An inversion theorem for the generalized Fourier-Bessel transform

[1]Afaf Laamimi and [2*]Mohamed El Hamma

Laboratoire Mathématiques Fondamentales et appliqués, Faculté des Sciences Aïn Chock, Université Hassan II, B.P 5366 Maarif, Casablanca, Maroc
[2*]E.mail: m_elhamma@yahoo.fr
[2*]corresponding author

Abstract: In this work, we give an inversion formula for the generalized Fourier-Bessel transform associated with Bessel operator. these results generalize an inversion theorem for Hankl transform due to Alan L. Schwartz [1].

Keywords: Bessel operator, Generalized Fourier-Bessel, Inversion formula, Bessel function.

AMS Mathemtics Subject Classification: 42B37.

1 Introduction and preliminaries

For $\alpha > \frac{-1}{2}$, the second-order singular differential operator on the half line defined by

$$\mathcal{B}f(x) = \frac{d^2 f(x)}{dx^2} + \frac{(2\alpha+1)}{x}\frac{df(x)}{dx} - \frac{4n(\alpha+n)}{x^2}f(x),$$

where $n = 0, 1, 2, \ldots$.
For $n = 0$, we obtain the classical Bessel operator

$$\mathcal{B}_\alpha f(x) = \frac{d^2 f(x)}{dx^2} + \frac{(2\alpha+1)}{x}\frac{df(x)}{dx}.$$

This operator is named after the mathematician Friedrich Wilhlm Bessel (1784-1846), it is important in mathematics and physics (see [6, 9, 10]).

Let M be the map defined by

$$Mf(x) = x^{2n} f(x), \quad n = 0, 1, ..$$

Let $L^p_{\alpha,n}, 1 \leqslant p < \infty$, be the class of measurable functions f on $[0, \infty[$ for which

$$\|f\|_{p,\alpha,n} = \left\| M^{-1} f \right\|_{p,\alpha+2n} < \infty,$$

where

$$\|f\|_{p,\alpha} = \left(\int_0^\infty |f(x)|^p x^{2\alpha+1} dx \right)^{1/p}$$

the following problem

$$\begin{cases} \mathcal{B}_\alpha(f)(x) = -\lambda^2 f(x); \lambda \in \mathbb{C}, \\ f(0) = 1, \\ f'(0) = 0, \end{cases}$$

admits a unique solution j_α, where j_α is the function defined by

$$j_\alpha(x) = \begin{cases} 2^\alpha \Gamma(\alpha+1) \frac{J_\alpha(x)}{x^\alpha} & \text{if} \quad x \neq 0, \\ 1 \quad \text{if} \quad x = 0, \end{cases}$$

and J_α is the Bessel function of the first kind and index α (see [3, 4]).

The Hankl transform, also known as the Fourier-Bessel transform, is an integral transform develepred by the mathematician Herman Hankl defined on $(0, +\infty)$

$$h_\alpha(f)(\lambda) = \int_0^\infty f(t) j_\alpha(\lambda t) dm_\alpha(t),$$

where

$$dm_\alpha(t) = \frac{t^{2\alpha+1}}{2^\alpha \Gamma(\alpha+1)} dt$$

Let $L^1_\alpha(0, +\infty)$, the space of measurable function on $(0, +\infty)$ such that $\int_0^\infty |f(x)| d\nu_\alpha(x)$ where

$$d\nu_\alpha(x) = \frac{x^{2\alpha+1}}{2^{\alpha+1} \Gamma(\alpha+1)} dx.$$

Theorem 1.1. *[1] Suppose $f \in L^1_\alpha((0, \infty))$ such that*
(i) f is of bounded variation in a neighborhood of $x > 0$.
(ii) $\int_0^1 |f(x)| x^{\alpha+\frac{1}{2}} dx < \infty$
Then

$$\lim_{A \longrightarrow +\infty} \int_0^A h_\alpha(f)(\lambda) j_\alpha(\lambda x) dm_\alpha(\lambda) = \frac{1}{2}[f(x+0) + f(x-0)]$$

There are many analogues of theorem 1.1, for the Hankl transform see [1] and the Dunkl transform [5]. This paper is intented to establish an analog of theorem 1.1 for the generalized Fourier-Bessel transform.

Definition 1.2. *The generalized Fourier-Bessel transform is defined on* $L^1_{\alpha,n}(0,+\infty)$ *by*

$$\forall \lambda \geqslant 0, \quad \mathcal{F}(f)(\lambda) = \int_0^\infty f(x)\varphi_\lambda(x)x^{2\alpha+1}dx$$

where

$$\varphi_\lambda(x) = x^{2n} j_{\alpha+2n}(\lambda x)$$

From [11], we have the two following theorems

Theorem 1.3. *Let* f *be in* $L^1_{\alpha,n}$ *such that the function* $\mathcal{F}(f)$ *belong to* $L^1_{\alpha+2n} = L^1\left([0,\infty[, x^{2\alpha+4n+1}dx\right)$, *then we have the following inversion formula for the transform* $\mathcal{F}$*:*

$$f(x) = \int_0^\infty \mathcal{F}f(\lambda)\varphi_\lambda(x)d\mu_{\alpha+2n}(\lambda)$$

where

$$d\mu_{\alpha+2n}(\lambda) = a_{\alpha+2n}\lambda^{2\alpha+4n+1}d\lambda, \quad a_\alpha = \frac{1}{4^\alpha(\Gamma(\alpha+1))^2}$$

Theorem 1.4. *For every* $f \in L^1_{\alpha,n} \cap L^2_{\alpha,n}$ *we have the Plancherel formula*

$$\int_0^{+\infty} |f(x)|^2 x^{2\alpha+1}dx = \int_0^{+\infty} |\mathcal{F}f(\lambda)|^2 \, d\mu_{\alpha+2n}(\lambda)$$

2 Main result

In this section, we give an inversion formula for the generalized Fourier-Bessel transform associated with Bessel operator.

Theorem 2.1. *suppose that* $f \in L^1_\alpha(0,\infty) \cap L^1_{\alpha,n}(0,\infty)$ *such that*
(i) f *is of bounded variation in a neighborhood of* $x > 0$.
(ii) $\int_0^1 |f(x)|x^{\alpha+\frac{1}{2}}dx < \infty$
Then

$$\lim_{A \longrightarrow +\infty} \int_0^A \mathcal{F}f(\lambda)\varphi_\lambda(x)d\mu_{\alpha+2n}(\lambda) = \frac{1}{2}[f(x+0) + f(x-0)]$$

Proof: Let $f \in L^1_\alpha(0,\infty) \cap L^1_{\alpha,n}(0,\infty)$.

$$\mathcal{F}f(\lambda) = \int_0^\infty f(x)\varphi_\lambda(x)x^{2\alpha+1}dx = \int_0^\infty f(x)j_{\alpha+2n}(\lambda x)x^{2\alpha+2n+1}dx,$$

on the other hand,

$$h_{\alpha+2n}(f)(\lambda) = \int_0^{\infty} f(t) j_{\alpha+2n}(\lambda t) \frac{t^{2\alpha+4n+1}}{2^{\alpha+2n}\Gamma(\alpha+2n+1)} dt.$$

Then,

$$h_{\alpha+2n}\left(\frac{f(t)}{t^{2n}}\right)(\lambda) = \int_0^{\infty} f(t) j_{\alpha+2n}(\lambda t) \frac{t^{2\alpha+2n+1}}{2^{\alpha+2n}\Gamma(\alpha+2n+1)} dt,$$

therefore,

$$\mathcal{F}f(\lambda) = 2^{\alpha+2n}\Gamma(\alpha+2n+1) h_{\alpha+2n}\left(\frac{f(t)}{t^{2n}}\right)(\lambda).$$

Now, let $A > 0$

$$\begin{aligned}\int_0^A \mathcal{F}f(\lambda)\varphi_\lambda(t) d\mu_{\alpha+2n}(\lambda) &= \int_0^A 2^{\alpha+2n}\Gamma(\alpha+2n+1) h_{\alpha+2n}\left(\frac{f(t)}{t^{2n}}\right)(\lambda) j_{\alpha+2n}(\lambda t) \frac{t^{2n}\lambda^{2\alpha+4n+1}}{4^{\alpha+2n}(\Gamma(\alpha+2n+1))^2} d\lambda \\ &= \frac{t^{2n}}{2^{\alpha+2n}\Gamma(\alpha+2n+1)} \int_0^A h_{\alpha+2n}(g)(\lambda) j_{\alpha+2n}(\lambda t) \lambda^{2\alpha+4n+1} d\lambda,\end{aligned}$$

where $g(t) = \frac{f(t)}{t^{2n}}$.

Thus,

$$\int_0^A \mathcal{F}f(\lambda)\varphi_\lambda(t) d\mu_{\alpha+2n}(\lambda) = t^{2n} \int_0^A h_{\alpha+2n}(g)(\lambda) j_{\alpha+2n}(\lambda t) dm_{\alpha+2n}(\lambda).$$

Hence, since

$$\int_0^1 |f(x)| x^{\alpha+2n+\frac{1}{2}} dx \leqslant \int_0^1 |f(x)| x^{\alpha+\frac{1}{2}} dx < +\infty,$$

then by theorem 1.1, we have

$$\begin{aligned}\lim_{A \longrightarrow +\infty} \int_0^A \mathcal{F}(f)(\lambda)\varphi_\lambda(t) d\mu_{\alpha+2n}(\lambda) &= t^{2n} \lim_{A \longrightarrow +\infty} \int_0^A h_{\alpha+2n}(g)(\lambda) j_{\alpha+2n}(\lambda t) dm_{\alpha+2n}(t) \\ &= \frac{t^{2n}}{2}(g(t+0) + g(t-0)) \\ &= \frac{t^{2n}}{2}\left(\frac{f(t+0)}{t^{2n}} + \frac{f(t-0)}{t^{2n}}\right) \\ &= \frac{1}{2}[f(t+0) + f(t-0)]\end{aligned}$$

and this end the proof □

In the following we will give an example that shows that the exponent $\alpha+\frac{1}{2}$ in (ii) cannot be increased.
Indeed suppose that the resulat in theorem 2.1 holds when we have

$$\int_0^1 |f(x)|x^{\alpha+\epsilon+\frac{1}{2}}dx<\infty, \quad \epsilon>0$$

Let

$$f(t)=\begin{cases} t^{-\left(\alpha+\frac{3}{2}\right)}, & 0<t\leqslant 1, \\ 0, & t>1. \end{cases}$$

It is clear that f is of bounded variation, and

$$\int_0^1 |f(t)|t^{\alpha+\epsilon+\frac{1}{2}}dt=\int_0^1 \frac{dt}{t^{1-\epsilon}}<\infty, \quad \epsilon>0$$

therefore, theorem 2.1 implies that

$$\lim_{A\to+\infty} I_A(x)=\lim_{A\to+\infty}\int_0^A \mathcal{F}f(\lambda)\varphi_\lambda(x)d\mu_{\alpha+2n}(\lambda) \;=\frac{1}{2}[f(x+0)+f(x-0)]=0,.$$

let $r_1,r_2,\ldots$ be the positive real zero of $J_{\alpha+2n}(x)$ in ascending order and $s_i=\frac{r_i}{x}$, $x>1$, such that $\lim\limits_{i\to+\infty} s_i=+\infty$. In addition,

$$\begin{aligned} I_{s_i}=\int_0^{s_i}\mathcal{F}f(\lambda)\varphi_\lambda(x)d\mu_{\alpha+2n}(\lambda) &= \int_0^{s_i}\int_0^1 \frac{t^{2\alpha+2n+1}j_{\alpha+2n}(\lambda t)}{t^{\alpha+\frac{3}{2}}}\varphi_\lambda(x)\;dt\;d\mu_{\alpha+2n}(\lambda) \\ &= \int_0^{s_i}\int_0^1 J_{\alpha+2n}(\lambda t)J_{\alpha+2n}(\lambda x)\lambda\frac{x^{-\alpha}}{\sqrt{t}}\;dt\;d\lambda \end{aligned}$$

then By Fubini theorem, we obtain

$$I_{s_i}(x)=x^{-\alpha}\int_0^1\left(\int_0^{s_i} J_{\alpha+2n}(\lambda t)J_{\alpha+2n}(\lambda x)\lambda d\lambda\right)\frac{dt}{\sqrt{t}}.$$

But,

$$\begin{aligned} \int_0^{s_i} J_{\alpha+2n}(\lambda t)J_{\alpha+2n}(\lambda x)\lambda d\lambda &= s_i\left(x^2-t^2\right)^{-1}\left(xJ_{\alpha+2n+1}\left(s_ix\right)J_{\alpha+2n}\left(s_it\right)-tJ_{\alpha+2n+1}\left(s_it\right)J_{\alpha+2n}\left(s_ix\right)\right) \\ &= s_i\left(x^2-t^2\right)^{-1}\left(xJ_{\alpha+2n+1}\left(s_ix\right)J_{\alpha+2n}\left(s_it\right)\right) \\ &= r_i\left(x^2-t^2\right)^{-1}J_{\alpha+2n+1}\left(r_i\right)J_{\alpha+2n}\left(s_it\right). \end{aligned}$$

Therefore

$$\begin{aligned} I_{s_i}(x) &= x^{-\alpha}\int_0^1 r_i\left(x^2-t^2\right)^{-1}J_{\alpha+2n+1}\left(s_ix\right)J_{\alpha+2n}\left(s_it\right)\frac{dt}{\sqrt{t}} \\ &= s_ix^{1-\alpha}J_{\alpha+2n+1}(r_i)\int_0^1\frac{J_{\alpha+2n}(s_it)}{x^2-t^2}\frac{dt}{\sqrt{t}} \end{aligned}$$

By change of variable $u = s_i t$, we get

$$\int_0^1 \frac{J_{\alpha+2n}(s_i t)}{(x^2 - t^2)} s_i \frac{dt}{\sqrt{t}} = \int_0^{s_i} \frac{J_{\alpha+2n}(u)}{\left(x^2 - \left(\frac{u}{s_i}\right)^2\right)\left(\frac{u}{s_i}\right)^{\frac{1}{2}}} du$$
$$= s_i^{\frac{1}{2}} \int_0^{s_i} \frac{J_{\alpha+2n}(u)}{\left(x^2 - \left(\frac{u}{s_i}\right)^2\right) u^{\frac{1}{2}}} du$$
$$= s_i^{\frac{1}{2}} x^{-2} \int_0^{s_i} \frac{J_{\alpha+2n}(u)}{\left(1 - \left(\frac{u}{r_i}\right)^2\right) u^{\frac{1}{2}}} du.$$

therefore,

$$\lim_{i\to+\infty} x^{-2} \int_0^{s_i} \frac{J_{\alpha+2n}(u)}{\left(1 - \left(\frac{u}{r_i}\right)^2\right) u^{\frac{1}{2}}} du = x^{-2} \int_0^{+\infty} J_{\alpha+2n}(u) u^{\frac{-1}{2}} du = \Gamma\left(\frac{(2\alpha+4n+1)/4}{2^{\frac{1}{2}}\Gamma((2\alpha+4n+3)/4)}\right)$$

From [2], it follows that

$$J_{\alpha+2n+1}(x) = (2/\pi x)^{\frac{1}{2}} \cos(x - \beta - (\pi/2)) + O\left(x^{\frac{-3}{2}}\right)$$
$$= (2/\pi x)^{\frac{1}{2}} \sin(x - \beta) + O\left(x^{\frac{-3}{2}}\right),$$

where $\beta = (2\alpha + 4n + 1)\pi/4$ and

$$J_{\alpha+2n}(x) = (2/\pi x)^{\frac{1}{2}} \cos(x - \beta) + O\left(x^{\frac{-3}{2}}\right).$$

Since $J_{\alpha+2n}(r_i) = 0$, we see that

$$|J_{\alpha+2n+1}(r_i)| \geq (\pi r_i)^{\frac{-1}{2}}$$

for n sufficiently large. Thus for some constant C we have

$$|I_{s_i}(\lambda)| \geq C x^{\frac{1}{2}-\alpha-2n}, \text{ for all } n \in \mathbb{N}.$$

Then

$$\lim_{i\to+\infty} |I_{s_i}(\lambda)| = 0 \geq C x^{\frac{1}{2}-\alpha-2n}.$$

so, it follows that the exponent $\alpha + \frac{1}{2}$ cannot be increased.

Compliance with ethical standards

Conflict of interest The authors declare that they have no confict of interest.

Human participants This article does not contain any studies with human participants or animals performed by the authors

References

[1] A. L. Schwartz, *An inversion formula for Hankel transform*, Proceeding of the american mathematical society, 22 (3), 713-717 (1969).

[2] E. C. Titchmarsh, *Introducion to the theory of Fourier integrals*, Oxford Univ. Press, London, (1959)

[3] N. M. Lebedev, *Special functions and their applications*, New-York: Dover Publications; (1972).

[4] G. N. Watson, *A treatise on the theory of Bessel functions*, (Cambridge Mathematical Library). Cambridge: Cambridge University Press; (1995).

[5] M. Dziri, S. Sahbani, *An inversion theorem for Dunkl transform*, Transactions of NAS of Azerbaijan, Issue Mathematics, 37 (4), 33–41 (2017).

[6] R. F. Al Subaie and M. A. Mourou, *The continuous wavelet transform for a Bessel type operator on the half line*, Mathematics and Statistics, 1 (4)(2013), 196-203.

[7] R. F. Al Subaie and M. A. Mourou, *Transmutation operators associated with a Bessel type operator on the half line and certain of their applications*, Tamsii. oxf. J. Inf. Math. Scien., 29 (3) (2013), 329-349.

[8] R. Daher, S. El Ouadih, M. El Hamma, *Dini Lipschitz Functions for the Generalized Fourier-Bessel Transform in the Space* $L^2_{\alpha,n}$, Journal of Mathematical Extension, Vol. 10, No. 2, (2016), 35-46.

[9] A. G. Sveshnikov, A. N. Bogolyubov and V. V. Kratsov, *Lectures on mathematical physics*, Nauka, Moscow, 2004.

[10] V. S. Vladimirov, *Equations of mathematical physics*, Marcel Dekker, New York, 1971.

[11] V. A. Abilov and F. V. Abilova, *Approximation of functions by Fourier Bessel sums*, Izv. Vyssh. Uchebn. Zaved., Mat., 8 (2001), 3-9.