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Polynomial identities for the Jordan algebra of 2×2 upper triangular matrices

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ABSTRACT

Let K be a field (finite or infinite) of $\mathrm{char}(K) \neq 2$ and let $UT_2(K)$ be the 2×2 upper triangular matrix algebra over K. If $\cdot$ is the usual product on $UT_2(K)$ then with the new product $a \circ b = (1/2)(a \cdot b + b \cdot a)$ we have that $UT_2(K)$ is a Jordan algebra, denoted by $UJ_2 = UJ_2(K)$. In this paper, we describe the set I of all polynomial identities of UJ_2 and a linear basis for the corresponding relatively free algebra. Moreover, if K is infinite we prove that I has the Specht property. In other words I, and every T-ideal containing I, is finitely generated as a T-ideal.

1. Introduction

Let A be an associative algebra over a field K, and denote the product in A by $a \cdot b$. One can attach to A a Lie algebra, denoted by $A^{(-)}$, by considering the usual

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Lie bracket $[a, b] = a \cdot b - b \cdot a$ on the vector space of A. If the characteristic of K is different from 2, one may consider the symmetric (or Jordan) product $a \circ b = (1/2)(a \cdot b + b \cdot a)$, and this makes the vector space A a Jordan algebra denoted by $A^{(+)}$. The well known Poincaré–Birkhoff–Witt theorem asserts that every Lie algebra is a subalgebra of some $A^{(-)}$. In the case of Jordan algebras there is no analogue of the PBW theorem. A Jordan algebra which is a subalgebra of some $A^{(+)}$ is called *special*, and it is *exceptional* otherwise. There exist exceptional Jordan algebras; the "smallest" example being the 27-dimensional Albert algebra. We shall not need here very many deep facts and results from the theory of Jordan algebras; the interested readers could find a wealth of information in the monographs by Jacobson [15], and by McCrimmon [23].

Let X be an infinite countable set, $X = \{x_1, x_2, \ldots\}$, and let $A(X)$ and $J(X)$ stand for the free associative and for the free Jordan algebra, freely generated over K by the set X. Thus if A is an arbitrary associative algebra and $a_i \in A$ then there exists a unique homomorphism $\varphi\colon A(X) \to A$ such that $\varphi(x_i) = a_i$ for every i, and analogously for $J(X)$ and an arbitrary Jordan algebra J. One can interpret $A(X)$ as the vector space over K with a basis consisting of all (associative) monomials in the x_i, and a product defined on the monomials by juxtaposition. The elements of $A(X)$ are called polynomials (associative albeit non-commutative). There is no such transparent description of $J(X)$; in fact a major open problem is describing a basis of the vector space of $J(X)$. Recall here that $J(X)$ is exceptional whenever $|X| > 2$. As in the associative case we shall call the elements of $J(X)$ *polynomials* as well.

In this paper we study polynomial identities in Jordan algebras. We recall that $f = f(x_1, \ldots, x_n) \in J(X)$ is a polynomial identity (or simply an identity) for the Jordan algebra J whenever $f(a_1, \ldots, a_n) = 0$ for every $a_i \in J$. In other words f lies in the kernel of each homomorphism $\varphi\colon J(X) \to J$. Likewise one defines identities for associative algebras. The set of all identities satisfied by a Jordan algebra J is denoted by $T(J)$, it is the T-ideal (or verbal ideal) of J. Clearly it is an ideal; moreover it is closed under endomorphisms. Every ideal T in $J(X)$ which is closed under endomorphisms is the T-ideal of some Jordan algebra, for example $J(X)/T$. We direct the interested readers to the monograph [9], and to the recent one [2]. Although these two books deal mainly with associative and Lie algebras the notions we need (as well as their proofs) are essentially the same in the case of Jordan algebras.

A major problem in the theory of algebras with polynomial identities is describing a basis of the identities satisfied by a given algebra. Recall that a basis of a T-ideal is a generating set of it as a T-ideal. This problem has been solved in very few cases. In the associative case the identities of the Grassmann algebra E are known, see [21], and [20]. The identities of the matrix algebra of order 2 were described in [25] and [8] (in characteristic 0), and in [18] (over an infinite field of characteristic $p \neq 2$). The identities of the tensor square $E \otimes E$ are also known in characteristic 0, [24]. Adding to this list the identities of the algebra $UT_n(K)$ of the upper triangular matrices of order n [22] one gets a more or less complete list of the associative algebras whose identities are known. Recall that the identities of $M_2(K)$ are not known in case K is infinite and of characteristic 2. The identities of $M_3(K)$ even in characteristic 0 are very far from our reach at present.

Turning to Jordan algebras the situation becomes even less clear. Bases of the identities of the Jordan algebras of a non-degenerate symmetric bilinear form are known, see [12] for the finite dimensional case in characteristic 0, and [32] for the general case. In contrast with the associative case, the identities of $UT_n(K)$ considered as a Jordan algebra, are known only when $n \le 2$, [19].

In 1950, W. Specht [27] posed a problem that has shaped a good part of the development of PI theory since then. He asked whether the T-ideals of associative PI algebras over a field of characteristic 0 are finitely generated. Clearly one may ask a similar question for varieties of nonassociative algebras: Lie, Jordan, alternative, and so on. One may also relax the requirement for the characteristic of the field.

In 1970, Vaughan-Lee proved that for Lie algebras in characteristic 2 the answer to the Specht problem is negative, [33,34]. Drensky extended this result to Lie algebras over an infinite field of characteristic $p > 2$, [7].

Around 1985, Kemer developed a sophisticated theory; it gave a classification of the T-ideals in the free associative algebra in characteristic 0. As a consequence of this theory Kemer obtained the positive answer to the Specht problem for associative algebras when $\operatorname{char} K = 0$, see [16,17]. Later on, by employing and adapting the methods developed by Kemer, similar results were obtained for ample classes of Lie algebras [14], Jordan algebras [31], alternative algebras [13]. On the other hand, around 2000, it turned out that in the case of associative algebras over infinite fields of positive characteristic, the answer to the Specht problem is negative. The first examples of T-ideals that do not admit finite bases were given simultaneously and independently in [4,10,26].

The Specht problem can be posed in the setting of graded algebras and the corresponding graded identities, algebras with involution, and so on. Concerning associative PI algebras over a field of characteristic 0, graded by a finite group, the positive solution to the Specht problem was obtained in [28,1], and in the f.g. associative involutive case in [29].

In this paper we study ideals of identities of Jordan algebras. We deal with the identities of the Jordan algebra $UJ_2 = UJ_2(K)$, over an arbitrary field of characteristic different from 2. Let us recall that the identities of UJ_2, in characteristic 0, were described in [19]. Also in characteristic 0, in [6] it was proved that the T-ideal of UJ_2 satisfies the Specht property. In other words, every T-ideal in $J(X)$ that contains $T(UJ_2)$ is finitely generated as T-ideal. More generally, the graded polynomial identities for UJ_2 were described in [5,19], and its Specht property in [6] when the field has characteristic 0.

Here we exhibit a basis for the T-ideal $T(UJ_2)$ over an arbitrary field (finite or infinite) of characteristic different from 2. This gives a new proof of the main theorem of [6]. Moreover, we prove that in case K is an infinite field, $T(UJ_2)$ satisfies the Specht property.

The methods we employ are combinatorial ones. We consider the case of finite and of infinite base field K separately, as the proofs in each case are rather different. Finally we make use of the technique of partially well-ordered sets [11] in order to deduce the Specht property.

2. Preliminaries

Throughout this paper, unless otherwise stated, K is a field (finite or infinite) with $\mathrm{char}(K) \neq 2$. Let $UT_n = UT_n(K)$ be the $n \times n$ upper triangular matrix algebra over K. The usual matrix multiplication $\cdot$ in UT_n makes it an associative algebra. Denote by $UJ_n = UJ_n(K)$ the vector space UT_n with the symmetric (or Jordan) product $\circ$ given by

$$u \circ v = (1/2)(u \cdot v + v \cdot u)$$

where u, $v \in UT_n$. Then UJ_n becomes a Jordan algebra. We recall that in this paper we are dealing with *linear* Jordan algebras, and that is why we need a field of characteristic different from 2.

Let e_{ij} be the matrix unit in UJ_2 whose (i,j)th entry equals 1 and all remaining entries equal 0. We denote

$$1 = e_{11} + e_{22}, \quad a = e_{11} - e_{22}, \quad \text{and} \quad b = e_{12}.$$

Note that $\{1, a, b\}$ is a basis for the vector space UJ_2,

$$a \circ a = 1 \quad \text{and} \quad a \circ b = b \circ b = 0.$$

For convenience, if u, $v \in UJ_2$ we will write $u \circ v = uv$.

Let $X = \{x_1, x_2, \ldots\}$ be an infinite countable set of variables. We denote by $J(X)$ the free Jordan algebra, freely generated by X, over K. If f_1, f_2, $\ldots$, $f_n \in J(X)$, we use the following convention:

$$f_1 f_2 \cdots f_n = f_1(f_2 \cdots f_n). \tag{1}$$

That is if no inner parentheses are given in a Jordan product it is understood to be right-normed. Moreover, (f_1, f_2, f_3) stands for the associator, that is,

$$(f_1, f_2, f_3) = (f_1 f_2) f_3 - f_1(f_2 f_3).$$

One defines the set Ω of the *long associators* in $J(X)$ as follows. It is the least subset of $J(X)$ such that for every f_1, f_2, $f_3 \in X \cup \Omega$ one has $(f_1, f_2, f_3) \in \Omega$. When dealing with long associators, we shall omit the inner parentheses when these are left-normed, that is

$$(f_1, f_2, f_3, f_4, \ldots, f_n) = ((f_1, f_2, f_3), f_4, \ldots, f_n)$$

when n is odd. Moreover, if $x_{i_1}, x_{i_2}, x_{i_3}, \ldots, x_{i_n} \in X$ and $n \geq 3$ is odd, then $(x_{i_1}, x_{i_2}, x_{i_3}, \ldots, x_{i_n})$ is called *regular associator.*

Note that

$$(w, v, u) = -(u, v, w) \quad \text{and} \quad (v, u, w) = (u, v, w) - (u, w, v). \tag{2}$$

If $W \subseteq J(X)$, we denote by $\langle W \rangle^T$ the T-ideal of $J(X)$ generated by W. We recall that an ideal I of $J(X)$ is a T-ideal if it is closed under all endomorphisms of $J(X)$. It is well known that I is a T-ideal if and only if it coincides with the ideal of all identities of some Jordan algebra over K.

If $f \in J(X)$ we denote by $\deg f$ its degree. If $w \in X$ is a variable (that is a free generator of $J(X)$) we denote by $\deg_w f$ the degree of f with respect to the variable w. By using the same argument as in [9, Proposition 4.2.3] we have the following:

Proposition 2.1. *Let K be a field (finite or infinite) of* $\operatorname{char}(K) \neq 2$ *and with $|K|$ elements. Let $f \in J(X)$, $w \in X$, $\deg_w f = d_w$, and*

$$f = \sum_{i=0}^{d_w} f^{(i)},$$

where $f^{(i)}$ is the homogeneous component of f with $\deg_w f^{(i)} = i$*. If $d_w < |K|$ then*

$$\langle f \rangle^T = \langle f^{(0)}, f^{(1)}, \ldots, f^{(d_w)} \rangle^T.$$

Let $K[x_1, \ldots, x_n]$ be the free associative and commutative algebra, freely generated by $x_1, \ldots, x_n$, over K. It is the usual polynomial algebra in the variables $x_1, \ldots, x_n$. The next lemma is a consequence of [9, Proposition 4.2.3].

Lemma 2.2. *Let K be a field (finite or infinite) such that $|K| \geq q$. Given $f \in K[x_1, \ldots, x_n]$, write*

$$f(x_1, \ldots, x_n) = \sum_{d_1=0}^{q-1} \cdots \sum_{d_n=0}^{q-1} \lambda_{(d_1,\ldots,d_n)} x_1^{d_1} \cdots x_n^{d_n},$$

where $\lambda_{(d_1,\ldots,d_n)} \in K$. If $f(\alpha_1, \ldots, \alpha_n) = 0$ for every choice of $\alpha_1, \ldots, \alpha_n \in K$ then $\lambda_{(d_1,\ldots,d_n)} = 0$ for all $(d_1, \ldots, d_n)$.

We will use the next lemma in the last section of this paper.

Lemma 2.3. *If p is a prime number and $1 \leq m < n$, then the multinomial coefficients*

$$\alpha = \binom{p^n - 1}{p^m - 1, \underbrace{p^m, \dots, p^m}_{p-1 \text{ factors}}, \underbrace{p^{m+1}, \dots, p^{m+1}}_{p-1 \text{ factors}}, \dots, \underbrace{p^{n-1}, \dots, p^{n-1}}_{p-1 \text{ factors}}} \neq 0 \pmod{p}$$

and

$$\beta = \binom{p^n}{\underbrace{p^m, \dots, p^m}_{p \text{ factors}}, \underbrace{p^{m+1}, \dots, p^{m+1}}_{p-1 \text{ factors}}, \dots, \underbrace{p^{n-1}, \dots, p^{n-1}}_{p-1 \text{ factors}}} = 0 \pmod{p}.$$

Proof. Write $(p^n)! = p^N b$, where $p \nmid b$. We recall that

$$N = \left\lfloor \frac{p^n}{p} \right\rfloor + \left\lfloor \frac{p^n}{p^2} \right\rfloor + \cdots + \left\lfloor \frac{p^n}{p^n} \right\rfloor = p^{n-1} + p^{n-2} + \cdots + 1 = \frac{p^n - 1}{p - 1},$$

where $\lfloor a \rfloor$ is the integer part of a. Thus,

$$(p^n - 1)! = p^{N-n} b.$$

By writing

$$(p^m - 1)!(p^m!)^{p-1}(p^{m+1}!)^{p-1} \cdots (p^{n-1}!)^{p-1} = p^M a,$$

where $p \nmid a$, it is easily shown that $M = N - n$. Therefore

$$\alpha = \frac{(p^n - 1)!}{(p^m - 1)!(p^m!)^{p-1}(p^{m+1}!)^{p-1} \cdots (p^{n-1}!)^{p-1}} = \frac{b}{a} \neq 0 \pmod{p}.$$

Since

$$\beta = \binom{p^n}{p^m} \alpha = p^{n-m} \alpha$$

and $n > m$, it follows that $p \mid \beta$. The lemma is proved. □

3. Identities of $UJ_2(K)$

Let $T(UJ_2)$ be the T-ideal of $UJ_2 = UJ_2(K)$, that is the set of all polynomial identities for $UJ_2(K)$. In this section we will describe $T(UJ_2)$ for an arbitrary field K such that $\mathrm{char}(K) \neq 2$.

Lemma 3.1. *Let $X_i = \alpha_i 1 + \beta_i a + \gamma_i b$, where α_i, β_i, $\gamma_i \in K$, $1 = e_{11} + e_{22}$, $a = e_{11} - e_{22}$ and $b = e_{12}$. If $n \geq 1$, then*

a) $(X_1, X_2, X_3, \ldots, X_{2n+1}) = [(-1)^{n-1}(\beta_1\gamma_3 - \gamma_1\beta_3)\beta_2\beta_4\beta_5 \cdots \beta_{2n+1}]b$;
b) $X_1 \cdots X_m(X_{i_1}, X_{i_2}, \ldots, X_{i_{2n+1}}) = [(-1)^{n-1}\alpha_1 \cdots \alpha_m(\beta_{i_1}\gamma_{i_3} - \gamma_{i_1}\beta_{i_3})\beta_{i_2}\beta_{i_4}\beta_{i_5} \cdots \beta_{i_{2n+1}}]b$.

Proof. a) We prove the lemma by induction on n. A direct verification shows that

$$(X_1, X_2, X_3) = [(\beta_1\gamma_3 - \gamma_1\beta_3)\beta_2]b = \det \begin{pmatrix} \beta_2 & 0 & 0 \\ 0 & \beta_1 & \gamma_1 \\ 0 & \beta_3 & \gamma_3 \end{pmatrix} b.$$

Thus, the case $n = 1$ is done. By the induction hypothesis it follows that

$$\begin{aligned}(X_1, \ldots, X_{2n+1}) &= ((X_1, \ldots, X_{2n-1}), X_{2n}, X_{2n+1}) \\ &= \det \begin{pmatrix} \beta_{2n} & 0 & 0 \\ 0 & 0 & (-1)^{n-2}(\beta_1\gamma_3 - \gamma_1\beta_3)\beta_2\beta_4 \cdots \beta_{2n-1} \\ 0 & \beta_{2n+1} & \gamma_{2n+1} \end{pmatrix} b \\ &= (-1)^{n-1}(\beta_1\gamma_3 - \gamma_1\beta_3)\beta_2\beta_4 \cdots \beta_{2n+1}b.\end{aligned}$$

b) Using the equalities $ab = bb = 0$, together with item a), we obtain the desired equality. □

Notation 3.2. *Let I be the T-ideal of $J(X)$ generated by the polynomials*

$$T(x_1, x_2, x_3, x_4) = (x_1x_2, x_3, x_4) - x_1(x_2, x_3, x_4) - x_2(x_1, x_3, x_4), \tag{3}$$

$$(x_1, (x_2, x_3, x_4), x_5), \tag{4}$$

$$(x_1, x_2, x_3)(x_4, x_5, x_6). \tag{5}$$

Define the equivalence relation $\equiv$ on $J(X)$ as follows: if f, $g \in J(X)$ then

$$f \equiv g \quad \text{if and only if} \quad f + I = g + I.$$

Lemma 3.3. *If I is the T-ideal defined above, then $I \subseteq T(UJ_2)$.*

Proof. By applying Lemma 3.1, we can prove that the three generators of I are polynomial identities for UJ_2. □

Lemma 3.4. *The following equivalences hold in $J(X)$:*

a) $(x_1x_2, x_3, x_4) \equiv (x_1, x_3, x_4)x_2 + (x_2, x_3, x_4)x_1$;
b) $(x_3, x_4, x_1x_2) \equiv (x_3, x_4, x_1)x_2 + (x_3, x_4, x_2)x_1$;
c) $(x_3, x_1x_2, x_4) \equiv (x_3, x_1, x_4)x_2 + (x_3, x_2, x_4)x_1$.

Proof. Since $T(x_1, x_2, x_3, x_4) \in I$ we obtain $T(x_1, x_2, x_3, x_4) \equiv 0$, that is

$$(x_1x_2, x_3, x_4) \equiv (x_1, x_3, x_4)x_2 + (x_2, x_3, x_4)x_1.$$

This proves the statement of (a).

By (2) we have $(x_3, x_4, x_1x_2) = -(x_1x_2, x_4, x_3)$. Now we use (a) and then (2) in order to deduce (b).

By (2) we have

$$(x_3, x_1x_2, x_4) = (x_1x_2, x_3, x_4) - (x_1x_2, x_4, x_3).$$

Now we apply (a) and afterwards (2) in order to obtain (c). □

Recall the convention (1): all Jordan products without parentheses are supposed to be right-normed. We also remember that if $x_{i_1}, x_{i_2}, x_{i_3}, \ldots, x_{i_n} \in X$ and $n \geq 3$ is odd, then

$$(x_{i_1}, x_{i_2}, x_{i_3}, \ldots, x_{i_n})$$

is called regular associator.

Lemma 3.5. *If $h \in \langle (x_1, x_2, x_3)\rangle^T$ then $h + I$ is a linear combination of elements of the form $fg + I$ where $f \in J(X)$ and g is a regular associator.*

Proof. If $h \in \langle (x_1, x_2, x_3)\rangle^T$, then it is a linear combination of elements

$$m_1m_2 \cdots m_t(u, v, w), \tag{6}$$

where $m_1, \ldots, m_t, u, v, w \in J(X)$ are monomials.

Claim 1. $h + I$ is a linear combination of elements

$$m_1m_2 \cdots m_tf + I, \tag{7}$$

where $m_1, m_2, \ldots, m_t \in J(X)$ are monomials, and $f \in J(X)$ is a regular associator.

Proof of Claim 1. By applying Lemma 3.4 several times to (u, v, w) of Eq. (6), we have the claim.

Therefore, in order to prove the lemma it is sufficient to suppose $h + I$ is as in (7).

Claim 2. Let $m_1, \ldots, m_t$ be monomials with $\deg m_1 + \cdots + \deg m_t = n$. If f is a regular associator then $m_1m_2 \cdots m_tf + I$ is a linear combination of elements $mg + I$, where m is a monomial with $\deg m \leq n$ and g is a regular associator.

Proof of Claim 2. We shall prove the claim by using induction in n. The case $n = 1$ is trivial. Suppose $n \geq 2$. By applying the induction hypothesis on $m_2 \cdots m_tf + I$ we obtain that $m_1m_2 \cdots m_tf + I$ is a linear combination of elements $m_1mg + I$, where m is a monomial with $\deg m \leq \deg m_2 + \cdots + \deg m_t$ and g is a regular associator.

Thus, for a polinomial $m_1 m g + I$ as above we have

$$\deg m_1 + \deg m \leq \deg m_1 + \deg m_2 + \cdots + \deg m_t = n.$$

If $\deg m_1 + \deg m < n$ then by using once again the induction hypothesis we have the claim. If $\deg m_1 + \deg m = n$ then

$$m_1 m g \equiv (m_1 m)g - (m_1, m, g) \equiv (m_1 m)g + (g, m, m_1). \tag{8}$$

By Lemma 3.4 we have that $(g, m, m_1) + I$ is a linear combination of elements $m'_1 \cdots m'_r g' + I$ where $m'_1, \ldots, m'_r$ are monomials with $\deg m'_1 + \cdots + \deg m'_r \leq n - 2$ and g' is a regular associator. By applying the induction hypothesis to $m'_1 \cdots m'_r g' + I$ we have by (8) that the claim is proved.

By Claim 1 and Claim 2 the proof is complete. □

Lemma 3.6. *If $y_1, \ldots, y_6 \in X$ then*

$$(x_1(y_1, y_2, y_3))(y_4, y_5, y_6) \equiv 0.$$

Proof. Since

$$(x_1, (x_2, x_3, x_4), x_5) \quad \text{and} \quad (x_1, x_2, x_3)(x_4, x_5, x_6)$$

belong to I, we have

$$(x_1(y_1, y_2, y_3))(y_4, y_5, y_6) = x_1((y_1, y_2, y_3)(y_4, y_5, y_6)) + (x_1, (y_1, y_2, y_3), (y_4, y_5, y_6)) \equiv 0$$

as desired. □

Notation 3.7. *Let $g = g(x_1, \ldots, x_n) \in J(X)$ be the monomial with $\deg_{x_i} g = d_i$, defined by*

$$g = x_1 \cdots x_1 x_2 \cdots x_2 \cdots x_n \cdots x_n.$$

We denote g by

$$g = x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}.$$

Lemma 3.8. *The quotient vector space $\langle (x_1, x_2, x_3) \rangle^T / I$ is spanned by the set of all polynomials*

$$(x_1^{a_1} \cdots x_n^{a_n})(x_{i_1}, x_{i_2}, \ldots, x_{i_{2k+1}}) + I,$$

where $n \geq 0$, $a_i \geq 0$ for every i and $k \geq 1$.

Proof. If $g \in \langle (x_1, x_2, x_3) \rangle^T$ then by Lemma 3.5 we have that $g+I$ is a linear combination of elements of the form $mh + I$ where $m \in J(X)$ is a monomial and h is a regular associator.

Claim 1. If $m = m(x_1, \dots, x_n) \in J(X)$ is a monomial of multidegree $(b_1, \dots, b_n)$ and h is a regular associator then

$$mh \equiv (x_1^{b_1} \cdots x_n^{b_n})h.$$

Proof of Claim 1. We have $m = x_1^{b_1} \cdots x_n^{b_n} + \widehat{h}$, for some $\widehat{h} \in \langle (x_1, x_2, x_3) \rangle^T$. Thus

$$mh = (x_1^{b_1} \cdots x_n^{b_n} + \widehat{h})h = (x_1^{b_1} \cdots x_n^{b_n})h + \widehat{h}h.$$

By Lemmas 3.5 and 3.6 it follows that $\widehat{h}h \in I$ and so

$$mh \equiv (x_1^{b_1} \cdots x_n^{b_n})h.$$

The claim is proved.

By Claim 1 we have that $g + I$ is a linear combination of the desired elements and the lemma is proved. □

The next lemma was proved in [19, Lemmas 16, 17]; we include it here for the sake of completeness.

Lemma 3.9. *If y_1, $y_2 \in X$ and $\sigma \in Sym(3)$ then*

$$(y_1, x_{\sigma(1)}, y_2, x_{\sigma(2)}, x_{\sigma(3)}) \equiv (y_1, x_1, y_2, x_2, x_3).$$

Proof. It suffices to prove that

a) $(y_1, x_1, y_2, x_2, x_3) \equiv (y_1, x_1, y_2, x_3, x_2)$;
b) $(y_1, x_1, y_2, x_2, x_3) \equiv (y_1, x_2, y_2, x_3, x_1)$.

By (2) and $(x_1, (x_2, x_3, x_4), x_5) \in I$, it follows that

$$\begin{aligned}(y_1, x_1, y_2, x_2, x_3) \equiv& (x_2, (y_1, x_1, y_2), x_3) - (x_2, x_3, (y_1, x_1, y_2)) \\ \equiv& -(x_2, x_3, (y_1, x_1, y_2)) \equiv (y_1, x_1, y_2, x_3, x_2).\end{aligned}$$

Therefore item a) is proved.

Write $f = (y_1, x_1, y_2, x_2, x_3) - (y_1, x_2, y_2, x_3, x_1)$. By definition of associator, $(x_1, (x_2, x_3, x_4), x_5) \in I$, and Lemma 3.4, item c), we have the following equivalences:

$$\begin{aligned}f \equiv& ((y_1, x_1, y_2)x_2)x_3 - (y_1, x_1, y_2)(x_2x_3) - ((y_1, x_2, y_2)x_3)x_1 + (y_1, x_2, y_2)(x_3x_1) \\ \equiv& ((y_1, x_1, y_2)x_2)x_3 - (y_1, x_1(x_2x_3), y_2) + (y_1, (x_2x_3), y_2)x_1 - ((y_1, x_2, y_2)x_3)x_1\end{aligned}$$

$$
\begin{aligned}
&+ (y_1, x_2(x_3x_1), y_2) - (y_1, (x_3x_1), y_2)x_2 \\
&\equiv ((y_1, x_1, y_2)x_2)x_3 + (y_1, (x_1, x_3, x_2), y_2) + ((y_1, x_2, y_2)x_3)x_1 + ((y_1, x_3, y_2)x_2)x_1 \\
&- ((y_1, x_2, y_2)x_3)x_1 - ((y_1, x_3, y_2)x_1)x_2 - ((y_1, x_1, y_2)x_3)x_2 \\
&\equiv ((y_1, x_1, y_2)x_2)x_3 + ((y_1, x_3, y_2)x_2)x_1 - ((y_1, x_3, y_2)x_1)x_2 - ((y_1, x_1, y_2)x_3)x_2 \\
&\equiv (x_2, (y_1, x_1, y_2), x_3) + (x_2, (y_1, x_3, y_2), x_1) \equiv 0.
\end{aligned}
$$

The proof of item b), and of the lemma, is complete. □

Lemma 3.10. *If $y_1,\ y_2 \in X$ and $\sigma \in Sym(2n+1)$ then*

$$(y_1, x_{\sigma(1)}, y_2, x_{\sigma(2)}, \ldots, x_{\sigma(2n+1)}) \equiv (y_1, x_1, y_2, x_2, \ldots, x_{2n+1})$$

for every $n \geq 1$.

Proof. Let $f = (y_1, x_{\sigma(1)}, y_2, x_{\sigma(2)}, \ldots, x_{\sigma(2n+1)})$. Note that it is enough to prove

$$f \equiv (y_1, x_{\sigma(i)}, y_2, x_{\sigma(2)}, \ldots, x_{\sigma(i-1)}, x_{\sigma(1)}, x_{\sigma(i+1)}, \ldots, x_{\sigma(2n+1)}), \tag{9}$$

for every $i = 2, \ldots, 2n+1$. We will prove this by induction on n. The case $n = 1$ is a consequence of Lemma 3.9. Suppose $n \geq 2$. By the induction hypothesis, (9) holds for every $i = 2, \ldots, 2n-1$. If $i = 2n$, by the induction hypothesis and Lemma 3.9 we obtain

$$
\begin{aligned}
f &\equiv (((y_1, x_{\sigma(1)}, y_2, x_{\sigma(2)}, \ldots, x_{\sigma(2n-3)}), x_{\sigma(2n-2)}, x_{\sigma(2n-1)}), x_{\sigma(2n)}, x_{\sigma(2n+1)}) \\
&\equiv (((y_1, x_{\sigma(2n-2)}, y_2, x_{\sigma(2)}, \ldots, x_{\sigma(2n-3)}), x_{\sigma(1)}, x_{\sigma(2n-1)}), x_{\sigma(2n)}, x_{\sigma(2n+1)}) \\
&\equiv (((y_1, x_{\sigma(2n-2)}, y_2, x_{\sigma(2)}, \ldots, x_{\sigma(2n-3)}), x_{\sigma(2n)}, x_{\sigma(2n-1)}), x_{\sigma(1)}, x_{\sigma(2n+1)}) \\
&\equiv (((y_1, x_{\sigma(2n)}, y_2, x_{\sigma(2)}, \ldots, x_{\sigma(2n-3)}), x_{\sigma(2n-2)}, x_{\sigma(2n-1)}), x_{\sigma(1)}, x_{\sigma(2n+1)}).
\end{aligned}
$$

If $i = 2n+1$, by Lemma 3.9 and the latter case it follows that

$$
\begin{aligned}
f &\equiv (((y_1, x_{\sigma(1)}, y_2, x_{\sigma(2)}, \ldots, x_{\sigma(2n-3)}), x_{\sigma(2n-2)}, x_{\sigma(2n-1)}), x_{\sigma(2n)}, x_{\sigma(2n+1)}) \\
&\equiv (((y_1, x_{\sigma(1)}, y_2, x_{\sigma(2)}, \ldots, x_{\sigma(2n-3)}), x_{\sigma(2n-2)}, x_{\sigma(2n-1)}), x_{\sigma(2n+1)}, x_{\sigma(2n)}) \\
&\equiv (((y_1, x_{\sigma(2n+1)}, y_2, x_{\sigma(2)}, \ldots, x_{\sigma(2n-3)}), x_{\sigma(2n-2)}, x_{\sigma(2n-1)}), x_{\sigma(1)}, x_{\sigma(2n)}) \\
&\equiv (((y_1, x_{\sigma(2n+1)}, y_2, x_{\sigma(2)}, \ldots, x_{\sigma(2n-3)}), x_{\sigma(2n-2)}, x_{\sigma(2n-1)}), x_{\sigma(2n)}, x_{\sigma(1)}).
\end{aligned}
$$

The proof is complete. □

Notation 3.11. *If $f,\ g \in J(X)$, $x \in X$ and $d \geq 1$, we denote*

$$(f, g, x^{(d)}) = (f, g, \underbrace{x, x, \ldots, x}_{d\ factors}).$$

Lemma 3.12. *Let S be the subset of $J(X)$ formed by all polynomials*

(a) $x_1^{m_1}x_2^{m_2}\cdots x_n^{m_n}$,
(b) $(x_1^{m_1}x_2^{m_2}\cdots x_n^{m_n})(x_t, x_u, x_l, x_u^{(s_u)}, x_{u+1}^{(s_{u+1})}, \ldots, x_n^{(s_n)})$,

where $m_1, \ldots, m_n \geq 0$; $t \leq u$ and $t < l$; $s_u, \ldots, s_n \geq 0$; $s_u + s_{u+1} + \cdots + s_n$ is even; $n \geq 0$. Then the quotient vector space $J(X)/I$ is spanned by the set of all elements $h + I$ where $h \in S$.

Proof. If $f \in J(X)$ is a monomial, then

$$f = x_1^{a_1}\cdots x_n^{a_n} + g$$

for some $g \in \langle (x_1, x_2, x_3)\rangle^T$ and $a_i \geq 0$ for every i.

By Lemma 3.8, Lemma 3.10 and (2) it follows that $f + I$ is a linear combination of the elements $h + I$ where $h \in S$. □

3.1. Identities of $UJ_2(K)$, when K is an infinite field

In [19, Theorem 19], the authors described $T(UJ_2(K))$ when K is an infinite field of $\mathrm{char}(K) \neq 2, 3$. In this subsection we deal with the remaining case. Since our proof works for every infinite field K such that $\mathrm{char}\,K \neq 2$ we do not consider separately the case $\mathrm{char}\,K = 3$ (which was left unsolved in [19]).

Theorem 3.13. *Let K be an infinite field of characteristic different from 2. If $T(UJ_2(K))$ is the T-ideal of the polynomial identities for the Jordan algebra $UJ_2(K)$, then $T(UJ_2(K))$ is generated, as a T-ideal, by the polynomials*

$$T(x_1, x_2, x_3, x_4), \quad (x_1, (x_2, x_3, x_4), x_5), \quad \text{and} \quad (x_1, x_2, x_3)(x_4, x_5, x_6).$$

Moreover, $I = T(UJ_2(K))$, and the set in Lemma 3.12 forms a basis for the quotient vector space $J(X)/I$.

Proof. By Lemma 3.3 we have $I \subseteq T(UJ_2)$.

Let S be the set in Lemma 3.12 and write $\overline{S} = \{g + T(UJ_2) \mid g \in S\}$. By Lemma 3.12 it follows that $J(X)/T(UJ_2) = span\overline{S}$. We shall prove that $\overline{S}$ is a linearly independent set. Let

$$f(x_1, \ldots, x_n) = \sum_{g \in S} \lambda_g g \in T(UJ_2), \qquad \lambda_g \in K.$$

Since K is an infinite field, by Proposition 2.1 we can suppose that f is a multihomogeneous polynomial. Let $d = (d_1, \ldots, d_n)$ be the multidegree of $f(x_1, \ldots, x_n)$. In this case,

$$f(x_1,\dots,x_n) = \lambda x_1^{d_1}\cdots x_n^{d_n} + \\ + \sum_m (x_1^{m_1}\cdots x_n^{m_n}) \underbrace{\left(\sum_{l=2}^{n} \lambda_{m,l}(x_t, x_u, x_l, x_u^{(s_u)}, x_{u+1}^{(s_{u+1})}, \dots, x_n^{(s_n)})\right)}_{f_m},$$

where $m = (m_1,\dots,m_n)$, $t \le u$, and $t < l$; $m_i + \deg_{x_i} f_m = d_i$ for every i. Since $f(1,\dots,1) = \lambda 1 = 0$ we have $\lambda = 0$. By using the same argument as in [9, Proposition 4.3.3], we obtain

$$\langle f\rangle^T = \langle f_m \mid m = (m_1,\dots,m_n),\ \text{and}\ m_1,\dots,m_n \ge 0\rangle^T. \tag{10}$$

Therefore

$$f_m(x_t, x_{t+1},\dots,x_n) = \sum_{l=2}^{n} \lambda_{m,l}(x_t, x_u, x_l, x_u^{(s_u)}, x_{u+1}^{(s_{u+1})}, \dots, x_n^{(s_n)}) \in T(UJ_2)$$

for every m. Fix $l > t$, $X_l = a + b$ and $X_i = a$ if $i \ne l$. Here, $a = e_{11} - e_{22}$ and $b = e_{12}$. By Lemma 3.1 we have

$$f_m(X_t, X_{t+1},\dots,X_n) = \pm\lambda_{m,l} b = 0,$$

and so $\lambda_{m,l} = 0$ as desired.

Since $\overline{S}$ is a linearly independent set of elements, and $I \subseteq T(UJ_2)$, by Lemma 3.12 it follows that $I = T(UJ_2)$. □

3.2. Identities of $UJ_2(K)$, when K is a finite field

In this subsection we describe the polynomial identities for UJ_2 when K is a finite field. Throughout this subsection, K stands for a finite field with $|K| = q$ elements, and $\mathrm{char}(K) \ne 2$.

Notation 3.14. *Let I' be the T-ideal of $J(X)$ generated by the polynomials (3), (4), (5) and*

$$(x_1^q - x_1)(x_2, x_3, x_4), \tag{11}$$

$$(x_1, x_2^q - x_2, x_3), \tag{12}$$

$$(x_1^q - x_1)(x_2^q - x_2), \tag{13}$$

$$(x_1, x_2, x_3, x_2^{(q-1)}) - (-1)^{\frac{q-1}{2}}(x_1, x_2, x_3), \tag{14}$$

$$(x_1, x_1, x_2, x_1^{(q-2)}, x_2^{(q-1)}, x_3) - (-1)^{\frac{q-1}{2}}(x_1, x_3, x_2^{(q)}) + (x_1, x_3, x_2) + \\ - (-1)^{\frac{q-1}{2}}(x_1, x_1, x_2, x_1^{(q-2)}, x_3). \tag{15}$$

Note that I' is generated, as a T-ideal, by 8 polynomials, and $I \subseteq I'$.

Lemma 3.15. *If I' is the T-ideal defined above, then $I' \subseteq T(UJ_2)$.*

Proof. Since the multiplicative group $K^* = (K - \{0\}, \cdot)$ is of order $q-1$, it follows that $\alpha^{q-1} = 1$ for every $\alpha \in K - \{0\}$. Thus,

$$\alpha^q = \alpha \quad \text{for every} \quad \alpha \in K.$$

The proof of the lemma is a direct consequence of this fact and Lemma 3.1. □

Lemma 3.16. *If $n \geq 0$ and $y_1, \ldots, y_n \in X$ then*

$$(x_1, y_1 \cdots y_n(x_2^q - x_2), x_3) \quad \textit{and} \quad (y_1 \cdots y_n(x_1^q - x_1))(x_2, x_3, x_4)$$

belong to I'.

Proof. Write $f = (x_1, y_1 \cdots y_n(x_2^q - x_2), x_3)$ and $g = (y_1 \cdots y_n(x_1^q - x_1))(x_2, x_3, x_4)$. We will prove the lemma by induction on n. If $n = 0$ (that is we have no product of the y_i in the polynomials f and g), then the lemma is a consequence of (11) and (12).

Suppose $n \geq 1$. Since $I \subseteq I'$, by Lemma 3.4 we obtain

$$f + I' = (x_1, y_1, x_3)(y_2 \cdots y_n(x_2^q - x_2)) + (x_1, y_2 \cdots y_n(x_2^q - x_2), x_3)y_1 + I'.$$

By applying the induction hypothesis, we have that $f \in I'$. For the polynomial g, by definition of associator, we have

$$g = y_1((y_2 \cdots y_n(x_1^q - x_1))(x_2, x_3, x_4)) + (y_1, y_2 \cdots y_n(x_1^q - x_1), (x_2, x_3, x_4)).$$

By applying the induction hypothesis again, we have that $g \in I'$ as required. □

Lemma 3.17. *The polynomial*

$$(x_1, x_2, x_3, x_4^{(q+1)}) - (-1)^{\frac{q-1}{2}}(x_1, x_2, x_3, x_4, x_4)$$

belongs to I'.

Proof. By Lemma 3.10 and (14), we have

$$\begin{aligned}(x_1, x_2, x_3, x_4^{(q+1)}) + I' &= (x_1, x_4, x_3, x_4^{(q-1)}, x_2, x_4) + I' \\ &= ((-1)^{\frac{q-1}{2}}(x_1, x_4, x_3), x_2, x_4) + I' \\ &= (-1)^{\frac{q-1}{2}}(x_1, x_2, x_3, x_4, x_4) + I',\end{aligned}$$

and the lemma is proved. □

Notation 3.18. *We denote by Λ_n the set of all elements $(a_1, \dots, a_n) \in \mathbb{Z}^n$ such that:*

a) $0 \leq a_1, \dots, a_n < 2q$;
b) *If $a_i \geq q$ for some i, then $a_j < q$ for all $j \neq i$.*

Lemma 3.19. *The quotient vector space $J(X)/I'$ is spanned by the set of all polynomials $g + I'$ such that*

(a) $g = x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$ *or*
(b) $g = (x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n})(x_t, x_u, x_l, x_u^{(s_u)}, x_{u+1}^{(s_{u+1})}, \dots, x_n^{(s_n)})$,

where $(m_1, \dots, m_n) \in \Lambda_n$, $0 \leq p_1, \dots, p_n < q$, $t \leq u$, and $t < l$; $0 \leq s_u < q - 1$, $0 \leq s_{u+1}, \dots, s_n < q$, and $s_u + s_{u+1} + \dots + s_n$ is even; and $n \geq 0$. Moreover, if $u = t$ and $s_u = q - 2$, then $0 \leq s_l < q - 1$.

Proof. Let $f(x_1, \dots, x_n) \in J(X)$ be a monomial, we shall prove that $f + I'$ is a linear combination of elements $g + I'$ where g is as in (a) or (b) in the statement.
Claim 1. $f + I'$ is a linear combination of elements

$$x_1^{m_1} \cdots x_n^{m_n} + g + I',$$

where $(m_1, \dots, m_n) \in \Lambda_n$ and $g \in \langle (x_1, x_2, x_3) \rangle^T$.
Proof of Claim 1: Denote by $H = \langle (x_1, x_2, x_3) \rangle^T + I'$. We have that

$$f + H = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} + H = (x_1^{a_1})(x_2^{a_2}) \cdots (x_n^{a_n}) + H, \tag{16}$$

where $a_i \geq 0$ for every i. Since $(x_i^q - x_i)(x_j^q - x_j) \in I' \subseteq H$ we obtain

$$(x_i^q)(x_j^q) + H = (x_i^q)x_j + x_i(x_j^q) - x_i x_j + H \quad \text{and} \quad x_i^{2q} + H = 2x_i^{q+1} - x_i^2 + H. \tag{17}$$

Thus, if there exists $a_j \geq 2q$ in (16) we can use the second equation of (17) and write $f + H$ as a linear combination of elements $m + H$, where $m = x_1^{b_1} \cdots x_n^{b_n}$ with $0 \leq b_1, \dots, b_n < 2q$. Moreover, if there exist $1 \leq j < k \leq n$ such that b_j, $b_k \geq q$ in $m + H$ then

$$m + H = x_1^{b_1} \cdots x_j^{b_j} \cdots x_k^{b_k} \cdots x_n^{b_n} + H = x_1^{b_1} \cdots x_j^{b_j - q} \cdots x_k^{b_k - q} \cdots x_n^{b_n}(x_j^q)(x_k^q) + H,$$

and by the first equation of (17) it follows that

$$\begin{aligned} m + H &= x_1^{b_1} \cdots x_j^{b_j - q} \cdots x_k^{b_k - q} \cdots x_n^{b_n}((x_j^q)x_k + x_j(x_k^q) - x_j x_k) + H \\ &= x_1^{b_1} \cdots x_j^{b_j} \cdots x_k^{b_k - q + 1} \cdots x_n^{b_n} + x_1^{b_1} \cdots x_j^{b_j - q + 1} \cdots x_k^{b_k} \cdots x_n^{b_n} \\ &\quad - x_1^{b_1} \cdots x_j^{b_j - q + 1} \cdots x_k^{b_k - q + 1} \cdots x_n^{b_n} + H. \end{aligned}$$

By applying several times this argument we can suppose that $f + H$ is a linear combination of elements $x_1^{m_1} \cdots x_n^{m_n} + H$, where $(m_1, \ldots, m_n) \in \Lambda_n$. Therefore $f + I'$ is a linear combination of elements $x_1^{m_1} \cdots x_n^{m_n} + g + I'$ where $(m_1, \ldots, m_n) \in \Lambda_n$ and $g \in \langle (x_1, x_2, x_3) \rangle^T$ as desired.

Claim 2. If $g \in \langle (x_1, x_2, x_3) \rangle^T$ then $g + I'$ is a linear combination of elements

$$(x_1^{p_1} \cdots x_n^{p_n})h + I',$$

where $0 \leq p_i < q$ for all i, and h is a regular associator.

Proof of Claim 2: Since $I \subseteq I'$ we have, by Lemma 3.5 that $g + I'$ is a linear combination of elements of the form $mh + I'$, where $m \in J(X)$ is a monomial and h is a regular associator. Denoting by $J = \langle (x_1, x_2, x_3) \rangle^T + \langle x_1^q - x_1 \rangle^T$ it follows that

$$m + J = x_1^{p_1} \cdots x_n^{p_n} + J,$$

where $0 \leq p_i < q$ for all i. Thus, there exist $u \in \langle (x_1, x_2, x_3) \rangle^T$ and $\widehat{u} \in \langle x_1^q - x_1 \rangle^T$ such that

$$m = x_1^{p_1} \cdots x_n^{p_n} + u + \widehat{u},$$

where $0 \leq p_i < q$ for every i. Therefore, if h is a regular associator we have

$$mh + I' = (x_1^{p_1} \cdots x_n^{p_n})h + uh + \widehat{u}h + I'.$$

By Lemmas 3.8 and 3.6 it follows that $uh \in I'$, and by Lemma 3.16 we have $\widehat{u}h \in I'$. Therefore, $g + I'$ is a linear combination of the elements

$$(x_1^{p_1} \cdots x_n^{p_n})h + I',$$

where $0 \leq p_i < q$ and h is a regular associator. Thus Claim 2 is proved.

Claim 3. If h is a regular associator, then $h + I'$ is a linear combination of elements of the form

$$(x_t, x_u, x_l, x_u^{(s_u)}, x_{u+1}^{(s_{u+1})}, \ldots, x_n^{(s_n)}) + I',$$

where $t \leq u$ and $t < l$, $0 \leq s_u < q - 1$, $0 \leq s_{u+1}, \ldots, s_n < q$, $s_u + s_{u+1} + \ldots + s_n$ is even, and $n \geq 0$. Moreover, if $u = t$ and $s_u = q - 2$, then $0 \leq s_l < q - 1$.

Proof of Claim 3: If h is a regular associator, then by Lemma 3.10 and by (2) it follows that

$$h + I' = (x_t, x_u, x_l, x_u^{(s_u)}, x_{u+1}^{(s_{u+1})}, \ldots, x_n^{(s_n)}) + I',$$

where $t \leq u$ and $t < l$, $s_u, \ldots, s_n \geq 0$, $s_u + s_{u+1} + \ldots + s_n$ is even, and $n \geq 0$. By (14) and Lemma 3.17, we have

$$(x_1, x_2, x_3, x_2^{(q-1)}) + I' = (-1)^{\frac{q-1}{2}}(x_1, x_2, x_3) + I',$$
$$(x_1, x_2, x_3, x_4^{(q+1)}) + I' = (-1)^{\frac{q-1}{2}}(x_1, x_2, x_3, x_4^{(2)}) + I'.$$

Thus we can suppose $0 \leq s_u < q-1$ and $0 \leq s_{u+1}, \ldots, s_n \leq q$. Moreover, by Lemma 3.10 and (14) once again we obtain

$$\begin{aligned}(x_1, x_2, x_3, x_4^{(q)}, x_5) + I' &= (x_1, x_4, x_3, x_4^{(q-1)}, x_2, x_5) + I' \\ &= (-1)^{\frac{q-1}{2}}(x_1, x_4, x_3, x_2, x_5) + I' \\ &= (-1)^{\frac{q-1}{2}}(x_1, x_2, x_3, x_4, x_5) + I'.\end{aligned}$$

Thus, we can suppose $0 \leq s_{u+1}, \ldots, s_n < q$.

Now, if $u = t$ and $s_u = q-2$, then by (15) we have that

$$\begin{aligned}(x_t, x_t, x_l, x_t^{(q-2)}, x_l^{(q-1)}, x_r) + I' = &-(x_t, x_r, x_l) + (-1)^{\frac{q-1}{2}}(x_t, x_r, x_l^{(q)}) \\ &+ (-1)^{\frac{q-1}{2}}(x_t, x_t, x_l, x_t^{(q-2)}, x_r) + I',\end{aligned}$$

where $r \neq t$, l, because $s_t < q-1$ and $s_l < q$. Since

$$(x_t, x_r, x_l^{(q)}) + I' = (x_t, x_l, x_l, x_l^{(q-2)}, x_r) + I',$$

we can suppose that $0 \leq s_l < q-1$. Therefore, $h + I'$ is a linear combination of the desired elements and the proof of the claim is complete.

By Claim 1, Claim 2, and Claim 3 we conclude that $f + I'$ is a linear combination of elements $g + I'$ where the g's are as in (a) or (b).

The lemma is proved. □

Theorem 3.20. *Let K be a finite field with $|K| = q$ elements and characteristic different from 2. If $T(UJ_2(K))$ is the T-ideal of the polynomial identities for the Jordan algebra $UJ_2(K)$, then $T(UJ_2) = I'$. Moreover, the set in Lemma 3.19 is a basis for the quotient vector space $J(X)/I'$.*

Proof. By Lemma 3.15 we have $I' \subseteq T(UJ_2)$.

Denote by S' the set of all polynomials g in Lemma 3.19 (a) and by S'' the set of all polynomials in Lemma 3.19 (b). Let $S = S' \cup S''$ and $\overline{S} = \{g + T(UJ_2) \mid g \in S\}$. Since $I' \subseteq T(UJ_2)$, by Lemma 3.19 it follows that $J(X)/T(UJ_2) = span\overline{S}$.

We shall prove that $\overline{S}$ is a linearly independent set. Let

$$f(x_1, \ldots, x_n) = \sum_{g \in S} \lambda_g g \in T(UJ_2), \quad \lambda_g \in K.$$

Write $f = f' + f''$ where

$$f' = \sum_{g\in S'} \lambda_g g \qquad \text{and} \qquad f'' = \sum_{g\in S''} \lambda_g g.$$

Denote by $UT_2^+ = span\{e_{11} + e_{22}, e_{12}\}$. Since $f''(a_1, \ldots, a_n) = 0$ for all $a_i \in UT_2^+$ we conclude that $f'(a_1, \ldots, a_n) = 0$ for all $a_i \in UT_2^+$.

Let $*$ be the involution on the associative algebra UT_2 defined by:

$$\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}^* = \begin{pmatrix} a_{22} & a_{12} \\ 0 & a_{11} \end{pmatrix}.$$

Note that the symmetric elements of UT_2 form the vector subspace UT_2^+. Moreover, if u, $v \in UT_2^+$ then

$$u \circ v = u \cdot v$$

where $\cdot$ is the usual product of UT_2. Thus

$$f' = f'(x_1, \ldots, x_n) = \sum_{m\in\Lambda_n} \lambda_m x_1^{m_1} \cdots x_n^{m_n},$$

where $m = (m_1, \ldots, m_n) \in \Lambda_n$ and $\lambda_m \in K$, is a $*$-polynomial identity for UT_2 in case x_1, …, x_n are symmetric variables. By [30, Lemma 5.8], we obtain $\lambda_m = 0$ for all $m \in \Lambda_n$.

In particular,

$$f(x_1, \ldots, x_n) = \sum_{g\in S''} \lambda_g g.$$

Denote by

$$g_{(p,t,u,l,s_u,s)} = (x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n})(x_t, x_u, x_l, x_u^{(s_u)}, x_{u+1}^{(s_{u+1})}, \ldots, x_n^{(s_n)}),$$

where $p = (p_1, \ldots, p_n)$ with $0 \leq p_1, \ldots, p_n < q$ and $s = (s_{u+1}, \ldots, s_n)$. Then

$$\begin{aligned} f(x_1, \ldots, x_n) = &\sum_p \sum_{t=1}^{n-1} \sum_{l>t} \sum_{s_t=0}^{q-3} \sum_{(s_{t+1},\ldots,s_n=0)}^{q-1} \lambda_{(p,t,t,l,s_t,s)} g_{(p,t,t,l,s_t,s)} \\ &+ \sum_p \sum_{t=1}^{n-1} \sum_{l>t} \sum_{(s_{t+1},\ldots,\widehat{s_l},\ldots,s_n=0)}^{q-1} \sum_{s_l=0}^{q-2} \lambda_{(p,t,t,l,q-2,s)} g_{(p,t,t,l,q-2,s)} \\ &+ \sum_p \sum_{t=1}^{n-1} \sum_{l>t} \sum_{u>t} \sum_{s_u=0}^{q-2} \sum_{(s_{u+1},\ldots,s_n=0)}^{q-1} \lambda_{(p,t,u,l,s_u,s)} g_{(p,t,u,l,s_u,s)}. \end{aligned}$$

Let $X_i = \alpha_i 1 + \beta_i a + \gamma_i b$, where α_i, β_i, $\gamma_i \in K$, $1 = e_{11} + e_{22}$, $a = e_{11} - e_{22}$, and $b = e_{12}$. By Lemma 3.1 we have

$$f(X_1, \ldots, X_n) = Bb = 0,$$

where $B = \sum_p \overline{B_p} \alpha_1^{p_1} \cdots \alpha_n^{p_n}$ with $p = (p_1, \ldots, p_n)$, $0 \leq p_1, \ldots, p_n < q$, and

$$\begin{aligned}\overline{B_p} &= \sum_{t=1}^{n-1} \sum_{l>t} \sum_{s_t=0}^{q-3} \sum_{(s_{t+1},\ldots,s_n=0)}^{q-1} \pm\lambda_{(p,t,t,l,s_t,s)} (\beta_t\gamma_l - \gamma_t\beta_l) \beta_t^{s_t+1} \beta_{t+1}^{s_{t+1}} \cdots \beta_n^{s_n} \\ &+ \sum_{t=1}^{n-1} \sum_{l>t} \sum_{(s_{t+1},\ldots,\widehat{s_l},\ldots,s_n=0)}^{q-1} \sum_{s_l=0}^{q-2} \pm\lambda_{(p,t,t,l,q-2,s)} (\beta_t\gamma_l - \gamma_t\beta_l) \beta_t^{q-1} \beta_{t+1}^{s_{t+1}} \cdots \beta_n^{s_n} \\ &+ \sum_{t=1}^{n-1} \sum_{l>t} \sum_{u>t} \sum_{s_u=0}^{q-2} \sum_{(s_{u+1},\ldots,s_n=0)}^{q-1} \pm\lambda_{(p,t,u,l,s_u,s)} (\beta_t\gamma_l - \gamma_t\beta_l) \beta_u^{s_u+1} \beta_{u+1}^{s_{u+1}} \cdots \beta_n^{s_n}.\end{aligned}$$

Since $B = 0$, $|K| = q$ and $\deg_{\alpha_i} B < q$ for all i, by Lemma 2.2 we have $\overline{B_p} = 0$ for all p. By using the fact that $\beta_t^q = \beta_t$ we obtain

$$\begin{aligned}0 = \overline{B_p} &= \sum_{t=1}^{n-1} \sum_{l>t} \sum_{s_t=0}^{q-3} \sum_{(s_{t+1},\ldots,s_n=0)}^{q-1} \pm\lambda_{(p,t,t,l,s_t,s)} (\beta_t\gamma_l - \gamma_t\beta_l) \beta_t^{s_t+1} \beta_{t+1}^{s_{t+1}} \cdots \beta_n^{s_n} \\ &+ \sum_{t=1}^{n-1} \sum_{l>t} \sum_{(s_{t+1},\ldots,\widehat{s_l},\ldots,s_n=0)}^{q-1} \sum_{s_l=0}^{q-2} \pm\lambda_{(p,t,t,l,q-2,s)} \gamma_l \beta_t \beta_{t+1}^{s_{t+1}} \cdots \beta_n^{s_n} \\ &+ \sum_{t=1}^{n-1} \sum_{l>t} \sum_{(s_{t+1},\ldots,\widehat{s_l},\ldots,s_n=0)}^{q-1} \sum_{s_l=0}^{q-2} \mp\lambda_{(p,t,t,l,q-2,s)} \gamma_t \beta_t^{q-1} \beta_{t+1}^{s_{t+1}} \cdots \beta_l^{s_l+1} \cdots \beta_n^{s_n} \\ &+ \sum_{t=1}^{n-1} \sum_{l>t} \sum_{u>t} \sum_{s_u=0}^{q-2} \sum_{(s_{u+1},\ldots,s_n=0)}^{q-1} \pm\lambda_{(p,t,u,l,s_u,s)} (\beta_t\gamma_l - \gamma_t\beta_l) \beta_u^{s_u+1} \beta_{u+1}^{s_{u+1}} \cdots \beta_n^{s_n}.\end{aligned} \tag{18}$$

Firstly suppose that $0 \leq s_1 \leq q-3$. We will show that the coefficients $\lambda_{(p,1,1,l,s_1,s)}$ are zeros for all $2 \leq l \leq n$ and for all s. We have that the homogeneous component M of $\overline{B_p}$ with $\deg_{\beta_1} M = s_1 + 2$ and $\deg_{\gamma_l} M = 1$ is

$$M = \sum_{(s_2,\ldots,s_n=0)}^{q-1} \pm\lambda_{(p,1,1,l,s_1,s)} \gamma_l \beta_1^{s_1+2} \beta_2^{s_2} \cdots \beta_n^{s_n}.$$

Since $\deg_{\beta_1} \overline{B_p} < q$ and $\deg_{\gamma_l} \overline{B_p} = 1 < q$ it follows that M is a polynomial identity for K. Thus, as $\deg_{\gamma_l} M = 1 < q$ and $\deg_{\beta_j} M < q$ for all j, we have by Lemma 2.2 that $\lambda_{(p,1,1,l,s_1,s)} = 0$ for all $2 \leq l \leq n$, $0 \leq s_1 \leq q-3$ and s.

In particular, by (18) we have

$$0 = \overline{B_p} = \sum_{t=2}^{n-1} \sum_{l>t} \sum_{s_t=0}^{q-3} \sum_{(s_{t+1},\dots,s_n=0)}^{q-1} \pm\lambda_{(p,t,t,l,s_t,s)}(\beta_t\gamma_l - \gamma_t\beta_l)\beta_t^{s_t+1}\beta_{t+1}^{s_{t+1}}\cdots\beta_n^{s_n} \tag{19}$$
$$+ \sum_{t=1}^{n-1} \sum_{l>t} \sum_{(s_{t+1},\dots,\widehat{s_l},\dots,s_n=0)}^{q-1} \sum_{s_l=0}^{q-2} \pm\lambda_{(p,t,t,l,q-2,s)}\gamma_l\beta_t\beta_{t+1}^{s_{t+1}}\cdots\beta_n^{s_n}$$
$$+ \sum_{t=1}^{n-1} \sum_{l>t} \sum_{(s_{t+1},\dots,\widehat{s_l},\dots,s_n=0)}^{q-1} \sum_{s_l=0}^{q-2} \mp\lambda_{(p,t,t,l,q-2,s)}\gamma_t\beta_t^{q-1}\beta_{t+1}^{s_{t+1}}\cdots\beta_l^{s_l+1}\cdots\beta_n^{s_n}$$
$$+ \sum_{t=1}^{n-1} \sum_{l>t} \sum_{u>t} \sum_{s_u=0}^{q-2} \sum_{(s_{u+1},\dots,s_n=0)}^{q-1} \pm\lambda_{(p,t,u,l,s_u,s)}(\beta_t\gamma_l - \gamma_t\beta_l)\beta_u^{s_u+1}\beta_{u+1}^{s_{u+1}}\cdots\beta_n^{s_n}.$$

Now, we will show that $\lambda_{(p,1,1,l,q-2,s)} = 0$ for all l, s. We have that the homogeneous component M of $\overline{B_p}$ with $\deg_{\beta_1} M = q-1$ and $\deg_{\gamma_1} M = 1$ is

$$M = \sum_{l=2}^{n} \sum_{(s_2,\dots,\widehat{s_l},\dots,s_n=0)}^{q-1} \sum_{s_l=0}^{q-2} \mp\lambda_{(p,1,1,l,q-2,s)}\gamma_1\beta_1^{q-1}\beta_2^{s_2}\cdots\beta_l^{s_l+1}\cdots\beta_n^{s_n}.$$

Since $\deg_{\beta_1}\overline{B_p} < q$ and $\deg_{\gamma_1}\overline{B_p} = 1 < q$ it follows that M is a polynomial identity for K. Thus, as $\deg_{\gamma_1} M = 1 < q$ and $\deg_{\beta_j} M < q$ for all j, we have by Lemma 2.2 that $\lambda_{(p,1,1,l,q-2,s)} = 0$ for all l, s.

In particular, by (19) we have

$$0 = \overline{B_p} = \sum_{t=2}^{n-1} \sum_{l>t} \sum_{s_t=0}^{q-3} \sum_{(s_{t+1},\dots,s_n=0)}^{q-1} \pm\lambda_{(p,t,t,l,s_t,s)}(\beta_t\gamma_l - \gamma_t\beta_l)\beta_t^{s_t+1}\beta_{t+1}^{s_{t+1}}\cdots\beta_n^{s_n} \tag{20}$$
$$+ \sum_{t=2}^{n-1} \sum_{l>t} \sum_{(s_{t+1},\dots,\widehat{s_l},\dots,s_n=0)}^{q-1} \sum_{s_l=0}^{q-2} \pm\lambda_{(p,t,t,l,q-2,s)}\gamma_l\beta_t\beta_{t+1}^{s_{t+1}}\cdots\beta_n^{s_n}$$
$$+ \sum_{t=2}^{n-1} \sum_{l>t} \sum_{(s_{t+1},\dots,\widehat{s_l},\dots,s_n=0)}^{q-1} \sum_{s_l=0}^{q-2} \mp\lambda_{(p,t,t,l,q-2,s)}\gamma_t\beta_t^{q-1}\beta_{t+1}^{s_{t+1}}\cdots\beta_l^{s_l+1}\cdots\beta_n^{s_n}$$
$$+ \sum_{t=1}^{n-1} \sum_{l>t} \sum_{u>t} \sum_{s_u=0}^{q-2} \sum_{(s_{u+1},\dots,s_n=0)}^{q-1} \pm\lambda_{(p,t,u,l,s_u,s)}(\beta_t\gamma_l - \gamma_t\beta_l)\beta_u^{s_u+1}\beta_{u+1}^{s_{u+1}}\cdots\beta_n^{s_n}.$$

Finally, we will show that $\lambda_{(p,1,u,l,s_u,s)} = 0$ for all $u > 1$, $l > 1$, s_u and s. The homogeneous component M of $\overline{B_p}$ with $\deg_{\beta_1} M = 1$ and $\deg_{\gamma_l} M = 1$ is

$$M = \sum_{u>1} \sum_{s_u=0}^{q-2} \sum_{(s_{u+1},\dots,s_n=0)}^{q-1} \pm\lambda_{(p,1,u,l,s_u,s)}\beta_1\gamma_l\beta_u^{s_u+1}\beta_{u+1}^{s_{u+1}}\cdots\beta_n^{s_n}.$$

Since $\deg_{\beta_1} \overline{B_p} = 1 < q$ and $\deg_{\gamma_l} \overline{B_p} = 1 < q$ it follows that M is a polynomial identity for K. Thus, as $\deg_{\gamma_l} M = 1 < q$ and $\deg_{\beta_j} B_{(p,1,u,l,s_u,s)} < q$ for all j it follows by Lemma 2.2 that $\lambda_{(p,1,u,l,s_u,s)} = 0$ for all $u > 1$, $l > 1$, $0 \le s_u \le q-2$ and $0 \le s_{u+1}, \ldots, s_n \le q-1$.

By (20) we have

$$\begin{aligned} 0 = \overline{B_p} = & \sum_{t=2}^{n-1} \sum_{l>t} \sum_{s_t=0}^{q-3} \sum_{(s_{t+1},\ldots,s_n=0)}^{q-1} \pm\lambda_{(p,t,t,l,s_t,s)}(\beta_t\gamma_l - \gamma_t\beta_l)\beta_t^{s_t+1}\beta_{t+1}^{s_{t+1}}\cdots\beta_n^{s_n} \\ & + \sum_{t=2}^{n-1} \sum_{l>t} \sum_{(s_{t+1},\ldots,\widehat{s_l},\ldots,s_n=0)}^{q-1} \sum_{s_l=0}^{q-2} \pm\lambda_{(p,t,t,l,q-2,s)}\gamma_l\beta_t\beta_{t+1}^{s_{t+1}}\cdots\beta_n^{s_n} \\ & + \sum_{t=2}^{n-1} \sum_{l>t} \sum_{(s_{t+1},\ldots,\widehat{s_l},\ldots,s_n=0)}^{q-1} \sum_{s_l=0}^{q-2} \mp\lambda_{(p,t,t,l,q-2,s)}\gamma_t\beta_t^{q-1}\beta_{t+1}^{s_{t+1}}\cdots\beta_l^{s_l+1}\cdots\beta_n^{s_n} \\ & + \sum_{t=2}^{n-1} \sum_{l>t} \sum_{u>t} \sum_{s_u=0}^{q-2} \sum_{(s_{u+1},\ldots,s_n=0)}^{q-1} \pm\lambda_{(p,t,u,l,s_u,s)}(\beta_t\gamma_l - \gamma_t\beta_l)\beta_u^{s_u+1}\beta_{u+1}^{s_{u+1}}\cdots\beta_n^{s_n}. \end{aligned}$$

By using the same argument in $\overline{B_p}$ for $t = 2, \ldots, n-1$, respectively, we obtain that $\lambda_{(p,t,u,l,s_t,s)} = 0$ for all (p, t, u, l, s_t, s).

Therefore, the set $\overline{S}$ is a basis for the quotient vector space $J(X)/T(UJ_2)$. Moreover, since $I' \subseteq T(UJ_2)$, by Lemma 3.19 we have $I' = T(UJ_2)$. □

4. Specht property

From now on, K is an infinite field of characteristic $p \neq 2$. In this section, we will prove that $T(UJ_2)$ satisfies the Specht property, that is, every T-ideal containing $T(UJ_2)$ is finitely generated as a T-ideal. We note that this result was obtained in [6] in characteristic 0. Our proof is based on a different approach that works in the case of $p > 2$ as well.

Notation 4.1. *If $f(x_1, \ldots, x_n) = (x_1, x_{l_1}, x_j, x_{l_2}, \ldots, x_{l_t})$, where $\deg_{x_i} f = d_i \ge 1$ for every i and $l_1 \le l_2 \le \ldots \le l_t$, we will denote f by*

$$f = f^{(j)}_{(d_1,\ldots,d_n)}.$$

Lemma 4.2. *Let K be an infinite field of $\mathrm{char}(K) \neq 2$. Let $f = f(x_1, \ldots, x_n) \notin T(UJ_2)$ be a polynomial given by*

$$f = \sum_{j=2}^{m} \alpha_j f^{(j)}_{(d_1,\ldots,d_n)},$$

where $\alpha_j \in K$, $1 \le d_1 = \cdots = d_m = d$ and $2 \le m \le n$.

(a) *If there exists* $2 \leq i \leq m$ *such that* $(\alpha_2 + \cdots + \alpha_{i-1} + 2\alpha_i + \alpha_{i+1} + \cdots + \alpha_m) \neq 0$, *then*

$$\langle f \rangle^T + T(UJ_2) = \langle f^{(2)}_{(d_1,\ldots,d_n)} \rangle^T + T(UJ_2).$$

(b) *If* $(\alpha_2+\cdots+\alpha_{i-1}+2\alpha_i+\alpha_{i+1}+\cdots+\alpha_m) = 0$ *for all* $2 \leq i \leq m$, *then* $\alpha_2 = \cdots = \alpha_m$. *In this case,*

$$\langle f \rangle^T + T(UJ_2) = \left\langle \sum_{j=2}^{m} f^{(j)}_{(d_1,\ldots,d_n)} \right\rangle^T + T(UJ_2).$$

Proof. Suppose $d \geq 2$ and $m \geq 3$. All remaining cases are analogous to this and we left them to the reader. For each $i = 2, \ldots, m$, denote by $v_i = v_i(x_1, \ldots, x_n)$ the polynomial

$$v_i = f(x_i, x_2, x_3, \ldots, x_{i-2}, x_{i-1}, x_1, x_{i+1}, x_{i+2}, \ldots, x_n).$$

We have the following equalities, modulo $T(UJ_2)$:

$$\begin{aligned} v_i = &+ \alpha_i(x_i, x_i, x_1, x_i^{(d-2)}, \ldots, x_1^{(d-1)}, \ldots)+ \\ &+ \sum_{\substack{j=2 \\ j\neq i}}^{m} \alpha_j(x_i, x_i, x_j, x_i^{(d-2)}, \ldots, x_1^{(d)}, \ldots, x_j^{(d-1)}, \ldots) \\ = &- \alpha_i(x_1, x_1, x_i, x_1^{(d-2)}, \ldots, x_i^{(d-1)}, \ldots)+ \\ &+ \sum_{\substack{j=2 \\ j\neq i}}^{m} \alpha_j(x_i, x_1, x_j, x_1^{(d-1)}, \ldots, x_i^{(d-1)}, \ldots, x_j^{(d-1)}, \ldots). \end{aligned}$$

Note that we used Lemma 3.10 and (2). By (2),

$$\begin{aligned} v_i = &- \alpha_i(x_1, x_1, x_i, x_1^{(d-2)}, \ldots, x_i^{(d-1)}, \ldots)+ \\ &- \sum_{\substack{j=2 \\ j\neq i}}^{m} \alpha_j(x_1, x_j, x_i, x_1^{(d-1)}, \ldots, x_i^{(d-1)}, \ldots, x_j^{(d-1)}, \ldots)+ \\ &+ \sum_{\substack{j=2 \\ j\neq i}}^{m} \alpha_j(x_1, x_i, x_j, x_1^{(d-1)}, \ldots, x_i^{(d-1)}, \ldots, x_j^{(d-1)}, \ldots). \end{aligned}$$

By Lemma 3.10 we have the following equality, modulo $T(UJ_2)$:

$$f - v_i = (\alpha_2+\cdots+\alpha_{i-1}+2\alpha_i+\alpha_{i+1}+\cdots+\alpha_m)(x_1, x_1, x_i, x_1^{(d-2)}, \ldots, x_i^{(d-1)}, \ldots). \quad (21)$$

If $(\alpha_2 + \cdots + \alpha_{i-1} + 2\alpha_i + \alpha_{i+1} + \cdots + \alpha_m) \neq 0$ for some $i = 2, \ldots, m$, then

$$\langle f\rangle^T + T(UJ_2) = \langle f - v_i\rangle^T + T(UJ_2) = \langle f^{(2)}_{(d_1,\ldots,d_n)}\rangle^T + T(UJ_2),$$

and the lemma is proved.

Suppose $(\alpha_2 + \cdots + \alpha_{i-1} + 2\alpha_i + \alpha_{i+1} + \cdots + \alpha_m) = 0$ for every $i = 2, \ldots, m$. In this case, we obtain the linear system

$$\begin{cases} 2\alpha_2 + \alpha_3 + \cdots + \alpha_m = 0 \\ \alpha_2 + 2\alpha_3 + \cdots + \alpha_m = 0 \\ \vdots \\ \alpha_2 + \alpha_3 + \cdots + 2\alpha_m = 0 \end{cases}, \tag{22}$$

with coefficient matrix

$$A = \begin{bmatrix} 2 & 1 & \ldots & 1 \\ 1 & 2 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 2 \end{bmatrix}.$$

Note that $\det A = m$.

If $p \nmid m$, then $\det A \neq 0$ and $\alpha_2 = \cdots = \alpha_m = 0$, which is an absurd.

If $p \mid m$, then $\det A = 0$. Since A is a square matrix of size $(m-1) \times (m-1)$, and has rank $m-2$, it follows that $\{(1, 1, \ldots, 1)\}$ is a basis for the vector space of the solutions. In this case, we obtain the statement of b). □

Theorem 4.3. *Let K be an infinite field of $\operatorname{char}(K) = p > 2$. If J is a T-ideal of $J(X)$ such that $T(UJ_2) \subseteq J$, then J is generated, as a T-ideal, by $T(UJ_2)$ and some polynomials of the form:*

1) $f^{(2)}_{(p^{r_1},\ldots,p^{r_n})}$, *where* $0 \leq r_1 = r_2 \leq r_3 \leq \cdots \leq r_n$ *and* $n \geq 2$*; or*
2) $\sum_{j=2}^{m} f^{(j)}_{(p^{r_1},\ldots,p^{r_n})}$, *where* $0 \leq r_1 = \cdots = r_m \leq r_{m+1} \leq \cdots \leq r_n$, $2 \leq m \leq n$ *and* $p \mid m$.

Proof. By (10), it follows that J is generated, as a T-ideal, by $T(UJ_2)$ and some multihomogeneous polynomials of multidegree $(d_1, \ldots, d_n)$ in the subspace

$$\operatorname{span}\{f^{(j)}_{(d_1,\ldots,d_n)} \mid 2 \leq j \leq n\}.$$

By using similar arguments as in Theorem 6 from [3, Section 4.2], Lemma 3.10 and (2), we have that J is generated by $T(UJ_2)$ and some multihomogeneous polynomials of multidegree $(p^{r_1}, \ldots, p^{r_n})$ in the subspace

$$\mathrm{span}\{f^{(j)}_{(p^{r_1},\dots,p^{r_n})} \mid 2 \le j \le n\}.$$

Renaming the variables if necessary, by Lemma 3.10 and (2) we can suppose $0 \le r_1 \le \dots \le r_n$.

Let $f(x_1,\dots,x_n) \in J - T(UJ_2)$ given by

$$f(x_1,\dots,x_n) = \sum_{j=2}^{n} \alpha_j f^{(j)}_{(p^{r_1},\dots,p^{r_n})},$$

where $\alpha_j \in K$ for all j, and $0 \le r_1 \le \dots \le r_n$.

Case 1. Suppose $r_1 < r_j$ and $\alpha_j \neq 0$ for some $j \ge 2$.

Without loss of generality, we will study the case $r_1 < r_2$ and $\alpha_2 \neq 0$. Write

$$p^{r_2} = p^{r_2-1}(p-1) + p^{r_2-2}(p-1) + \dots + p^{r_1+1}(p-1) + p^{r_1}p.$$

By exchanging the variable x_2 of $f(x_1,x_2,\dots,x_n)$ by a sum of distinct variables in $X - \{x_1,\dots,x_n\}$, as below,

$$x_2 =: y_1 + \dots + y_p + \sum_{i=1}^{r_2-r_1-1}\left[\sum_{l=1}^{p-1} y_{(i,l)}\right],$$

we obtain a new polynomial $\overline{f}$. Denote by h the multihomogeneous component of $\overline{f}$ such that $\deg_{x_i} h = p^{r_i}$ if $i \neq 2$, $\deg_{y_i} h = p^{r_1}$ if $1 \le i \le p$, $\deg_{y_{(i,l)}} h = p^{r_1+i}$ if $1 \le i \le r_2 - r_1 - 1$, and $1 \le l \le p-1$. Write

$$h = h(x_1, y_1, \dots, y_p, y_{(1,1)}, \dots, y_{(1,p-1)}, y_{(2,1)}, \dots, y_{(2,p-1)}, \dots, y_{(r_2-r_1-1,p-1)}, x_3, \dots, x_n),$$

and denote

$$d = (p^{r_1}, p^{r_1}, \dots, p^{r_1}, p^{r_1+1}, \dots, p^{r_1+1}, p^{r_1+2}, \dots, p^{r_1+2}, \dots, p^{r_2-1}, p^{r_3}, \dots, p^{r_n}).$$

By Lemmas 3.10 and 2.3 the polynomial h is equal, modulo $T(UJ_2)$, to

$$h = \alpha\alpha_2\left(\sum_{i=2}^{p+1} f^{(i)}_d(x_1, y_1, \dots, y_p, y_{(1,1)}, \dots, y_{(2,1)}, \dots, y_{(r_2-r_1-1,p-1)}, x_3, \dots, x_n)\right),$$

where the multinomial coefficient

$$\alpha = \binom{p^{r_2}-1}{p^{r_1}-1, \underbrace{p^{r_1},\dots,p^{r_1}}_{(p-1)\text{ factors}}, \underbrace{p^{r_1+1},\dots,p^{r_1+1}}_{(p-1)\text{ factors}}, \dots, \underbrace{p^{r_2-1},\dots,p^{r_2-1}}_{(p-1)\text{ factors}}} \neq 0.$$

By Lemma 4.2 a), it follows that

$$\langle f_d^{(2)}\rangle^T + T(UJ_2) = \langle h\rangle^T + T(UJ_2) \subseteq \langle f\rangle^T + T(UJ_2),$$

where $f_d^{(2)} = f_d^{(2)}(x_1, y_1, \ldots, y_p, y_{(1,1)}, \ldots, y_{(2,1)}, \ldots, y_{(r_2-r_1-1,p-1)}, x_3, \ldots, x_n)$. In this case,

$$\langle f\rangle^T + T(UJ_2) = \langle \widehat{f},\ f_d^{(2)}\rangle^T + T(UJ_2),$$

where

$$\widehat{f}(x_1, \ldots, x_n) = \sum_{j=3}^{n} \alpha_j f^{(j)}_{(p^{r_1},\ldots,p^{r_n})}.$$

Now one can apply the same argument on $\widehat{f}$ and, after several steps, renaming the variables if necessary, and using Lemma 3.10, one will obtain that

$$\langle f\rangle^T + T(UJ_2) = \langle F\rangle^T + T(UJ_2),$$

where F is a set formed by multihomogeneous polynomials that are linear combination of elements in item 1) of the statement.
Case 2. Suppose that there is no $j \geq 2$ such that $r_1 < r_j$ and $\alpha_j \neq 0$.

In this case, renaming the variables if necessary and using Lemma 3.10, we can suppose

$$f(x_1, \ldots, x_n) = \sum_{j=2}^{m} \alpha_j f^{(j)}_{(p^{r_1},\ldots,p^{r_n})},$$

where $0 \leq r_1 = \cdots = r_m \leq r_{m+1} \leq \cdots \leq r_n$, $2 \leq m \leq n$, and $\alpha_j \neq 0$ for every $j = 2$, $\ldots$, m. By Lemma 4.2, it follows that

$$\langle f\rangle^T + T(UJ_2) = \langle f^{(2)}_{(p^{r_1},\ldots,p^{r_n})}\rangle^T + T(UJ_2)$$

or

$$\langle f\rangle^T + T(UJ_2) = \left\langle \sum_{j=2}^{m} f^{(j)}_{(p^{r_1},\ldots,p^{r_n})} \right\rangle^T + T(UJ_2),\ \text{and } p \mid m.$$

In this way we finish the proof of this theorem. □

A partially ordered set $(P, \leq)$ is called a *partially well-ordered set* if, given an arbitrary nonempty subset Q of P, there exist $q_1, \ldots, q_n \in Q$ with the following property: if $q \in Q$, then $q_i \leq q$ for some $1 \leq i \leq n$.

Partially well-ordered sets were first studied systematically from an algebraic point of view by Higman [11]. Later on the principal results from [11] have been rediscovered several times by different authors.

We need some basic properties of the partially well-ordered sets. We refer the reader to [3, Chapter 5] for more information and applications to the study of PI algebras.

The next lemma is the corollary to Proposition 4 in [3, Section 5.2].

Lemma 4.4. *Let $(P_1, \leq_1)$ and $(P_2, \leq_2)$ be partially well-ordered sets. If $P = P_1 \times P_2$, define the relation $\leq$ on P as follows: if p_1, $p_1' \in P_1$ and p_2, $p_2' \in P_2$, then*

$$(p_1, p_2) \leq (p_1', p_2') \text{ if and only if } p_1 \leq_1 p_1' \text{ and } p_2 \leq_2 p_2'.$$

Then $(P, \leq)$ is a partially well-ordered set.

The next theorem is a consequence of Theorem 4 in [3, Section 5.2].

Theorem 4.5. *Consider the usual order on the positive integers $\mathbb{N}$. Let $D(\mathbb{N})$ denote the set of all finite sequences $(a_1, \ldots, a_r)$ where the components are in $\mathbb{N}$ and $r \geq 1$. Define the following order $\preceq$ on $D(\mathbb{N})$: $(a_1, \ldots, a_r) \preceq (b_1, \ldots, b_s)$ if and only if there exists an injection $\psi\colon \mathbb{N} \to \mathbb{N}$ such that:*

(i) *ψ preserves the order, that is, if $u \leq v$ then $\psi(u) \leq \psi(v)$;*
(ii) *$\psi(r) \leq s$;*
(iii) *$a_i \leq b_{\psi(i)}$ for every $i = 1, \ldots, r$.*

Then $(D(\mathbb{N}), \preceq)$ is a partially well-ordered set.

Theorem 4.6. *Let K be an infinite field of $\mathrm{char}(K) = p > 2$. If J is a T-ideal of $J(X)$ such that $T(UJ_2) \subseteq J$, then J is finitely generated as a T-ideal.*

Proof. Consider the partially well-ordered set $D(\mathbb{N})$ in Theorem 4.5, and define the following order $\leq$ on $\mathbb{N} \times D(\mathbb{N})$: if l, $l' \in \mathbb{N}$ and d, $d' \in D(\mathbb{N})$, then

$$(l, d) \leq (l', d') \text{ if and only if } l \leq l' \text{ and } d \leq d'. \tag{23}$$

By Lemma 4.4, it follows that $\mathbb{N} \times D(\mathbb{N})$ is a partially well-ordered set.

If J is a T-ideal of $J(X)$ such that $T(UJ_2) \subseteq J$, then by Theorem 4.3 there exist subsets A, $B \subseteq J$ such that

$$J = T(UJ_2) + \langle A \cup B \rangle^T,$$

where:

(1) A is formed by some polynomials

$$f^{(2)}_{(p^{r_1}, \ldots, p^{r_n})},$$

with $0 \leq r_1 = r_2 \leq r_3 \leq \cdots \leq r_n$ and $n \geq 2$.

(2) B is formed by some polynomials

$$\sum_{j=2}^{m} f^{(j)}_{(p^{r_1},\ldots,p^{r_n})},$$

with $0 \leq r_1 = \cdots = r_m \leq r_{m+1} \leq \cdots \leq r_n$, $2 \leq m \leq n$, and $p \mid m$.

Given an element in A, denote

$$\eta\left(f^{(2)}_{(p^{r_1},\ldots,p^{r_n})}\right) = (p^{r_1}, (p^{r_3}, \ldots, p^{r_n})).$$

Note that $p^{r_1} \in \mathbb{N}$ and $(p^{r_3}, \ldots, p^{r_n}) \in D(\mathbb{N})$, that is,

$$\eta\left(f^{(2)}_{(p^{r_1},\ldots,p^{r_n})}\right) \in \mathbb{N} \times D(\mathbb{N}).$$

By Lemma 3.10, it follows that

$$f^{(2)}_{(p^{r_1+1},p^{r_1+1},p^{r_3},\ldots,p^{r_n})} = \left(f^{(2)}_{(p^{r_1},p^{r_1},p^{r_3},\ldots,p^{r_n})}, x_1^{((p-1)p^{r_1})}, x_2^{((p-1)p^{r_1})}\right)$$

modulo $T(UJ_2)$, and also

$$f^{(2)}_{(p^{r_1},p^{r_1},\ldots,p^{r_j+1},\ldots,p^{r_n})} = \left(f^{(2)}_{(p^{r_1},p^{r_1},\ldots,p^{r_j},\ldots,p^{r_n})}, x_j^{((p-1)p^{r_j})}\right)$$

modulo $T(UJ_2)$. Therefore, by (23), if $f,\ f' \in A$ and $\eta(f) \leq \eta(f')$, then $f' \in \langle f\rangle^T + T(UJ_2)$. Since the set $\mathbb{N} \times D(\mathbb{N})$ is partially well-ordered, it follows that the subset $\eta(A) = \{\eta(f)\colon f \in A\}$ has minimal elements $\eta(f_1), \ldots, \eta(f_k)$ for some $f_1, \ldots, f_k \in A$. Therefore

$$J = T(UJ_2) + \langle f_1, \ldots, f_k\rangle^T + \langle B\rangle^T.$$

Given an element in B, write

$$h_{(m,p^{r_1},p^{r_{m+1}},\ldots,p^{r_n})} = \sum_{j=2}^{m} f^{(j)}_{(p^{r_1},\ldots,p^{r_n})},$$

where $0 \leq r_1 = \cdots = r_m \leq r_{m+1} \leq \cdots \leq r_n$, $2 \leq m \leq n$, and $p \mid m$. Moreover, denote

$$\xi\left(h_{(m,p^{r_1},p^{r_{m+1}},\ldots,p^{r_n})}\right) = (m, (p^{r_1}, (p^{r_{m+1}}, \ldots, p^{r_n}))).$$

Note that $m \in \mathbb{N}$ and $(p^{r_1}, (p^{r_{m+1}}, \ldots, p^{r_n})) \in \mathbb{N} \times D(\mathbb{N})$, that is

$$\xi\left(h_{(m,p^{r_1},p^{r_{m+1}},\ldots,p^{r_n})}\right) \in \mathbb{N} \times (\mathbb{N} \times D(\mathbb{N})).$$

We define the following order $\leq$ on $\mathbb{N}\times(\mathbb{N}\times D(\mathbb{N}))$: if m, m', l, $l'\in\mathbb{N}$ and d, $d'\in D(\mathbb{N})$, then

$$(m,(l,d))\leq(m',(l',d')) \text{ if and only if } m\leq m',\ l\leq l',\ \text{and } d\leq d'. \tag{24}$$

By Lemma 4.4, it follows that $\mathbb{N}\times(\mathbb{N}\times D(\mathbb{N}))$ is a partially well-ordered set.

As before, we have

$$h_{(m,p^{r_1+1},p^{r_{m+1}},\ldots,p^{r_n})}\in\langle h_{(m,p^{r_1},p^{r_{m+1}},\ldots,p^{r_n})}\rangle^T+T(UJ_2),$$

and also

$$h_{(m,p^{r_1},p^{r_{m+1}},\ldots,p^{r_j+1},\ldots,p^{r_n})}\in\langle h_{(m,p^{r_1},p^{r_{m+1}},\ldots,p^{r_j},\ldots,p^{r_n})}\rangle^T+T(UJ_2).$$

Fix an integer $m\geq 1$ such that $p\mid m$, and denote by Y the set of all m-tuples below:

$$\begin{gathered}(x_1,x_2,x_3,\ldots,x_{m-1},x_m),\quad (x_1,x_3,x_4,\ldots,x_m,x_{m+1}),\ldots,\\ (x_1,x_{p+2},x_{p+3},\ldots,x_{m+p-1},x_{m+p}),(x_1,x_{p+3},x_{p+4},x_{m+p},x_2),\\ (x_1,x_{p+4},x_{p+5},\ldots,x_2,x_3),\ldots,(x_1,x_{m+p},x_2,\ldots,x_{m-2},x_{m-1}).\end{gathered}$$

Note that Y has $m+p-1$ elements. By Lemma 3.10, we have that the element $h_{(m+p,p^{r_1},p^{r_{m+p+1}},\ldots,p^{r_n})}$ equals, modulo $T(UJ_2)$

$$\frac{1}{m+p-1}\sum_{y\in Y}\left(h_{(m,p^{r_1},p^{r_{m+p+1}},\ldots,p^{r_n})}(y_1,\ldots,y_m,x_{m+p+1},\ldots,x_n),y_{m+1}^{(p^{r_1})},\ldots,y_{m+p}^{(p^{r_1})}\right),$$

where $y=(y_1,\ldots,y_m)\in Y$, $y_{m+1}<\cdots<y_{m+p}$ and $\{y_1,\ldots,y_{m+p}\}=\{x_1,\ldots,x_{m+p}\}$. In other words

$$h_{(m+p,p^{r_1},p^{r_{m+p+1}},\ldots,p^{r_n})}\in\langle h_{(m,p^{r_1},p^{r_{m+p+1}},\ldots,p^{r_n})}\rangle^T+T(UJ_2).$$

Therefore, by (24), if h, $h'\in B$ and $\xi(h)\leq\xi(h')$ then $h'\in\langle h\rangle^T+T(UJ_2)$. Since the set $\mathbb{N}\times(\mathbb{N}\times D(\mathbb{N}))$ is partially well-ordered, it follows that the subset $\xi(B)=\{\xi(h)\mid h\in B\}$ has minimal elements $\xi(h_1),\ldots,\xi(h_t)$ for some $h_1,\ldots,h_t\in B$. Therefore

$$J=T(UJ_2)+\langle f_1,\ldots,f_k\rangle^T+\langle h_1,\ldots,h_t\rangle^T,$$

as desired. □

Now we give another proof of [6, Theorem 15].

Theorem 4.7. *Let K be a field of $\operatorname{char}(K)=0$. If J is a T-ideal of $J(X)$ such that $T(UJ_2)\subsetneqq J$, then*

$$J = T(UJ_2) + \langle (x_1, x_2, x_3, \ldots, x_m) \rangle^T$$

for some odd integer $m \geq 3$. Moreover, J is finitely generated, as a T-ideal.

Proof. Since $\mathrm{char}(K) = 0$ we have that J is generated by its multilinear elements. By (10), J is generated, as a T-ideal, by $T(UJ_2)$ and some polynomials

$$f = \sum_{j=2}^{n} \alpha_j (x_1, x_2, x_j, x_3, \ldots, \widehat{x_j}, \ldots, x_n),$$

where $\alpha_j \in K$ for every j, and $n \geq 3$ is an odd integer. By Lemma 4.2, it follows that

$$\langle f \rangle^T + T(UJ_2) = \langle (x_1, x_2, x_3, \ldots, x_n) \rangle^T + T(UJ_2).$$

Therefore, if $m = \min\{n \mid (x_1, x_2, x_3, \ldots, x_n) \in J\}$ then

$$J = T(UJ_2) + \langle (x_1, x_2, x_3, \ldots, x_m) \rangle^T.$$

By Theorem 3.13, it follows that J is finitely generated as a T-ideal. □

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