ON THE NUMBER OF COUNTABLE MODELS OF CONSTANT AND UNARY PREDICATES EXPANSIONS OF THE DENSE MEET-TREE THEORY

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Abstract: In the paper, we investigate Ehrenfeucht theories, that is, theories which have finitely many countable models but which are not countably categorical. More precisely, we count all possible numbers of countable models of the theory DMT of dense meet-trees expanded by several sequences of constants including decreasing ones and by unary predicates with finite realizations. Also, we study the realizations of models over a certain set of formulas based on the Rudin-Keisler preorders on models.

Keywords: Constant expansion, Ehrenfeucht theory, the number of countable models, the number of limit models, the number of prime models, small theory, Rudin-Keisler preorder.

1 Introduction

Since Andrzej Ehrenfeucht had constructed his example of a theory with three non-isomorphic countable models this field of Model Theory, called Ehrenfeucht theories, is one of most important in Model Theory. One of the main problems here is to build new Ehrenfeucht theories, which are not based on a dense linear ordering. One of the first such examples was constructed by M. Peretyat'kin in [14]. Alistair Lachlan posed the problem if there exists a stable Ehrenfeucht theory. Sergey Sudoplatov solved this problem by having constructed such a theory [17]. Some other papers related to the topic are [1]–[11], [13], [15], [16], [18]–[20]. However, in this paper, we focus on the example by M. Peretyat'kin, where he considered a partially densely ordered set, the so-called Dense Meet Tree [12], DMT for short, expanded by a countable sequence of constants. While M. Peretyat'kin considered only one increasing sequence of constants, we consider several sequences, including decreasing ones, as well as expansions by unary predicates with finite realizations. We give all possible numbers of countable models of such expansions of DMT (Theorem 4).

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So, we give the axioms of DMT. A *dense meet-tree* $\mathcal{M} = \langle M; \leq, \sqcap\rangle$ is a lower semilattice (that is, for each elements a and b there exists their greatest lower bound, which we denote by $a \sqcap b$ and call the *meet* of a and b) without the least and greatest elements such that:

(a) for each pair of incomparable elements, their join does not exist;

(b) for each pair of distinct comparable elements, there is an element between them;

(c) for each element a there exist infinitely many pairwise incomparable elements greater than a, whose infimum is equal to a.

Note that x and y are incomparable if $x \not\leq y$ and $y \not\leq x$. We denote it by $x \parallel y$.

We study the realizations of models over a certain set of formulas based on the Rudin-Keisler preorders on models. The next set of definitions is taken from [17].

A model $\mathcal{M}$ is *prime over a type* p if there is a tuple of elements $\bar{a}$ in $\mathcal{M}$ such that $\bar{a}$ is a realization of p and $\mathcal{M}$ is prime over $\bar{a}$. We denote a prime model over a type p by $\mathcal{M}_p$. A model $\mathcal{M}$ is *almost prime* if it is prime over a realization of some type. If a model is not almost prime, we call it a *limit model*.

Definition 1. Let p and q be types in $S(T)$. We say that the type p *is dominated by the type* q, or p *does not exceed* q *under the Rudin-Keisler preorder* (written $p \leq_{\mathrm{RK}} q$), if $\mathcal{M}_q \models p$, that is, $\mathcal{M}_p$ is an elementary submodel of $\mathcal{M}_q$ (written $\mathcal{M}_p \preceq \mathcal{M}_q$).

Besides, we say that a model $\mathcal{M}_p$ *is dominated by a model* $\mathcal{M}_q$, or $\mathcal{M}_p$ *does not exceed* $\mathcal{M}_q$ *under the Rudin-Keisler preorder*, and write $\mathcal{M}_p \leq_{\mathrm{RK}} \mathcal{M}_q$.

Definition 2. Types p and q are said to be *domination-equivalent*, *realization-equivalent*, *Rudin-Keisler equivalent*, or RK-*equivalent* (written $p \sim_{\mathrm{RK}} q$) if $p \leq_{\mathrm{RK}} q$ and $q \leq_{\mathrm{RK}} p$.

Models $\mathcal{M}_p$ and $\mathcal{M}_q$ are said to be *domination-equivalent, Rudin-Keisler equivalent*, or RK-*equivalent* (written $\mathcal{M}_p \sim_{\mathrm{RK}} \mathcal{M}_q$).

2 Constant expansions of the DMT theory

In this section, we expand T_{dmt} with the signature $\mathcal{L}_{\mathrm{dmt}} = \{\leq, \sqcap\}$ by countable sequences of constants and find all possible values $I(T, \omega)$ for these expansions $T \supseteq T_{\mathrm{dmt}}$.

Let us extend $\mathcal{L}_{\mathrm{dmt}}$ to $\mathcal{L}_0$ by constants $c_k^{(0)}$, $k \in \omega$, and extend the theory T_{dmt} to T_0, so that constants $c_k^{(0)}$, $k \in \omega$, form a strictly increasing sequence. In this case, the signature of T_0 is $\mathcal{L}_0 = \mathcal{L}_{\mathrm{dmt}} \cup \{c_k^{(0)} \mid k \in \omega\}$ and $T_0 = T_{\mathrm{dmt}} \cup \{c_k^{(0)} < c_{k+1}^{(0)} \mid k \in \omega\}$. Note that the theory T_0 was constructed by Peretyat'kin in [14], where he proved that it is Ehrenfeucht, namely, T_0 has exactly three countable models: the prime model, the saturated model, and the prime model over the realization of the powerful type $p_0(x)$, isolated by the set of formulas $\{c_k^{(0)} < x \mid k \in \omega\}$. In Figure 1, we illustrate all possible realizations of the type p_0 (a.) and represent the Hasse diagram of Rudin-Keisler preorder $\leq_{\mathrm{RK}}$ on the set of countable models (up to isomorphisms) of T_0 (b.).

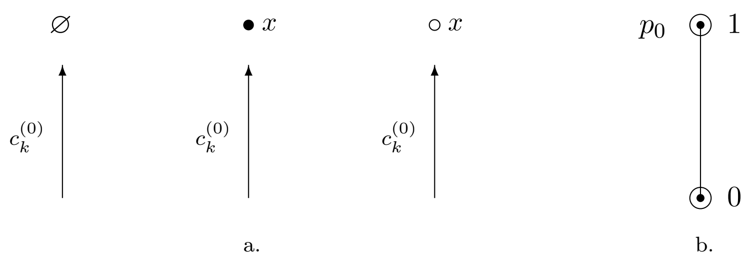

FIGURE 1. From left to right: a prime model, where there is no element greater than all $c_k^{(0)}$, a prime model, where among the elements of large $c_k^{(0)}$ there is the smallest element, and a limit model where among the elements of large $c_k^{(0)}$ there is no the smallest element (a.); $\mathrm{RK}(T_0)$ is a linear order of two elements. In Hasse diagrams, we draw the realization of prime models over a certain set of formulas using the character $\bullet$ based on the Rudin-Keisler preorder $\leq_{\mathrm{RK}}$ on the models. The circled circle $\bigcirc$ above the $\bullet$ means the possibility of realization of limit models over these sets of formulas. It should be noted here that the total number of characters $\bullet$ is the total number of prime models, and the numbers next to $\odot$ mean the number of limit models (b.).

Now we construct a theory T_1. To do this, we expand $\mathcal{L}_0$ with a strictly decreasing sequence of constants $c_k^{(1)}$, $k \in \omega$. We put $\mathcal{L}_1 = \mathcal{L}_0 \cup \{c_k^{(1)} \mid k \in \omega\}$ and $T_1 = T_0 \cup \{c_k^{(1)} > c_{k+1}^{(1)} \mid k \in \omega\} \cup \{c_k^{(0)} < c_k^{(1)} \mid k \in \omega\}$.

It is straightforward to show that there exist exactly two non-principal 1-types over an empty set in T_1:

$$p(x) = p_0(x) \cup p_1(x) = \{c_k^{(0)} < x \mid k \in \omega\} \cup \{x < c_k^{(1)} \mid k \in \omega\}$$

and

$$\bar{\bar{p}}(x) = p_0(x) \cup \bar{\bar{p}}_1(x) = \{c_k^{(0)} < x \mid k \in \omega\} \cup \{x \parallel c_k^{(1)} \mid k \in \omega\}$$

Here, $p_1(x) = \{x < c_k^{(1)} \mid k \in \omega\}$ and $\bar{\bar{p}}_1(x) = \{x \parallel c_k^{(1)} \mid k \in \omega\}$.

We can write three completions of $p(x) \cup p(y)$. They are defined by the following sets of formulas:

$$\begin{aligned} q_1(x, y) &= p(x) \cup p(y) \cup \{x < y\}; \\ q_2(x, y) &= p(x) \cup p(y) \cup \{y < x\}; \\ q_3(x, y) &= p(x) \cup p(y) \cup \{x = y\}. \end{aligned}$$

The reason for the existence of exactly three types involving two variables is the following: due to the quantifier elimination result and obvious logical equivalences of negation of these formulas, there are just four formulas in two variables: $x < y$, $y < x$, $x = y$, and $x \parallel y$. Note that here the types

$q_1(x,y)$ and $q_2(x,y)$ are the same up to a permutation of variables, the type $q_3(x,y)$ is logically equivalent to $p(x)$, and, as it is easy to check that $p(x)\cup p(y)\cup\{x\parallel y\}$ is inconsistent. Indeed, let $(a,b)\models p(x)\cup p(y)\cup\{x\parallel y\}$. Then both a and b are less than $c_1^{(1)}$. By Axiom 1 of T_{dmt} the elements a and b are comparable, for a contradiction.

Similarly, there are two completions of $p(x)\cup\bar{\bar{p}}(y)$. They are defined by the following sets of formulas:

$$q_4(x,y) = p(x)\cup\bar{\bar{p}}(y)\cup\{x<y\};$$
$$q_5(x,y) = p(x)\cup\bar{\bar{p}}(y)\cup\{x\parallel y\}.$$

Remark. It holds that $p(x)\sim_{\mathrm{RK}} q_4(x,y)$. Indeed, let $a_1\models p(x)$. By Axiom 3 of T_{dmt} there exist infinitely many pairwise incomparable elements b_1, b_2, $\dots$, b_n, $\dots$, such that $b_i\sqcap b_j=a_1$ for any $1\le i<j<\omega$. Assume that b_i and b_j are less than $c_k^{(1)}$ for some i, j, and $k\in\omega$. Then b_i and b_j are comparable by Axiom 1 of T_{dmt}, for a contradiction. So, there is some b_i such that $b_i>a_1$ and $b_i\parallel c_k^{(1)}$. Then $(a_1,b_i)\models q_4(x,y)$. Hence, $q_4(x,y)\le_{RK} p(x)$.

Since $p(x)\subseteq q_4(x,y)$, so $p(x)\le_{RK} q_4(x,y)$. Thus, $p(x)\sim_{RK} q_4(x,y)$.

Remark. It holds that $q_1\sim_{\mathrm{RK}} q_5$. Indeed, let $(a_1,a_2)\models q_1(x,y)$. As above, there exists b such that $a_1<b$ and $b\parallel a_2$. Then $(a_2,b)\models q_5$ and $q_5\le_{RK} q_1$.

Conversely, let $(a_2,b)\models q_5$ and let $a_1=a_2\cap b$. Then $(a_1,b)\models q_1$, so $q_1\le_{RK} q_5$. Thus, $q_1\sim_{RK} q_5$.

We have the following pairwise non-isomorphic countable models of T_1:

- a prime model;
- a prime model over a realization of $p(x)$, with a unique realization of this type;
- a prime model over the realization of $q_1(x,y)$ forming a closed interval $[a,b]$ with $c_k^{(0)}<a\le b<c_k^{(1)}$;
- three limit models over the type $q_1(x,y)$, in which the sets of realizations of $p(x)$ are $[a,b),(a,b],(a,b)$, correspondingly.

In Figure 2, we illustrate all possible realizations of the type $p(x)$ and represent the Hasse diagram of Rudin-Keisler preorders $\le_{\mathrm{RK}}$ on the set of countable models (up to isomorphisms) of the theory T_1, respectively.

So, the next is clear.

Theorem 1. *The theory T_1 has exactly 6 countable models up to isomorphism.*

Starting with the theory T_2, the situation looks a little different. First, we define the signature $\mathcal{L}_2$ for T_2 as $\mathcal{L}_1\cup\{c_k^{(2)}\mid k\in\omega\}$, where $c_k^{(2)}, k\in\omega$ is also a strictly decreasing sequence of constants on the tree, additionally $c_k^{(0)}, k\in\omega$ is comparable with $c_k^{(1)}, k\in\omega$ and $c_k^{(2)}, k\in\omega$, but $c_k^{(1)}, k\in\omega$ and $c_k^{(2)}, k\in\omega$ are incomparable. Therefore, the theory T_2 has the following form

$$T_2 = T_1\cup\{c_k^{(2)}>c_{k+1}^{(2)}\mid k\in\omega\}\cup\{c_k^{(0)}<c_k^{(2)}\mid k\in\omega\}\cup\{c_k^{(1)}\parallel c_k^{(2)}\mid k\in\omega\}.$$

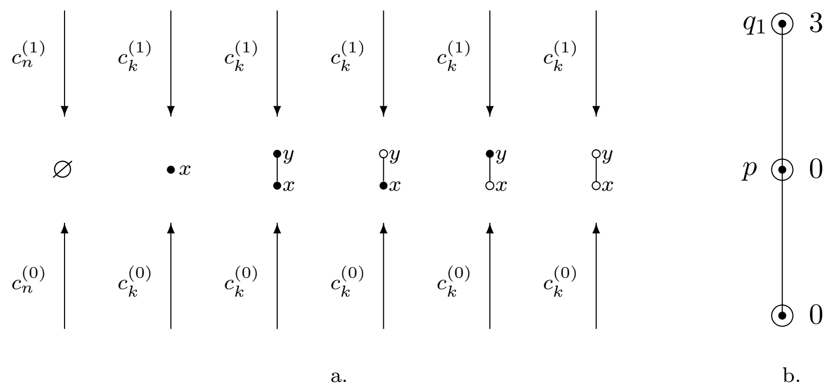

FIGURE 2. Realizations of $p(x)$ (a.); and Hasse diagram of $\mathrm{RK}(T_1)$ (b.)

Lemma 1. *The meet of incomparable elements of the strictly decreasing sequences of constants* $(c_k^{(1)})_{k\in\omega}$, $(c_k^{(2)})_{k\in\omega}$ *does not depend on their representative constants, that is,* $c_l^{(1)} \sqcap c_m^{(2)} = c_i^{(1)} \sqcap c_j^{(2)}$ *for any pairs* (l, m) *and* (i, j).

Proof. Since $c_i^{(k)} > c_{i+1}^{(k)}$ for both k, we obtain that

$$c_i^{(1)} \sqcap c_i^{(2)} \geq c_{i+1}^{(1)} \sqcap c_{i+1}^{(2)}$$

Assume that $d_{12} \triangleq c_i^{(1)} \sqcap c_i^{(2)} > c_{i+1}^{(1)} \sqcap c_{i+1}^{(2)}$. So, $\neg(d_{12} \leq c_{i+1}^{(1)} \wedge d_{12} \leq c_{i+1}^{(2)})$ holds. Both d_{12} and $c_{i+1}^{(k)}$ are less than $c_i^{(k)}$, so they are comparable, and $\neg(d_{12} \leq c_{i+1}^{(k)})$ is equivalent $d_{12} > c_{i+1}^{(k)}$, for both k. Then

$$\neg(d_{12} \leq c_{i+1}^{(1)} \wedge d_{12} \leq c_{i+1}^{(2)}) \Leftrightarrow (d_{12} > c_{i+1}^{(1)} \vee d_{12} > c_{i+1}^{(2)})$$

If $d_{12} > c_{i+1}^{(1)}$ then by transitivity we obtain that $c_i^{(2)} > c_{i+1}^{(1)}$, because $c_i^{(2)} > d_{12}$, for a contradiction. The case $d_{12} > c_{i+1}^{(2)}$ is similar.

By mathematical induction, we obtain that $c_i^{(1)} \sqcap c_i^{(2)} = c_j^{(1)} \sqcap c_j^{(2)}$ for all i and $j < \omega$. Now, let $i < j$. Then

$$c_i^{(1)} \sqcap c_i^{(2)} \geq c_i^{(1)} \sqcap c_j^{(2)} \geq c_j^{(1)} \sqcap c_j^{(2)} = c_i^{(1)} \sqcap c_i^{(2)}$$

that proves the lemma. □

Let $d_{1,2} = c_k^{(1)} \sqcap c_k^{(2)}$, $k \in \omega$. The element $d_{1,2}$ exists in any model of T_2. So, we redefine our types as follows

$$p_0(x_0) = \{c_k^{(0)} < x_0 \mid k \in \omega\} \cup \{x_0 < d_{1,2}\};$$
$$p_1(x_0) = \{x_0 < c_k^{(1)} \mid k \in \omega\} \cup \{d_{1,2} < x_0\};$$
$$p_2(x_0) = \{x_0 < c_k^{(2)} \mid k \in \omega\} \cup \{d_{1,2} < x_0\}.$$

Note that $p_0(x_0) \cup p_1(x_1) \cup p_2(x_2)$ defines a complete type. Then we can consider each type separately, by analogy with the theory T_0. Therefore, according to the orthogonality of the types p_0, p_1, and p_2, two prime models and one limit model arise over the realization of each type $p_i(x_i)$ for any $i \in \{0,1,2\}$. In total, $I(T_2,\omega) = 3^3 = 27$, where $2^3 = 8$ of them are prime models (over some set).

In Figure 3, we illustrate all possible realizations of the types p_0, p_1 and p_2 (a.) and represent the Hasse diagram of Rudin-Keisler preorder $\leq_{\mathrm{RK}}$ on the set of countable models (up to isomorphisms) of T_2 (b.).

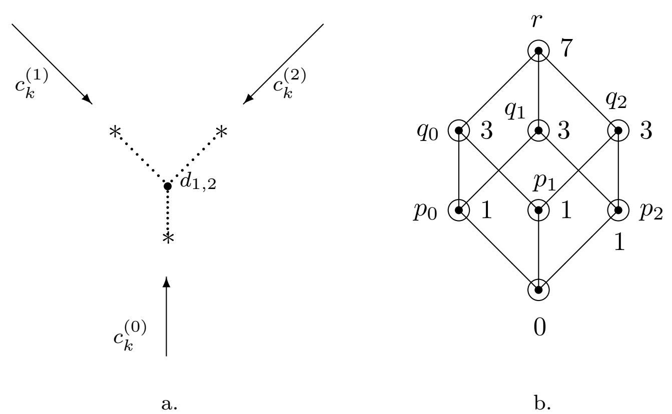

FIGURE 3. Here, $d_{1,2}$ is the meet of strictly decreasing sequences of constants $(c_k^{(1)})_{k\in\omega}$ and $(c_k^{(2)})_{k\in\omega}$, and the character $*$ means one of the $\{\varnothing, \bullet, \circ\}$ (a.); $\mathrm{RK}(T_2)$ is a Hasse diagram with boolean of $\{p_0,p_1,p_2\}$, here the type $q_0 = p_0 \cup p_1$, $q_1 = p_0 \cup p_2$, $q_2 = p_1 \cup p_2$, $r = p_0 \cup p_1 \cup p_2$ (b.).

But in the theory

$$T_3 = T_2 \cup \{c_k^{(3)} > c_{k+1}^{(3)} \mid k \in \omega\} \cup \{c_k^{(1)} \parallel c_k^{(3)} \mid k \in \omega\} \cup \cup \{c_k^{(0)} < c_k^{(3)} \mid k \in \omega\} \cup \{c_k^{(2)} \parallel c_k^{(3)} \mid k \in \omega\}$$

with the signature $\mathcal{L}_3 = \mathcal{L}_2 \cup \{c_k^{(3)} \mid k \in \omega\}$ there are three meets:

$$d_{1,2} = c_k^{(1)} \sqcap c_k^{(2)},\ d_{1,3} = c_k^{(1)} \sqcap c_k^{(3)},\ \text{and } d_{2,3} = c_k^{(2)} \sqcap c_k^{(3)} \text{ for any } k \in \omega.$$

Note that any two of $d_{1,2}$, $d_{1,3}$, $d_{2,3}$ are comparable. Since $d_{1,2} < c_k^{(1)}$ and $d_{1,3} < c_k^{(1)}$, then $d_{1,2}$ and $d_{1,3}$ are comparable. A similar situation is repeated for pairs $d_{1,2}$ and $d_{2,3}$; $d_{1,3}$ and $d_{2,3}$.

Let $d_{1,2} \leq d_{2,3}$. Since $d_{1,2} < c_k^{(1)}$ and $d_{1,2} \leq d_{2,3} < c_k^{(3)}$ for any $k \in \omega$, we have $d_{1,2} \leq d_{1,3}$. If $d_{2,3} \leq d_{1,3}$, then $d_{2,3} \leq d_{1,2}$, which contradicts $d_{1,2} \leq d_{2,3}$, because $d_{2,3} < c_k^{(2)}$ and $d_{2,3} \leq d_{1,3} < c_k^{(1)}$. Hence, $d_{1,3} < d_{2,3}$. By the fact $d_{1,3} < c_k^{(1)}$ and $d_{1,3} \leq d_{2,3} < c_k^{(2)}$, we have $d_{1,3} \leq d_{1,2}$. Thus, there are two cases:

Case 1. Assume that $d_{1,2} = d_{1,3} = d_{2,3}$. Then $T_3 \cup \{d_{1,2} = d_{1,3} = d_{2,3}\}$ is complete. Then the types of this theory are defined by the following sets of formulas for every $i \in \{1, 2, 3\}$:

$$p_0(x_0) = \{c_k^{(0)} < x_0 \mid k \in \omega\} \cup \{x_0 < d_{1,2}\};$$
$$p_i(x_0) = \{x_0 < c_k^{(i)} \mid k \in \omega\} \cup \{d_{1,2} < x_0\}.$$

Since the union of types $p_0(x_0) \cup p_1(x_1) \cup p_2(x_2) \cup p_3(x_3)$ defines a complete type and these types are orthogonal, we obtain $3^4 = 81$ countably pairwise non-isomorphic models over the realizations of these types, where $2^4 = 16$ of them are prime models.

Case 2. Assume now that $d_{1,2} = d_{1,3} < d_{2,3}$. As in the previous case, $T_3 \cup \{d_{1,2} = d_{1,3} < d_{2,3}\}$ is complete and we define the types of this theory:

$$p_0(x_0) = \{c_k^{(0)} < x_0 \mid k \in \omega\} \cup \{x_0 < d_{1,2}\};$$
$$p_1(x_0) = \{x_0 < c_k^{(1)} \mid k \in \omega\} \cup \{d_{1,2} < x_0\}$$
$$p_i(x_0) = \{x_0 < c_k^{(i)} \mid k \in \omega\} \cup \{d_{2,3} < x_0\}, \text{ for } i = 2, 3.$$

Therefore, we obtain the 3^4 of countable models over the realizations of these types as in the first case.

Note that, in the each cases $d_{1,2} = d_{2,3} < d_{1,3}$ and $d_{1,3} = d_{2,3} < d_{1,2}$, we also obtain 3^4 countable models.

Below we illustrate the above-described cases for elements $d_{1,2}$, $d_{1,3}$ and $d_{2,3}$ in Figure 4.

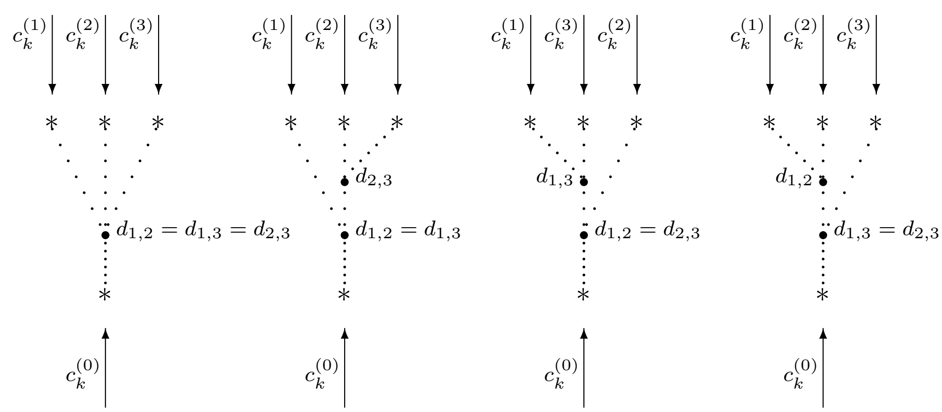

FIGURE 4.

Let us proceed to consider the case of arbitrary $n \geq 2$. The signature $\mathcal{L}_n = \mathcal{L}_{n-1} \cup \{c_k^{(n)} \mid k \in \omega\}$ and the theory T_n is given as

$$T_n = T_{n-1} \cup \{c_k^{(n)} > c_{k+1}^{(n)} \mid k \in \omega\} \cup \{c_k^{(0)} < c_k^{(n)} \mid k \in \omega\} \cup \\ \cup \{c_i^{(n)} \parallel c_j^{(m)} \mid m < n, \forall i, j \in \omega\}.$$

In this theory, we have $C(n,2)$ meets: $d_{1,2}$, $d_{1,3}$, $\ldots$, $d_{n-1,n}$.

Here it is enough to consider the case when $d_{1,2} = d_{1,3} = \cdots = d_{n-1,n}$, since other cases give the same number of models. The list of complete types is defined as follows (here, $i \in \{1, 2, \ldots, n\}$):

$$p_0(x) = \{c_k^{(0)} < x \mid k \in \omega\} \cup \{x < d_{1,2}\};$$
$$p_i(x) = \{x < c_k^{(i)} \mid k \in \omega\} \cup \{d_{1,2} < x\}.$$

Thus, we have the following:

Theorem 2. *Let T_n, where $n \geq 2$, be a countable constant expansion of the dense meet-tree theory T_{dmt} with an increasing sequence of constants $(c_k^{(0)})_{k\in\omega}$ and n decreasing sequences of constants $(c_k^{(1)})_{k\in\omega}, \ldots, (c_k^{(n)})_{k\in\omega}$, so that $c_k^{(0)} < c_k^{(1)}, \ldots, c_k^{(0)} < c_k^{(n)}$, $k \in \omega$ and $c_k^{(j)} \parallel c_k^{(t)}$ for each $1 \leq j \neq t \leq n$. Then T has exactly 3^{n+1} countable models, where 2^{n+1} of them are prime models.*

Note that the number of limit models is equal to subtraction of the number of prime models from the total number of countable models, i.e. $3^{n+1} - 2^{n+1}$.

Since the complete type $p_0(x_0) \cup p_1(x_1), \ldots p_n(x_n)$ is powerful, then the Rudin-Keisler preorder $\leq_{\text{RK}}$ is a Boolean of the $(n+1)$-element set.

If there are several sequences that are increasing, but pairwise incomparable, and each one has several sequences that are decreasing from above, then this can be considered as a disjunctive union. Then the following theorem will be true.

Theorem 3. *Let T be a countable constant expansion of the dense meet-tree theory T_{dmt} with n increasing sequences of constants $(c_k^{(i,0)})_{k\in\omega}$ such that*

$$\sup_{k\in\omega} c_k^{(i,0)} \neq \sup_{k\in\omega} c_k^{(j,0)} \text{ for all } i \neq j.$$

Let for each i there are $\tau(i)$ decreasing sequences $(c_k^{(1,i)})_{k\in\omega}$, $\ldots$, $(c_k^{(\tau(i),i)})_{k\in\omega}$, such that $c_k^{(i,0)} < c_k^{(1,i)}, \ldots, c_k^{(i,0)} < c_k^{(\tau(i),i)}$ for all $k \in \omega$, and $c_k^{(j,i)} \parallel c_k^{(t,i)}$ for each $1 \leq j \neq t \leq \tau(i)$. We define $d(m)$ as $|\{i \mid \tau(i) = m\}|$ and we put $I = \{i : 1 \leq i \leq n \wedge d(i) \neq 1\}$. Then T has exactly

$$6^{d(1)} \cdot \prod_{i\in I} 3^{\tau(i)+1}$$

countable models, where

$$3^{d(1)} \cdot \prod_{i\in I} 2^{\tau(i)+1}$$

of them are prime models.

Note that T from Theorem 3 has as many countable models as some finite disjoint union of theories of the form T_0, T_1, and T_n, for $n \geq 2$.

3 On unary predicates expansions of DMT

Let T_n be the complete constant expansion of T_{dmt} as we have considered above. We replace L_n with a new language $\mathcal{L}_n^P = \mathcal{L}_0 \cup \{P_k \mid k \in \omega\}$, where we interpret the predicates P_k as the set of constants $\{c_k^{(1)}, c_k^{(2)}, \ldots c_k^{(n)}\}$, for each $k \in \omega$. Let $T_n^P = \text{Th}\{(\mathcal{M}, \mathcal{L}_n^P) \mid (\mathcal{M}, L_n) \models T_n\}$. We aim to count the number of countable models of T_n^P. As T_n is not complete, so is T_n^P, that is why we consider various completions of T_n.

The difference between models of T_n and T_n^P is that in countable models of T_n^P there are automorphisms which make some permutation of $P_k(\mathcal{M})$. So, we have the following schemes of axioms:

$1)_k$ $(\exists^{!n} x) P_k(x)$;

$2)_k$ $\forall x (P_k(x) \to \exists! y (P_{k+1}(y) \wedge y < x))$;

$3)_k$ $\forall x \forall y (P_k(x) \wedge P_k(y) \wedge x \neq y \to x \parallel y)$;

We work simultaneously in models of T_n and T_n^P. Since each model of T_n after replacing the constants with the predicates becomes a model of T_n^P, and vice versa, each model of T_n^P after a suitable replacement of the predicates with the constants becomes a model of T_n.

If $\mathcal{M} = (M, \mathcal{L}_n)$ is a model of T_n, we denote the corresponding model of T_n^P by $\mathcal{M}_P = (M, \mathcal{L}_n^P)$. Since the universes of $\mathcal{M}$ and $\mathcal{M}^P$ are the same, $p_k(\mathcal{M})$ being a type of T_n, defines a subset of M, which is also a subset of $\mathcal{M}^P$.

The first case is $c_1^{(i)} \sqcap c_1^{(j)} = c_1^{(s)} \sqcap c_1^{(t)}$ for all $i \neq j$ and $s \neq t$. We denote this theory by $T_{n,=}^P$.

In T_n we have 3 possible realizations of each type p_k, where $k \in \{1, \ldots n\}$. Because of automorphisms we can permute realizations of $p_k(\mathcal{M})$ and $p_m(\mathcal{M})$. Then the number of non-isomorphic countable models of $T_{n,=}^P$ is equal to the number of combinations of a three-element set with repetitions. Hence we have $3 \cdot C(3+n-1, n) = 3 \cdot \frac{(n+2)(n+1)}{2}$ countable models. Note that the number of non-isomorphic prime countable models of $T_{n,=}^P$ is equal to the number of combinations of a two-element set with repetitions, so the number is $2 \cdot C(2+n-1, n) = 2 \cdot (n+1)$.

Now we consider T_3 and its possible completions different from the considered above. We consider the case that $d_{1,2} = d_{1,3} < d_{2,3}$, where $d_{i,j} = c_0^{(i)} \sqcap c_0^{(j)}$ (see Figure 4). Then there is an automorphism of $\mathcal{M}^P$ which moves $p_2(M)$ to $p_3(M)$ and $p_3(M)$ to $p_2(M)$. So, we have 3 kinds of realizations for each of p_0 and p_1, and by the above formula $C(3+n-1, n)$ realizations for p_2 and p_3, where $n = 2$. Thus, the number of countable models is $3^2 \cdot 6$ and the number of countable prime models is $2^2 \cdot 3$, where the number 3 is the result of the formula $C(2+n-1, n) = n+1$ for $n = 2$. Since other completions of T_3 give the same T_3^P, we are done.

We describe Hasse diagrams for the completions $T_{3,=}^P$ and $T_{3,<}^P$ of T_3^P by $d_{1,2} = d_{1,3} = d_{2,3}$ and by $d_{1,2} = d_{1,3} < d_{2,3}$, respectively.

Let's describe the realizations of types in this theory. For this, we set $d_{1,2}$ as a unique element which satisfies the following formula

$$\exists x_1 \exists x_2 (P_k(x_1) \wedge P_k(x_2) \wedge x_1 \parallel x_2 \wedge y = x_1 \sqcap x_2).$$

As we showed this above, $d_{1,2}$ does not depend on the choice of k.

First we consider $T^P_{3,=}$. We desrcibe all types up to Rudin-Keisler preorder:

$$\begin{aligned}
p_{1,0}(x_0) &= \{c_k^{(0)} < x_0 \mid k \in \omega\} \cup \{x_0 < d_{1,2}\} \\
p_{0,1}(x_1) &= \{d_{1,2} < x_1\} \cup \{\exists t (P_k(t) \wedge x_1 < t) \mid k \in \omega\} \\
p_{0,2}(x_1, x_2) &= p_{0,1}(x_1) \cup p_{0,1}(x_2) \cup \{x_1 \parallel x_2\} \\
p_{0,3}(x_1, x_2, x_3) &= p_{0,2}(x_1, x_2) \cup p_{0,2}(x_1, x_3) \cup p_{0,2}(x_2, x_3) \\
p_{1,1}(x_0, x_1) &= p_{1,0}(x_0) \cup p_{0,1}(x_1) \\
p_{1,2}(x_0, x_1, x_2) &= p_{1,1}(x_0, x_1) \cup p_{0,2}(x_1, x_2) \\
p_{1,3}(x_0, x_1, x_2, x_3) &= p_{1,1}(x_0, x_1) \cup p_{0,3}(x_1, x_2, x_3)
\end{aligned}$$

Hasse diagram of Rudin-Keisler preorder for the types of $T^P_{3,=}$ is given in the Figure 5.

Now we consider $T^P_{3,<}$. As in the previous case, $T^P_{3,<} \cup \{d_{1,2} = d_{1,3} < d_{2,3}\}$ is complete and we desrcibe all types up to Rudin-Keisler preorder:

$$\begin{aligned}
p_{1,0,0}(x_0) &= \{c_k^{(0)} < x_0 \mid k \in \omega\} \cup \{x_0 < d_{1,2}\} \\
p_{0,1,0}(x_1) &= \{d_{1,2} < x_1\} \cup \{\exists t (P_k(t) \wedge x_1 < t) \mid k \in \omega\} \\
p_{0,0,1}(x'_1) &= \{d_{1,2} < x'_1\} \cup \{\exists t (P_k(t) \wedge x'_1 < t) \mid k \in \omega\} \\
p_{0,2,0}(x_1, x_2) &= p_{0,1,0}(x_1) \cup p_{0,1,0}(x_2) \cup \{x_1 \parallel x_2\} \\
p_{0,1,1}(x_1, x'_1) &= p_{0,1,0}(x_1) \cup p_{0,0,1}(x'_1) \cup \{x_1 \parallel x'_1\} \\
p_{0,2,1}(x_1, x_2, x'_1) &= p_{0,2,0}(x_1, x_2) \cup p_{0,1,1}(x_1, x'_1) \\
p_{1,1,0}(x_0, x_1) &= p_{1,0,0}(x_0) \cup p_{0,1,0}(x_1) \\
p_{1,2,0}(x_0, x_1, x_2) &= p_{1,1,0}(x_0, x_1) \cup p_{0,2,0}(x_1, x_2) \\
p_{1,0,1}(x_0, x'_1) &= p_{1,0,0}(x_0) \cup p_{0,0,1}(x'_1) \\
p_{1,1,1}(x_0, x_1, x'_1) &= p_{1,1,0}(x_0, x_1) \cup p_{1,0,1}(x_0, x'_1) \cup p_{0,1,1}(x_1, x'_1) \\
p_{1,2,1}(x_0, x_1, x_2, x'_1) &= p_{1,2,0}(x_0, x_1, x_2) \cup p_{1,1,1}(x_0, x_1, x'_1) \cup p_{0,2,1}(x_1, x_2, x'_1)
\end{aligned}$$

Hasse diagram of Rudin-Keisler preorder for the types of $T^P_{3,<}$ is given in the Figure 5.

It is turn to consider possible completions on T^P_4 via completions of T_4, where some of $d_{i,j}$ are not the same. There are 5 such completions of T^P_4 (see Figure 6, in this figure we omit the case when all $d_{i,j}$ are equal).

Case 1. $d_{1,2} = d_{1,3} = d_{2,3} > d_{1,4} = d_{2,4} = d_{3,4}$. There are 3 realizations for each of p_0 and p_4. The number of realization of p_1, p_2, and p_3 is expressed by $C(3 + n - 1, n)$, where $n = 3$. So we have $3^2 \cdot 10$ countable models. The number of countable prime models is $2^2 \cdot C(2 + n - 1, n) = 2^2 \cdot 4$.

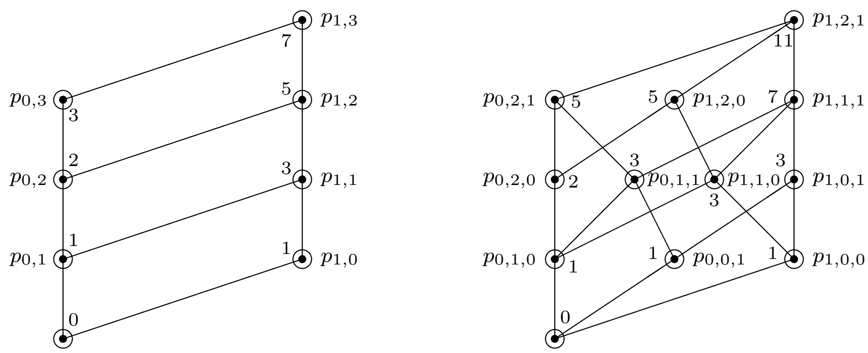

FIGURE 5. Hasse diagrams for $T^P_{3,=}$ and $T^P_{3,<}$

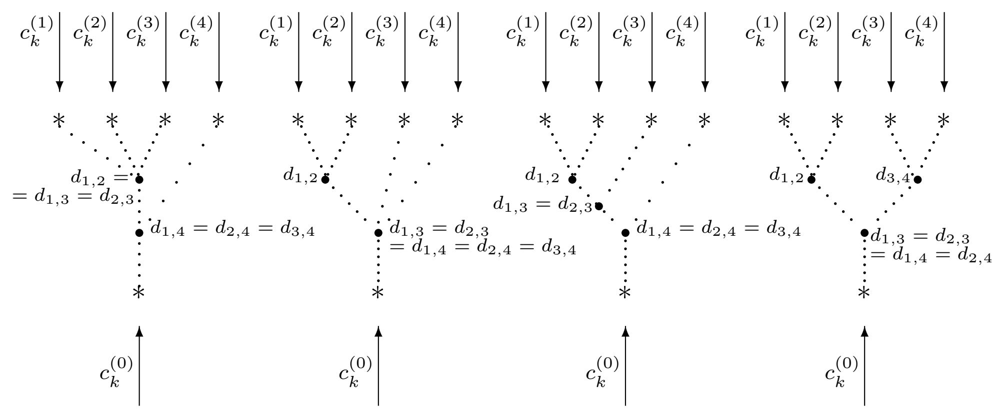

FIGURE 6.

Case 2. $d_{1,2} > d_{1,3} = d_{2,3} = d_{1,4} = d_{2,4} = d_{3,4}$. The number of realizations of p_1 and p_2 is expressed by $C(3 + n - 1, n)$, where $n = 2$, as well as the number of realizations of p_3 and p_4. So we have $3 \cdot 6^2$ countable models. The number of prime countable models is $2 \cdot (C(2 + n - 1, n))^2 = 2 \cdot 4^2$.

Case 3. $d_{1,2} > d_{1,3} = d_{2,3} > d_{1,4} = d_{2,4} = d_{3,4}$. There are 3 realizations for each of p_0, p_3, and p_4. The number of realization of p_1 and p_2 is expressed by $C(3 + n - 1, n)$, where $n = 2$. So we have $3^3 \cdot 6$ countable models. The number of countable prime models is $2^3 \cdot C(2 + n - 1, n) = 2^3 \cdot 3$.

Case 4. $d_{1,2} > d_{1,3} = d_{2,3} = d_{1,4} = d_{2,4}$, $d_{3,4} > d_{1,4}$, and $d_{1,2} \parallel d_{3,4}$. This is the most interesting case here. There are automorphisms of $\mathcal{M}^P$ which swap $p_1(\mathcal{M})$ and $p_2(\mathcal{M})$, $p_3(\mathcal{M})$ and $p_4(\mathcal{M})$, and $p_1(\mathcal{M}) \cup p_2(\mathcal{M})$ and $p_3(\mathcal{M}) \cup p_4(\mathcal{M})$.

First, we calculate the number of countable prime models. It is $2 \cdot 6 = 12$. The number 6 is obtained by the following calculation. For each of p_i, we have 2 possibilities: either the realization is empty, or not. We denote it by 0 and 1, correspondingly. In order to code all possible variants, we use multisets. Since we have 2 elements: 0 and 1, there are exactly 3 multisets of cardinality 2: $\{0,0\}$, $\{0,1\}$, and $\{1,1\}$. Now we calculate the number of combinations of three-element set with repetitions, when we take 2 elements. It is equal to $C(4,2)=6$. Below, we list all possible choices for realization of p_1, p_2, p_3, and p_4:

$$\{\{0,0\},\{0,0\}\},\ \{\{0,0\},\{0,1\}\},\ \{\{0,0\},\{1,1\}\}$$

$$\{\{0,1\},\{0,1\}\},\ \{\{0,1\},\{1,1\}\},\ \{\{1,1\},\{1,1\}\}$$

Now we calculate the number of all countable models. We denote by 2 the limit realization of a type. Since we have 3 elements: 0, 1, and 2, there are exactly 6 multisets of cardinality 2: $\{0,0\}$, $\{0,1\}$, $\{0,2\}$, $\{1,1\}$, $\{1,2\}$, and $\{2,2\}$. Now we calculate the number of combinations of six-element set with repetitions, when we take 2 elements. It is equal to $C(7,2)=21$. Recall, that p_0 has 3 kinds of realizations. Thus, the number of countable models is $3\cdot 21=63$.

We express the general way for calculating the number of countable models. Let $\mathcal{T}$ be a finite rooted tree. Let G be the group of all automorphisms of $\mathcal{T}$ as a rooted tree. Now we color each leaf of $\mathcal{T}$ into k colors, where $k\in\{2,3\}$. Two colorings of $\mathcal{T}$ are said to be equivalent if there is a $g\in G$, which takes the first coloring to the second one. Each completion of T_n correspond to some tree with n leaves, where each vertex has at least two sons or it is a leaf. Then the number of countable models of a completion of T_n is equal to the number of non-equivalent colorings of the corresponding tree, where the number of colors is equal to 3 modulo the number of realizations of the type p_0. So, the total number of countable models is 3 times the number of non-equivalent colorings of the corresponding tree. For counting the number of prime models of T_n we take the number of colors to be 2.

We consider some example of $\mathcal{T}$. Assume that the root of $\mathcal{T}$ has m_3 sons, each son has m_2 sons, and each grandson of the root has m_1 sons. Then there are $\gamma_1=C(3+m_1-1,m_1)$ ways to color the sons of some grandson. Now there are $\gamma_2=C(\gamma_1+m_2-1,m_2)$ ways to color the grandsons with their children. So, there are $\gamma_3=C(\gamma_2+m_3-1,m_3)$ ways to color the sons with their sons and grandson. Also, we have 3 coloring of the root. Totally, there are

$$3\cdot C(C(C(3+m_1-1,m_1)+m_2-1,m_2)+m_3-1,m_3)$$

countable models of the corresponding theory.

Recall that the height of a vertex in a rooted tree is the length of the longest downward path to a leaf from that vertex. Let a and b be two parents of two leaves, that is, of height 1. Then there exists an automorphism that swaps a and b, swaps sons of a and sons of b and fixes the rest of the tree if and only if they have the same number of sons. So, we say that a vertex of the height 1 which has exactly m sons is of the type m.

Let c be a vertex of height 2. We say that c is of the type

$$((m_0, 0), (m_1, k_1), \ldots, (m_s, k_s))$$

if it has exactly $m_0 + m_1 + \cdots + m_s$ sons, moreover, it has exactly m_0 sons which are leaves, and exactly m_i sons of the type k_i, for each i.

Let d be a vertex of height $p+1$, and let K_i be a type of vertex c_i of the height p. We say that d is of the type

$$((m_1, K_i), \ldots, (m_s, K_s))$$

if it has exactly $m_1 + \cdots + m_s$ sons, moreover, it has exactly m_i sons of the type K_i for each i.

It is straightforward to prove that for any two vertices, there is an automorphism which swaps them and their descendants and fixes the other part of the tree if and only if the types of these elements are the same.

Let a be a vertex which is fixed by each automorphism of $\mathcal{T}$ and which has at least two sons of the same type. Let b_0 be a leaf that is a descendant of a. Let b_1 be the parent of b_0 of type m_1^0, that is, b_1 has exactly m_1^0 sons.

Let b_2 be the parent of b_1 and have exactly m_2^0 sons of the same type with b_1. And so on, let $b_k = a$ be the parent of b_{k-1} and let it have exactly m_k^0 sons of the same type as b_{k-1} has. Let

$$\gamma_1^0 = C(3 + m_1^0 - 1, m_1^0) \text{ and } \gamma_{i+1}^0 = C(\gamma_i^0 + m_i^0 - 1, m_i^0),$$

$$\delta_1^0 = C(2 + m_1^0 - 1, m_1^0) \text{ and } \delta_{i+1}^0 = C(\delta_i^0 + m_i^0 - 1, m_i^0).$$

Let $B_0 = G(b_0)$, where G is the group of all automorphisms of the rooted tree $\mathcal{T}$. Then similarly to the example above we can prove that there exist γ_k^0 colorings of B_0 into 3 colors and δ_k^0 colorings of B_0 into 2 colors. Given a set B_0, we denote γ_k^0 by Γ_0 and δ_k^0 by Δ_0.

From the above considerations, Theorem 4 follows.

Theorem 4. *Let T be a completion of T_n^P and let $\mathcal{T}$ be the corresponding rooted tree. Let B_0, ..., B_w be a partition of the set of leaves of $\mathcal{T}$, where each B_i is the orbit of some leaf under the action of the group of automorphisms of $\mathcal{T}$. Then the number of countable models of T is equal to*

$$3 \cdot \prod_{i \le w} \Gamma_i$$

and the number of countable prime models is equal to

$$2 \cdot \prod_{i \le w} \Delta_i.$$

Now we give the diagram for $T_{n,=}^P$ in Figure 7, where $T_{n,=}^P$ is a such completion of T_n^P where all meets of the descending sequences are the same. We omit the Hasse diagram of the Rudin-Keisler preorders $\le_{\mathrm{RK}}$ for the other theories since the drawing these diagrams seems cumbersome.

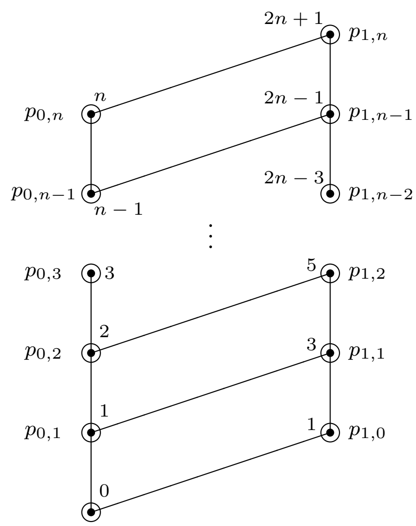

FIGURE 7. Hasse diagram for $T^P_{n,=}$

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