, \tag{2.1}$$

where the right-hand side contains a surface integral over the hyperplane $Y(w,p)$. Using Fubini's theorem for any $\omega \in \Omega$, we have:

$$\int_G f(y)dy = \int\limits_{-\infty}^{+\infty} \int\limits_{y\cdot\omega=p} f(y)d_y\sigma dp = \int\limits_{-\infty}^{+\infty} [Rf](w,p)dp \tag{2.2}$$

In the following, we will assume that $f \in F$.

Similarly to identity (2.2), for any continuous function $\psi(s)$, $s \in (0, \infty)$, we write the equalities:

$$\int_G f(y)\psi\Big((y-x)\cdot w\Big)dy = \int\limits_{-\infty}^{\infty} \int\limits_{(y-x)\cdot\omega=p} f(y)\psi(p)d_y\sigma dp = \int\limits_{-\infty}^{+\infty} \psi(p)[Rf](\omega, p + x\cdot\omega)dp \tag{2.3}$$

The following problem is investigated in this work:

Problem: Given the function $[Rf](\omega,p)$ for $\omega \in \Omega$, $p \in (-\infty, +\infty)$, find $f(y)$, $y \in \mathbb{E}_n \backslash \partial G_0$, where $f \in F$.

3 The case of odd n in the Radon transform inversion problem $(n = 2m+1)$

Here, we will need the following identities, as provided, for example, in [2].

$$\int\limits_{\Omega} |\xi\cdot\omega| d\omega = \beta_{1,n}(\xi), \quad \xi \in \mathbb{E}_n, \tag{3.1}$$

$$\beta_{1,n} = \frac{2\sqrt{\pi^{n-1}}}{\Gamma\left(\frac{n+1}{2}\right)}$$

$$(\Delta_x)^{\frac{n-1}{2}} \, |y-x| = \beta_{2,n} |\, y-x|^{2-n}, \quad n \geqslant 3, \tag{3.2}$$

$$\beta_{2,n} = \frac{2^n \Gamma(1.5)\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n}{2}\right)}{(2-n)\pi}(-1)^{\frac{n-1}{2}},$$

$$\Delta_x \int\limits_G \frac{f(y)}{|y-x|^{n-2}} dy = \beta_{3,n} f(x), x \in \mathbb{E}_n \backslash \partial G_0, \quad n \geqslant 3 \tag{3.3}$$

$$\beta_{3,n} = \frac{(2-n)\sqrt{\pi^n} \cdot 2}{\Gamma\left(\frac{n}{2}\right)}.$$

Note that these identities hold true for both even and odd n.

Theorem 1. *If $f \in F$, $n = 2m+1$, $m = 1, \ldots$, then for all $x \in \mathbb{E}_n \backslash \partial G_0$, the equality holds:*

$$\beta_n \cdot f(x) = (\Delta_x)^{\frac{n+1}{2}} \int\limits_{\Omega} \int\limits_{-\infty}^{+\infty} |p| [Rf](\omega, p + x \cdot \omega) dp d\omega, \quad \beta_n \neq 0 \tag{1}$$

Доказательство. Consider the function

$$U_1(x) = \int\limits_{\Omega} \int\limits_{G} f(y) |(y-x) \cdot \omega| dy dw, \quad x \in \mathbb{E}_n \backslash \partial G_0 \tag{3.5}$$

Using the identities (2.3) for the function $\psi(s) = s$, $s \in (0, +\infty)$, we obtain

$$U_1(x) = \int\limits_{\Omega} \int\limits_{-\infty}^{\infty} |p| [Rf](\omega, p + x \cdot \omega) dp d\omega \tag{3.6}$$

Now, let's transform $U_1(x)$ in a different way. Changing the order of integration in the right-hand side of equality (3.5) and considering the identity (3.1), we can write

$$U_1(x) = \beta_{1,n} \int\limits_G f(y) |y-x| dy.$$

Next, we apply the operator $(\Delta_x)^{\frac{n-1}{2}}$ to this equality. Using identity (3.2), we derive the property

$$(\Delta_x)^{\frac{n-1}{2}} U_1(x) = \beta_{1,n} \cdot \beta_{2,n} \int\limits_G \frac{f(y)}{|y-x|^{n-2}} dy \tag{3.7}$$

Now, applying the operator Δ_x to both sides of the latter equality and using property (3.3), we obtain

$$(\Delta_x)^{\frac{n+1}{2}} U_1(x) = \beta_{1,n} \cdot \beta_{2,n} \cdot \beta_{3,n} f(x) \tag{3.8}$$

In (3.8), let $\beta_n = \beta_{1,n} \cdot \beta_{2,n} \cdot \beta_{3,n}$, and represent the function $U_1(x)$ using expression (3.6). As a result, we obtain equality (3.4). The theorem is proven. $\square$

Remark 1. *It is worth clarifying that the presented proof is analogous to a segment of the proof of the inversion formula in [2]. In our notation, the formula in [2] takes the form*

$$2(2\pi i)^{n-1} f(x) = (\Delta_x)^{\frac{n-1}{2}} \int\limits_{\Omega} [Rf](\omega, x \cdot \omega) d\omega. \tag{3.9}$$

Now let's compare formulas (3.4) and (3.9). We can see that equality (3.9) contains one less integral than equality (3.4). The power of the Laplace operator in (3.9) is also smaller than in (3.4). In other words, finding the function $f(x)$ using (3.9) is a simpler operation than a similar deduction using (3.4). However, equality (3.9) is proven for smooth functions, while equality (3.4) holds true for discontinuous functions as well. Thus, it can be said that formulas (3.4) and (3.9) have their own merits and drawbacks.

4 Inversion of the Radon transform for even n $(n = 2m)$

Here we will use the following identities [2].

$$\int\limits_{\Omega} \ln |(y - x) \cdot \omega| d\omega = \gamma_{1,n}(\ln |y - x| + \gamma_{0,n}), \quad x, y \in \mathbb{E}_n, \quad \omega \in \Omega, \tag{4.1}$$

$$\gamma_{1,n} = \frac{2\sqrt{\pi^n}}{\Gamma\left(\frac{n}{2}\right)}, \quad \gamma_{0,n} \neq 0,$$

$$(\Delta_x)^{\frac{n-2}{2}} \ln |y - x| = \gamma_{2,n} |y - x|^{2-n}, \tag{4.2}$$

$$\gamma_{2,n} = \frac{2^{n-2}\Gamma^2\left(\frac{n}{2}\right)}{2 - n}(-1)^{\frac{n-2}{2}}, \quad n \geqslant 3, \quad x, y \in \mathbb{E}_n,$$

$$\Delta_x \int\limits_G f(y) \ln |y - x| dy = 2\pi f(x), \quad x \in \mathbb{E}_n \backslash \partial G_0, \quad n = 2. \tag{4.3}$$

Theorem 2. *If $f \in F$, $n = 2m$, $m = 1, \ldots$, then for all $x \in \mathbb{E}_n \backslash \partial G_0$, the equality holds:*

$$\gamma_n \cdot f(x) = (\Delta_x)^{\frac{n}{2}} \int\limits_{\Omega} \int\limits_{-\infty}^{+\infty} \ln |p| [Rf](\omega, p + x \cdot \omega) dp d\omega, \quad \gamma_n \neq 0. \tag{4.4}$$

Доказательство. We define the function:

$$U_2(x) = \int\limits_G \int\limits_{\Omega} f(y) \ln |(y - x) \cdot \omega| d\omega dy, \quad x \in \mathbb{E}_n \backslash \partial G_0 \tag{4.5}$$

We change the order of integration in the right-hand side of the last equality and use identities (2.3) with $\psi(s) = \ln s$. Then we obtain:

$$U_2(x) = \int_{\Omega} \int_{G} f(y) \ln |(y-x)\cdot\omega| dy d\omega = \int_{\Omega} \int_{-\infty}^{\infty} \ln |p| [Rf](\omega, p + x\cdot\omega) dp d\omega \tag{4.6}$$

For the same function $U_2(x)$, we give another representation. We transform the inner integral in equality (4.5) using identity (4.1):

$$U_2(x) = \int_{G} f(y)\left(\gamma_{1,n} \ln |y-x| + \gamma_{0,n}\right) dy \tag{4.7}$$

Next, the cases $n = 2$ and $n > 2$ are considered separately. For $n > 2$ we apply the operator $(\Delta_x)^{\frac{n-2}{2}}$ to equality (4.7) and using identity (4.2), we obtain

$$(\Delta_x)^{\frac{n-2}{2}} U_2(x) = \gamma_{1,n} \cdot \gamma_{2,n} \int_{G} \frac{f(y)}{|y-x|^{n-2}} dy \tag{4.8}$$

From here, using identity (3.3), we derive

$$(\Delta_x)^{\frac{n}{2}} U_2(x) = \gamma_{1,n} \cdot \gamma_{2,n} \cdot \gamma_{3,n} f(x), \gamma_{3,n} = \beta_{3,n}. \tag{4.9}$$

Let's denote $\gamma_n = \gamma_{1,n} \cdot \gamma_{2,n} \cdot \gamma_{3,n}$ and compare equalities (4.9) and (4.6). As a result, we obtain the identity (4.4).

Now, let $n = 2$. Applying the operator Δ_x to (4.7) and using identity (4.3), we write:

$$\Delta_x U_2(x) = \gamma_{1,2} \cdot 2\pi f(x) \tag{4.10}$$

Let's denote $\gamma_{1,2} \cdot 2\pi = \gamma_2$ and compare (4.10) and (4.6). As a result, we obtain the equality (4.4) for the particular case $n = 2$. The theorem is proven. □

Remark 2. *For even n, the classical inversion formula takes the form:*

$$(2\pi i)^n f(x) = (\Delta_x)^{\frac{n-2}{2}} \int_{\Omega} \int_{-\infty}^{\infty} \ln |p - x\cdot\omega| \frac{\partial^2}{\partial p^2} [Rf](\omega, p) dp dw \tag{4.11}$$

Let's compare the inversion formulas (4.4) and (4.11). Each of them involves the same order of integration and differentiation. However, to derive the equality (4.11), it is necessary to assume that the function $f(y)$ has continuous and bounded derivatives up to second order. For formula (4.4), it is sufficient to require that $f(y)$ is piecewise continuous in the Holder sense. Hence, we can conclude that formula (4.4) is more valuable for applications than (4.11).

The work was carried out within the framework of the state task of the SB RAS (project FWNF-2022-0009).

References

[1] R. Courant, Partial differential equations, *Interscience, France*, (1962).

[2] F. John, Plane waves and spherical means applied to partial differential equations, *Springer, New York*, (1981).

[3] F. Natterer, *The Mathematics of Computerized Tomography*, Wiley, Hoboken, NJ, USA, (1986).

[4] A.M. Akhmetshin, V.V.Tripolskaya, "Improving the accuracy of signature identification based on invariant moments of the Radon transform.," *Bulletin of Zaporozhye State University.* **1**, (2002).

[5] V.S. Flanden , A.Yu. Poroikov , D.A. Golentsov , "Reconstruction of Flame Local Temperature Field Using Tomographic Approach by Spectral Ratio Pyrometer," *Conference: 29th International Conference on Computer Graphics, Image Processing and Computer Vision, Visualization Systems and the Virtual Environment GraphiCon'2019* **1**, 108–112 (2019).

[6] S.A. Klestov, "Calibration of a 3D X-ray microtomograph," *III International Conference "Cognitive Robotics"within the framework of the International Forum "Intelligent Systems of the 4th Industrial Revolution November 21-23, 2018* 10–13 (2019).

[7] S.I. Elesina , S.S. Kudryashov, "Using the Radon transform in correlation-extremal navigation systems," *Methods and means of processing and storing information: Interuniversity. Sat. scientific tr. / Ed. VC. Zlobina Ryazan: RGRTU* 92–100 (2010).

[8] D. S. Anikonov, E. Yu. Balakina, D. S. Konovalova, An inverse problem for generalized Radon transformation, *St. Petersburg Polytechnical State University Journal. Physics and Mathematics* **15**(1) (2022). DOI: https://doi.org/10.18721/JPM.

[9] D. S. Anikonov and D. S. Konovalova, A Problem of Integral Geometry for a Family of Curves with Incomplete Data, *Doklady Mathematics* **92**(2) (2015), 221пїS224.

[10] E. Y. Derevtsov, S.V. Maltseva, S. V. Svetov, Determination of discontinuities of a function in a domain with refraction from its attenuated ray transform, *J. Appl. Ind. Math.* **12** (4) (2018), 619–641.

[11] Kalnin T. G., Ivonin D. A., Abrosimov K. N., et al., Analysis of tomographic images of the soil pore space structure by integral geometry methods, *Eurasian Soil Sci. 54 (9) (2021) 1400–1409.*

[12] Lyakhov L. N., Lapshina M. G., Roshchupkin S. A., Support theorem for the Radon–Kipriyanov K - transform, *Itogi Nauki i Tekhn. Ser. Sovrem. Mat. Pril. Temat. Obz. 171 (2019) 118–124 (in Russian).*

[13] Simonov E. N., Prokhorov A. V., Akintseva A. V., Mathematical modelling of reconstruction of volumetric images in X-ray computed tomography using holographic methods, *YuUrGU. Ser. Mat. Model. Progr.* **12**(3) (2019) 102–114 (in Russian).

[14] Temirgaliyev N., Abikenova Sh. K., Azhgaliyev Sh. U., Taugynbayeva G. E., The Radon transform in the scheme C(N)D-inverstigations and the quasi-Monte Carlo theory, *Math. Iz. VUZ.* **64** (3) (2020) 87–92.

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