SPHERICALLY ORDERABLE GROUPS

S.V. SUDOPLATOV [ID]

Communicated by P.P. PETROV

Abstract: We introduce and study the class of spherically ordered groups. The notions of spherically orderable groups and their spectra of spherical orderability are defined. Values of these spectra are found for a series of natural groups.

Keywords: spherical order, group, spectrum of spherical orderability.

1 Introduction

Well known linearly ordered groups and circularly ordered groups are both deeply investigated and described [1, 2] and admit various generalizations and modifications for partial, left and right orderings [1, 3], betweenness and separation groups [4, 5, 6] and semigroups [7].

We continue to study n-spherical orders introducing and investigating n-spherically ordered groups which generalize linearly and cyclically ordered groups, with $n = 2$ and $n = 3$, respectively. The notions of spherically ordered groups and their spectra of spherical orderability are introduced. Values of these spectra are found for a series of natural groups.

2 Spherically orderable groups and their spherical spectra

Recall the following generalization of linear and circular orders.

Definition [8, 9, 10]. An n-ary relation K_n is called n-*ball*, or n-*spherical*, or n-*circular* order relation, for $n \geq 2$, if it satisfies the following conditions:
(nso1)[1] for any even permutation σ on $\{1, 2, \ldots, n\}$,

$$\forall x_1, \ldots, x_n \left(K_n(x_1, x_2, \ldots, x_n) \to K_n \left(x_{\sigma(1)}, x_{\sigma(2)} \ldots, x_{\sigma(n)} \right) \right);$$

(nso2) $\forall x_1, \ldots, x_n \bigg((K_n(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) \wedge$

$$\wedge K_n(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n)) \leftrightarrow \bigvee_{1 \leq k < l \leq n} x_k = x_l \bigg)$$

for any $1 \leq i < j \leq n$;

(nso3) $\forall x_1, \ldots, x_n \bigg(K_n(x_1, \ldots, x_n) \to$

$$\to \forall t \left(\bigvee_{i=1}^{n} K_n(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n) \right) \bigg);$$

(nso4) $\forall x_1, \ldots, x_n(K_n(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) \vee$

$$\vee K_n(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n)), \ 1 \leq i < j \leq n.$$

A structure $\mathcal{M}$ provided with a n-spherical order is called n-*spherically ordered*.

The axioms above produce all possible linear orders K_2 and circular orders K_3. Here (nso2) gives the reflexivity: $\forall x K_2(x, x)$, and the antisymmetry:

$$\forall x_1, x_2(K_2(x_1, x_2) \wedge K_2(x_2, x_1) \to x_1 \approx x_2),$$

(nso2) and (nso3) give the transitivity:

$$\forall x_1, x_2, x_3(K_2(x_1, x_2) \wedge K_2(x_2, x_3) \to K_2(x_1, x_3)),$$

and the axiom (nso4) gives the linearity: $\forall x_1, x_2(K_2(x_1, x_2) \vee K_2(x_2, x_1))$. For the transitivity it suffices to take pairwise distinct elements a, b, c with $(a, b) \in K_2$ and $(b, c) \in K_2$. By (nso3) we have $(a, c) \in K_2$ or $(c, b) \in K_2$, and $(b, a) \in K_2$ or $(a, c) \in K_2$. But the cases $(c, b) \in K_2$ and $(b, a) \in K_2$ are impossible in view of (nso2), implying the required $(a, c) \in K_2$.

---

[1]Here we admit cyclic permutations iff n is odd. In view of this circumstance the item (nso1) is corrected with respect to the previous version.

Similarly to linear orders any n-spherical order K_n on a set Z has the dual one consisting of all n-tuples in $Z^n \setminus K_n$ united with the set of all n-tuples in Z^n with some repeated coordinates. We denote this dual order by $\overline{K_n}$.

Definition. A group G is called *n-spherically ordered*, or *n-s-ordered*, if G is provided with a n-spherical order K_n such that for any $(x_1, \ldots, x_n) \in K_n$ and any $y \in G$ the tuples $(x_1 y, \ldots, x_n y)$ and $(y x_1, \ldots, y x_n)$ belong to K_n.

A group G is called *n-spherically orderable*, or *n-s-orderable*, if G has a n-spherically ordered expansion. A group G is called *spherically orderable* if it is n-spherically orderable for some n.

For a group G we define its *spectrum* $\mathrm{Sp}_{\mathrm{so}}$ *of spherical orderability*, or *spherical spectrum*, as follows:

$$\mathrm{Sp}_{\mathrm{so}}(G) = \{n \in \omega \setminus \{0, 1\} \mid G \text{ is } n\text{-spherically orderable}\}.$$

A group G is called *totally spherically orderable*, or *totally s-orderable*, if G has maximal spectrum of spherical orderability, i.e. $\mathrm{Sp}_{\mathrm{so}}(G) = \omega \setminus \{0, 1\}$.

A group G is called *almost totally spherically orderable*, or *almost totally s-orderable*, if $\mathrm{Sp}_{\mathrm{so}}(G)$ is a cofinite subset of ω.

A group G is (*almost*) *not s-orderable in any way* if $\mathrm{Sp}_{\mathrm{so}}(G)$ is empty (respectively, finite).

The notions above for the spherical orderability and its spectra can be naturally spread for an arbitrary structure. Besides, the spherical orderability admits similar variations of orderability as for linear (bi-)orderability such as left-orderability and right-orderability [3, 11].

A natural **problem** arises on description of spherical spectra for groups and related structures.

By the definition a group G is linearly ordered iff G is 2-spherically ordered, and G is cyclically ordered iff G is 3-spherically ordered. Here $2 \in \mathrm{Sp}_{\mathrm{so}}(G)$ and $3 \in \mathrm{Sp}_{\mathrm{so}}(G)$, respectively.

Again by the definition a group G is spherically orderable iff $\mathrm{Sp}_{\mathrm{so}}(G) \neq \emptyset$.

Remark 1. If a group G is n-spherically orderable then each subgroup of G is n-spherically orderable, too, since any restriction $\mathcal{M}$ of a n-spherically ordered structure $\mathcal{N}$, with a n-spherical order K_n, is again n-spherically ordered, with the n-spherical order $K_n \cap M^n$.

In view of Remark 1 we have the following *Monotonicity property* for the spectrum of spherical orderability:

Proposition 1. *For any groups G_1, G_2 if $G_1 \leqslant G_2$ then $\mathrm{Sp}_{\mathrm{so}}(G_1) \supseteq \mathrm{Sp}_{\mathrm{so}}(G_2)$.*

Proposition 1 immediately implies:

Corollary 1. *If G is an (almost) totally s-orderable group then any its subgroup is also (almost) totally s-orderable.*

Corollary 2. *If G is (almost) not s-orderable in any way then any its supergroup is also (almost) not s-orderable in any way.*

Proposition 1 and Corollary 1 can be also deduced from the following criterion of n-spherical orderability:

Theorem 1. *A group G is n-spherically ordered by a n-spherical order K_n iff for any n-tuple $(a_1, \ldots, a_n) \in K_n$ with pairwise distinct coordinates and for any $b \in G$ the tuples $(a_1 b, \ldots, a_n b)$ and $(b a_1, \ldots, b a_n)$ are even permutations, i.e. are not odd permutations of some tuples in K_n.*

Proof. By the definition of n-spherical order K_n it consists of all n-tuples with some repeated coordinates and of all even permutations of given n-tuples in K_n such that for any n-tuple, in the universe, with pairwise distinct coordinate either this tuple or all its odd permutations belong to K_n. Now if G is n-spherically ordered by K_n it is forbidden to include into K_n its odd permutations of forms $(a_1 b, \ldots, a_n b)$ and $(b a_1, \ldots, b a_n)$, and conversely permitted to include its even permutations of forms $(a_1 b, \ldots, a_n b)$ and $(b a_1, \ldots, b a_n)$.

Remark 2. Theorem 1 shows that the only obstacle for a n-spherical order K_n on a group G to produce this group to be n-spherical ordered is the possibility for the multiplications $(a_1 b, \ldots, a_n b)$ and $(b a_1, \ldots, b a_n)$ to produce an odd permutation for a tuple $(a'_1, \ldots, a'_n)$ in K_n with pairwise distinct coordinates. In particular, elements of stabilizers for n-element sets $\{a_1, \ldots, a_n\} \subseteq G$ can not produce odd permutations of tuples $(a_1, \ldots, a_n) \in K_n$. Here $b \neq e$ and therefore the maps $a_i \mapsto a_i b$ and $a_i \mapsto b a_i$ do not have fixed elements.

In general, K_n is uniquely defined by its subrelation K_n^0 consisting of all n-tuples in K_n with pairwise distinct coordinates, since $K_n = K_n^0 \dot\cup K_n^1$, where K_n^1 consists of all tuples in G^n with repeated coordinates. The subrelation K_n^0 produces an algebra $\mathcal{K}_n^0$ with $\dfrac{n!}{2}$ unary operations of even permutations forming the alternating group A_n, and unary operations r_b and l_b, for $b \in G$, carrying out the maps $a_i \mapsto a_i b$ and $a_i \mapsto b a_i$, respectively. Since all these operations are invertible, with $(r_b)^{-1} = r_{b^{-1}}$ and $(l_b)^{-1} = l_{b^{-1}}$, and form a *derivative* group $S = S(G, n)$ with the identical even permutation which is equal both to r_e and l_e, we obtain an S-act $\mathcal{K}_n^0 = \langle K_n^0, s \rangle_{s \in S}$, for $|G| \geq n$. Here actions r_b and l_b can be even permutations if, for instance, $|G| = n$, which implies $S = A_n$, They may not belong to A_n, if $b \neq e$ and G is torsion-free. And they coincide if b belongs to the center of G. The group S is generated by its two subgroups A_n and $S' = \{r_b, l_b \mid b \in G\}$ and elements in $f \in A_n$ and in $s \in S'$ commute: $fs = sf$. Besides, $r_b = l_b$ and $\alpha_{bb'} = \alpha_b \alpha_{b'}$ for any $\alpha \in \{r, l\}$ and $b, b' \in G$, if G is commutative. Since A_n is commutative iff $n \leq 3$, the algebra $\mathcal{K}_n^0$ has the commutative derivative group S iff $n \leq 3$ and G is commutative.

Clearly, if G is a group n-spherically ordered by an order K_n then G is n-spherically ordered by the dual order $\overline{K_n} = (G^n \setminus K_n) \cup K_n^1$, too.

In view of Theorem 1 and Remark 2 we have the following:

Corollary 3. *A group G is linearly ordered by a 2-spherical order K_2 iff for any pair $(a_1, a_2) \in K_2$ with $a_1 \neq a_2$ and for any $b \in G$ the pairs $(a_1 b, a_2 b)$ and (ba_1, ba_2) are not transpositions of some pairs in K_2.*

Since all odd permutations for the group S_3 are transpositions we additionally have:

Corollary 4. *A group G is circularly ordered by a circular order K_3 iff for any triple $(a_1, a_2, a_3) \in K_3^0$ and for any $b \in G$ the triples $(a_1 b, a_2 b, a_3 b)$ and (ba_1, ba_2, ba_3) are not transpositions of some triples in K_3^0.*

Remark 3. Each orbit O of the group S on K_n^0 connects all even permutations of tuples in O and possibly tuples based on distinct n-element sets if some r_b or l_b are not permutations of these sets.

The number r of these orbits O is called the *rank of generation* of K_n^0 with respect to S and denoted by $\mathrm{rk}_S(K_n^0)$. This rank is said to be the rank of generation for K_n, denoted by $\mathrm{rk}_S(K_n)$.

If the group G is abelian then $\mathrm{rk}_S(K_n^0)$ is finite iff G is finite, since for tuples $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ based on distinct n-element sets the link $r_b = l_b$ is defined by $b = y_1 - x_1$. For an infinite non-abelian group one may put defining relations connecting n-tuples into finitely many orbits via chains of tuples. Indeed, the connection of tuples $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ can be organized via intermediate tuples $(z_1, \ldots, z_n)$ formed by new defining elements and with appropriate r_b and l_b, where the elements b are composed with some new defining elements. Thus there are infinite non-abelian groups with finite $\mathrm{rk}_S(K_n^0)$. Moreover, finite values $\mathrm{rk}_S(K_n^0)$ can be realized arbitrarily in $\omega \setminus \{0\}$ using defining relations connecting step-by-step orbits by r_b and l_b with new defining elements b.

In view of Remark 3 for an abelian group G the operators $r_b = l_b$ on G^3 either fix all coordinates of a triple or move all its coordinates, i.e. can not generate odd permutations (transpositions). Thus Corollary 4 immediately implies:

Corollary 5. *For any abelian group G, $3 \in \mathrm{Sp}_{\mathrm{so}}(G)$.*

The following construction based on Theorem 1 shows how the spectrum $\mathrm{Sp}_{\mathrm{so}}(G)$ can be reduced till the empty one, i.e. with G which is not s-orderable in any way.

For a required group G we take a generating set $\{a_n \mid n \in \omega \setminus \{0\}\} \cup \{b, c\}$. Now we consider the following defining relations: $ba_1 c = a_2$, $ba_2 c = a_1$, $ba_n c = a_n$ for $n \geq 3$. These relations show that any m-tuple $(ba_{i_1} c, ba_{i_2} c, \ldots, ba_{i_m} c)$ consisting of pairwise distinct elements and with some $i_j = 1$ and $i_k = 2$ is a transposition of $(a_{i_1}, a_{i_2}, \ldots, a_{i_m})$. Therefore m-spherical orders on G coordinated with left and right group actions can not be formed for any m. Hence we have the following:

Theorem 2. *There exists a group G such that $\mathrm{Sp}_{\mathrm{so}}(G) = \emptyset$.*

Definition. Let $<$ be a strict linear order on a group G, C be the *cyclification* of $<$ consisting of all tuples (x, y, z) with $x < y < z \vee y < z < x \vee z < x < y$. A n-ary relation K_n^0 on G, for $n \geq 3$, is called $<$-*coordinated* if K_n^0 consists of all n-tuples $(a_0, \ldots, a_{n-1})$ with $a_0 < \ldots < a_{n-1}$, and for any $b \in G$ the tuples $(a_0 b, \ldots, a_{n-1} b)$ and $(b a_0, \ldots, b a_{n-1})$ satisfy $C\left(a_i b, a_{(i+1)\,(\mathrm{mod}\,n)} b, a_{(i+2)\,(\mathrm{mod}\,n)} b\right)$ and $C\left(b a_i, b a_{(i+1)\,(\mathrm{mod}\,n)}, b a_{(i+2)\,(\mathrm{mod}\,n)}\right)$, $i = 0, \ldots, n - 1$.

Remark 4. By Remark 2 any $<$-coordinated relation K_n^0 is uniquely extensible to its closure K_n under even permutations of tuples and addition of all tuples in G^n with repeated coordinates. In view of Remark 2 this closure satisfies the axioms of n-spherical order iff even permutations of tuples $(a_1, \ldots, a_n)$ in K_n do not meet their odd permutations under group actions $(a_1' b, \ldots, a_n' b)$ and $(b a_1'', \ldots, b a_n'')$ in K_n, where $(a_1', \ldots, a_n'), (a_1'', \ldots, a_n'') \in K_n$, $b \in G$.

Remark 5. If G is a finite group with $|G| = m$ and $m < n$ then G is n-spherically orderable. Indeed, in such a case the relation K_n consisting of all tuples with repeated coordinates satisfies the axioms of n-spherical order implying that $K_n = K_n^1$ and $\langle G, K_n \rangle$ is a n-spherically ordered group.

In view of Remark 5 we have the following:

Proposition 2. *If G is a group with $|G| = m \in \omega$ then $\mathrm{Sp}_{\mathrm{so}}(G) \supseteq \{n \in \omega \mid n > m\}$, in particular, G is almost totally s-orderable.*

The following assertion is a reformulation of well-known folklore fact that nonunit linearly ordered groups are infinite.

Proposition 3. *A finite group G is 2-spherically orderable iff $|G| = 1$.*

Proof. The one-element group G is 2-spherically orderable in view of Remark 5. If $|G| > 1$ then G contains a nonunit element g of finite order. Assuming that G is 2-spherically orderable with the order K_2 we take the unit e and obtain $(e, g) \in K_2$ or $(g, e) \in K_2$. By the definition of 2-spherically ordered group we obtain $(g^k, g^{k+1}) \in K_2$ for any k or $(g^{k+1}, g^k) \in K_2$ for any k. For the first case we take $k_0 > 0$ with $g^{k_0+1} = e$ and obtain $(g^{k_0}, e) \in K_2$, $g^{k_0} \neq e$, which contradicts the transitivity. In the second case we also have a contradiction to the transitivity in view of $(e, g^{k_0}) \in K_2$, $g^{k_0} \neq e$.

The arguments for Proposition 3 imply the following:

Corollary 6. *Any 2-spherically orderable group G is torsion-free, that is, if G contains an element of finite positive order then $2 \notin \mathrm{Sp}_{\mathrm{so}}(G)$.*

Remark 6. Since each group $\mathbb{Z}_m$ is circularly orderable [1], i.e. $3 \in \mathrm{Sp}_{\mathrm{so}}(\mathbb{Z}_m)$, then Proposition 2 and Corollary 6 imply that $\mathrm{Sp}_{\mathrm{so}}(\mathbb{Z}_2) = \mathrm{Sp}_{\mathrm{so}}(\mathbb{Z}_3) = \omega \setminus \{0, 1, 2\}$.

The groups $\mathbb{Z}_m$, for even m, are not m-spherically orderable since the definition of m-spherically orderable group implies that K_m is closed both

under even and odd permutations of m pairwise distinct elements: odd permutations are obtained by actions of elements in $\mathbb{Z}_m$ on m-tuples of pairwise distinct elements of $\mathbb{Z}_m$. It contradicts the axiom (nso2). Thus $m \notin \mathrm{Sp}_{\mathrm{so}}(\mathbb{Z}_m)$ for each even m. In particular, by Proposition 2 and Corollary 6, $\mathrm{Sp}_{\mathrm{so}}(\mathbb{Z}_4) = \omega \setminus \{0, 1, 2, 4\}$.

The arguments above show that any group G containing a subgroup $\mathbb{Z}_m$, for even m, can not be m-spherically ordered by a relation K_m since, in view of (nso4), it should contain a tuple $(i_1, \ldots, i_m)$, where $\{i_1, \ldots, i_m\} = \{0, \ldots, m-1\}$. In particular, if G contains subgroups $\mathbb{Z}_m$, for each even m, then $\mathrm{Sp}_{\mathrm{so}}(G)$ is contained in the set of odd numbers.

At the same time the groups $\mathbb{Z}_m$, for odd $m \geq 5$, are m-spherically orderable since the definition of m-spherically orderable group implies even permutations of elements in K_m, generated by the tuple $(0, 1, \ldots, m-1)$ only. Thus $m \in \mathrm{Sp}_{\mathrm{so}}(\mathbb{Z}_m)$ for each odd $m \geq 5$.

In view of Proposition 1 and Remark 6 each element a of even order m in a given group G implies $m \notin \mathrm{Sp}_{\mathrm{so}}(G)$. Thus we have the following:

Proposition 4. *For any group G, $\mathrm{Sp}_{\mathrm{so}}(G)$ does not contain even numbers which are equal to orders of elements in G.*

In particular, we have the following:

Corollary 7. *If a group G contains elements of each even order then*
$$\mathrm{Sp}_{\mathrm{so}}(G) \cap 2\mathbb{Z} = \emptyset.$$

Proposition 5. *For any natural m, n with $2 < n < m$, and $n \nmid m$ if n is even, then the group $\mathbb{Z}_m$ is n-spherically orderable.*

Proof. We form the n-spherical order K_n on the universe $\mathbb{Z}_m$ adding to the set of n-tuples with some repeated coordinates all even permutations of tuples $(k_1(\mathrm{mod}\,m), \ldots, k_n(\mathrm{mod}\,m))$ with $k_1 < \ldots < k_n$ and even permutations of tuples $(k_1 + q(\mathrm{mod}\,m), \ldots, k_n + q(\mathrm{mod}\,m))$ with $q \in \mathbb{Z}_m$, i.e. K_n is generated by its $<$-coordinated subrelation K_n^0, where $<$ is the natural strict order on $\mathbb{Z}_m$.

Since $n \nmid m$ for even n, $\mathbb{Z}_m$ does not contain a subgroup $\mathbb{Z}_n$ violating the n-spherical orderability as in Remark 6. Thus even permutations of tuples $(k_1 + q(\mathrm{mod}\,m), \ldots, k_n + q(\mathrm{mod}\,m))$ are coordinated with even permutations of tuples $(k_1, \ldots, k_n)$, i.e. they do not produce odd permutations. Since these permutations cover all possibilities for tuples with even permutations only, Theorem 1 guarantees that $\mathbb{Z}_m$ is n-spherically orderable by K_n.

Propositions 2, 3, 5 and Remark 6 immediately imply the following description of spherical spectra for the groups $\mathbb{Z}_m$:

Theorem 3. *Let $m \in \omega \setminus \{0, 1\}$. Then*
$$\mathrm{Sp}_{\mathrm{so}}(\mathbb{Z}_m) = \omega \setminus (\{0, 1, 2\} \cup \{n \mid n | m \text{ and } n \text{ is even}\}).$$

Proposition 6. *The group $\mathbb{Z}$ is totally s-orderable.*

Proof. Let $n \in \omega \setminus \{0, 1\}$. We form the n-spherical order K_n on the universe $\mathbb{Z}$ adding to the set of n-tuples with some repeated coordinates all even permutations of tuples $(k_1, \ldots, k_n)$ with $k_1 < \ldots < k_n$, i.e. generate K_n by the $<$-coordinated relation K_n^0 with the natural order $<$. Clearly, for any $m \in \mathbb{Z}$, $(k_1 + m, \ldots, k_n + m)$ preserves the set of these tuples and satisfies the axioms of n-spherical orders in view of Theorem 1, as required. Thus, $\mathrm{Sp_{so}}(\mathbb{Z}) = \omega \setminus \{0, 1\}$, i.e. $\mathbb{Z}$ is totally s-orderable.

It is known that any torsion-free abelian group can be lexicographically ordered with respect to its generators. Therefore the arguments for Proposition 6 imply the following:

Theorem 4. *Any torsion-free abelian group is totally s-orderable.*

Remark 7. The group confirming Theorem 2 is torsion-free and it is not s-orderable at all. It illustrates that the commutativity of a group is essential in Theorem 4.

References

[1] L. Fuchs, *Partially Ordered Algebraic Systems*, Dover books on mathematics, International series of monographs in pure and applied mathematics, Courier Corporation, 2011. 229 p.

[2] A.I. Kokorin, V.M. Kopytov *Fully ordered groups*, New Work, J. Wiley, 1974. 147 p.

[3] V.M. Kopytov, N.Ya. Medvedev *The theory of lattice-ordered groups*, Dordrecht, Springer, 1994. 400 p.

[4] J.A.H. Shepperd, *Separation and the definition of betweenness and separation groups*, Journal of the London Mathematical Society, **s1-31**:2 (1956), 240–248.

[5] J.A.H. Shepperd, *Betweenness groups*, Journal of the London Mathematical Society, **32** (1957), 277–285.

[6] J.A.H. Shepperd, *Separation groups*, Proceedings of the London Mathematical Society, **s3-7**:1 (1957), 518–548.

[7] J. Gilder, *Betweenness and order in semigroups*. Mathematical Proceedings of the Cambridge Philosophical Society, **61**:1 (1965), 13–28.

[8] S.V. Sudoplatov, *Arities and aritizabilities of first-order theories*, Siberian Electronic Mathematical Reports, **19**:2 (2022), 889–901.

[9] B.Sh. Kulpeshov, S.V. Sudoplatov, *Spherical orders, properties and countable spectra of their theories*, Siberian Electronic Mathematical Reports, **20**:2 (2023), 588–599.

[10] S.V. Sudoplatov, *Minimality conditions, topologies and ranks for spherically ordered theories*, Siberian Electronic Mathematical Reports, **20**:2 (2023), 600–615.

[11] K. Muliarchyk, *Free products of bi-orderable groups*, arXiv:2403.14779 [math.GR], 2024. 15 p.

SERGEY VLADIMIROVICH SUDOPLATOV
SOBOLEV INSTITUTE OF MATHEMATICS,
ACADEMICIAN KOPTYUG AVENUE, 4
630090, NOVOSIBIRSK, RUSSIA
E-mail address: sudoplat@math.nsc.ru