# THE COERCIVE UNIFORM ESTIMATE FOR SOME NONLOCAL DIFFERENTIAL OPERATOR EQUATIONS

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**Abstract**

In this paper we study the maximal regularity properties of the Cauchy problem for the abstract nonlocal parabolic equation with parameters in weighted spaces.

**Keywords**: Nonlocal equations, Cauchy problem, sectorial operators

## 1.Introduction

In recent years, the maximal regularity properties of abstract differential equations, especially for elliptic and parabolic types have been studied extensively, e.g. in [1],[2],[4],[5],[11] and the references therein. Moreover, the nonlocal differential equations have been treated e.g. in [8]. Convolution operators in Banach–valued function spaces studied e.g. in [6],[11]. However, the nonlocal differential operator equations are relatively less investigated subjects. The parabolic type nonlocal differential equation with bounded operator coefficients was investigated in [3]. The main aim of the present paper is to establish maximal regularity properties of the Cauchy problem for the following parabolic nonlocal differential operator equations with parameter

$$\frac{\partial u}{\partial t} + \sum_{|\alpha| \leq l} \varepsilon_\alpha a_\alpha * D^\alpha u + A * u = f(t, x),\ t \in (0, T),\ x \in R^n, \tag{1}$$

$$u(0, x) = 0,\ x \in R^n,\ 0 < T < \infty,$$

in $E$-valued mixed $L_{\mathbf{p},\gamma}$ spaces, where $a_\alpha = a_\alpha(x)$ are complex-valued functions, $l$ is a natural number, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\alpha_k$ are nonnegative integers, $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$, $\varepsilon_\alpha = \prod_{k=1}^n \varepsilon_k{}^{\frac{\alpha_k}{l}}$, $\varepsilon_k$ are positive parameter and $A = A(x)$ is a linear operator in a Banach space $E$. Here, the convolutions $a_\alpha * D^\alpha u, A * u$ are defined in the distribution sense (see e.g. [2]).

## 2.Notations and background

Let $E$ be a Banach space and $\gamma=\gamma(x),\ x=(x_1,x_2,\ldots,x_n)$ be a positive measurable weighted function on a measurable subset $\Omega\subset R^n$. Let $L_{p,\gamma}(\Omega;E)$ denote the space of all strongly measurable $E$-valued functions that are defined on $\Omega$ with the norm

$$\|f\|_{L_{p,\gamma}}=\|f\|_{L_{p,\gamma}(\Omega;E)}=\left(\int_{\Omega}\|f(x)\|_E^p\,\gamma(x)dx\right)^{\frac{1}{p}},\ 1\leq p<\infty,$$

for $\gamma(x)\equiv 1$, the space $L_{p,\gamma}(\Omega;E)$ will be denoted by $L_p=L_p(\Omega;E)$

$$\|f\|_{\ L_{\infty,\gamma}(\Omega;E)}=ess\sup_{x\in\Omega}\left[\gamma(x)\,\|f(x)\|_E\right].$$

The weight function $\gamma=\gamma(x)$ is said to satisfy an $A_p$ condition, i.e., $\gamma(x)\in A_p$, $1<p<\infty$ if there is a positive constant $C$ such that

$$\sup_Q\left(\frac{1}{|Q|}\int_Q\gamma(x)dx\right)\left(\frac{1}{|Q|}\int_Q\gamma^{-\frac{1}{p-1}}(x)dx\right)^{p-1}\leq C$$

for all compacts $Q\subset R^n$ (see [7, Ch.9].

Let $C$ be the set of complex numbers and

$$S_\varphi=\{\lambda:\lambda\in C,\ |arg\lambda|\leq\varphi\}\cup\{0\},\ 0\leq\varphi<\pi.$$

Let $E_1$ and $E_2$ be two Banach spaces and let $B\left(E_1,E_2\right)$ denote the space of bounded linear operators from $E_1$ to $E_2$. For $E_1=E_2=E$ we denote $B(E,E)$ by $B(E)$.

A closed linear operator $A$ is said to be $\varphi$-sectorial in Banach space $E$ with bound $M>0$ if $KerA=\{0\}$, $D(A)$ and $R(A)$ are dense on $E$ and

$$\left\|(A+\lambda I)^{-1}\right\|_{B(E)}\leq M|\lambda|^{-1}$$

for all $\lambda\in S_\varphi,\ \varphi\in[0,\pi)$ , where $I$ is an identity operator in $E$. It is known that the fractional powers of the operator $A$ are well defined. Let $E\left(A^\theta\right)$ denote the space $D\left(A^\theta\right)$ with the graph norm

$$\|u\|_{E(A^\theta)}=\left(\|u\|_E^p+\left\|A^\theta u\right\|_E^p\right)^{\frac{1}{p}},\ 1\leq p<\infty,\ -\infty<\theta<\infty.$$

Let $S=S\left(R^n;E\right)$ denotes the Schwartz class, i.e., the space of $E-$ valued rapidly decreasing smooth functions on $R^n$, equipped with its usual topology generated by seminorms. Here, $S'\left(R^n;E\right)$ denotes the space of all continuous linear operators $L:S\left(R^n;E\right)\rightarrow E$, equipped with the bounded convergence topology. Recall $S\left(R^n;E\right)$ is norm dense in $L_{p,\gamma}\left(R^n;E\right)$ when $1<p<\infty,\ \ \gamma\in A_p$.

Let $F$ denotes the Fourier transform defined by

$$\widehat{u}(\xi)=Fu=\int_{R^n}e^{-ix\xi}u(x)dx\qquad for\ u\in S\left(R^n;E\right)$$

and $x, \xi \in R^n$. It is known that

$$F\left(D_x^\alpha f\right) = (i\xi_1)^{\alpha_1} \dots (i\xi_2)^{\alpha_n} \widehat{f},\ D_\xi^\alpha\left(F(f)\right) = F\left[(-ix_1)^{\alpha_1} \dots (-ix_2)^{\alpha_n} f\right]$$

for all $f \in S^{'}\left(R^n; E\right)$.

The inverse Fourier transform

$$F^{-1}u = (2\pi)^{-n} \int_{R^n} e^{ix\xi} \widehat{u}(\xi) d\xi.$$

Let $E$ be Banach space. The function $u \to Tu : R^n \to B(E)$ is called a Fourier multiplier $L_{p,\gamma}\left(R^n; E\right)$ for $p \in (1, \infty)$ if

$$\left\|F^{-1}TFu\right\|_{L_{p,\gamma}(R^n;E)} \leq C\left\|u\right\|_{L_{p,\gamma}(R^n;E)}, u \in S\left(R^n; E\right).$$

The space of all Fourier multipliers from $L_{p,\gamma}\left(R^n; E\right)$ will be denoted $M_{p,\gamma}^{p,\gamma}(E)$.

A Banach space $E$ is called a $UMD$ space (see e.g. [3], [10]) if the Hilbert operator

$$(Hf)(x) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} dy$$

is initially defined on $S\left(R;E\right)$ and is bounded in $L_p\left(R;E\right), p \in (1, \infty)$ ([5]).

A family of operators $\mathcal{T} \subset B\left(E_1, E_2\right)$ is called $R-$ bounded if there is a constant $C > 0$ such that for all $T_1, T_2, \dots, T_k \in \mathcal{T}$ and $u_1, u_2, \dots, u_m \in E_1$ and for all independent, symmetric $\{-1; 1\}$ valued random variables $\mu_j$ on $[0; 1]$

$$\int_0^1 \left\|\sum_{j=1}^m \mu_j(y) T_j u_j\right\|_{E_2} dy \leq C \int_0^1 \left\|\sum_{j=1}^m \mu_j(y) u_j\right\|_{E_1} dy$$

is valid. The smallest $C$ is called the $R$-bound of $\mathcal{T}$ and denoted by $R\left(\mathcal{T}\right)$.

A Banach space $E$ is said to be a space satisfying the multiplier condition with respect to weighted function $\gamma$ and $p \in (1, \infty)$ if for any $\Psi \in C^{(n)}\left(R^n \backslash \{0\}; B(E)\right)$ the $R$-boundedness of the set

$$\left\{|\xi|^{|\beta|} D_\xi^\beta \Psi(\xi) : \xi \in R^n \backslash \{0\}, \beta = \left(\beta_1, \beta_2, \dots, \beta_n\right)\ \beta_k \in \{0, 1\}\right\}$$

implies that $\Psi$ is a Fourier multiplier in $L_{p,\gamma}\left(R^n; E\right)$, i.e. $\Psi \in M_{p,\gamma}^{p,\gamma}(E)$.

A sectorial operator $A(x),\ x \in R^n$ is said to be uniformly $R-$ sectorial in a Banach space $E$ if there exists $\varphi \in [0, \pi\ )$ such that

$$x \in R^n sup\ R\left\{\left[A(x)\left(A(x) + \xi I\right)^{-1}\right] : \xi \in S_\varphi\right\} \leq M.$$

Let $A = A(x), x \in R^n$ be closed linear operator in $E$ with domain $D(A)$ independent $x$. The Fourier transformation of $A(x)$ is a linear operator with the domain $D(A)$ defined

$$\widehat{A}(\xi) u(\varphi) = A(x) u\left(\widehat{\varphi}\right) \quad for\ u \in S^{'}\left(R^n; E\right),\ \varphi \in S\left(R^n\right).$$

Let $A=A(x)$ be a linear operator with domain $D(A)$ independent on $x \in R^n$ such that $Au \in L^{'}\left(R^n;E\right)$ for $u \in S\left(R^n;D(A)\right)$. The convolution $A*u$ of $A$ and $u \in S\left(R^n;D(A)\right)$ is defined as

$$A*u=\int_{R^n} A(x)u(x-\xi)d\xi \quad for\ u \in S\left(R^n;D(A)\right).$$

Let $E_0$ and $E$ be two Banach spaces where $E_0$ is continuously and densely embedded into $E$. Let $l$ be a natural number. $W^l_{p,\gamma}\left(R^n;E_0,E\right)$ denotes the space of all functions from $S^{'}\left(R^n;E_0\right)$ such that $u \in L_{p,\gamma}\left(R^n;E_0\right)$ and the generalized derivatives $D^l_k u=\frac{\partial^l u}{\partial x^l_k} \in L_{p,\gamma}\left(R^n;E\right)$ with the norm

$$\|u\|_{W^l_{p,\gamma}(R^n;E_0,E)}=\|u\|_{L_{p,\gamma}(R^n;E_0)}+\sum_{k=1}^{n}\left\|D^l_k u\right\|_{L_{p,\gamma}(R^n;E)}<\infty.$$

### 3.Nonlocal differential operator equations

Consider the following nonlocal differential operator equation

$$\sum_{|\alpha|\le l}\varepsilon_\alpha a_\alpha * D^\alpha u + A*u=f(x),\ x\in R^n, \tag{2}$$

where $A=A(x)$ is a linear operator in a Banach space $E$ for $x \in R^n$, $a_\alpha=a_\alpha(x)$ are complex − valued functions.

We defined sufficient conditions for the separability of a linear problem which are the followings.

**Condition 1.** Suppose the followings are satisfied:

(1) $L_\varepsilon(\xi)=\sum\limits_{|\alpha|\le l}\varepsilon_\alpha\widehat{a_\alpha}(\xi)(i\xi)^\alpha \in S_{\varphi_1},\ \varphi_1\in[0,\pi) \quad for\ \xi \in R^n,$

$$\left|L_\varepsilon(\xi)\right| \ge C\sum_{k=1}^{n}\varepsilon_k\left|\widehat{a}_{\alpha(l,k)}\right|\left|\xi_k\right|^l,$$

$\alpha(l,k)=(0,0,\ldots,l,0,0,\ldots,0),\ i.e.\ \alpha_i=0,\ \ i\ne k,\ \alpha_k=l,\ i=1,2,\ldots,n;$

(2) $\widehat{a}_\alpha \in C^{(n)}\left(R^n\right)$ and

$$|\xi|^{|\beta|}\left|D^{(\beta)}\widehat{a_\alpha}(\xi)\right|\le C_1,\ \beta_k\in\{0,1\},\ 0\le|\beta|\le n;$$

(3) for $0\le|\beta|\le n,\ \xi,\xi_0\in R^n\setminus\{0\}$:

$$\left[D^\beta\widehat{A}(\xi)\right]\widehat{A}^{-1}\left(\xi_0\right)\in C\left(R^n;B(E)\right),\ |\xi|^{|\beta|}\left\|\left[D^\beta\widehat{A}(\xi)\right]\widehat{A}^{-1}\left(\xi_0\right)\right\|_{B(E)}\le C_2.$$

Here $\widehat{A}(\xi)$ is a uniformly $\varphi$-sectorial operator in $E$ with $\varphi\in[0,\pi)$ .

Consider operator functions

$$\sigma_{0\varepsilon}(\xi,\lambda)=\lambda D_{\varepsilon}(\xi,\lambda),\ \sigma_{1\varepsilon}(\xi,\lambda)=\widehat{A}(\xi)D_{\varepsilon}(\xi,\lambda),$$

$$\sigma_{2\varepsilon}(\xi,\lambda)=\sum_{|\alpha|\leq l}\varepsilon_{\alpha}|\lambda|^{1-\frac{|\alpha|}{l}}\widehat{a_{\alpha}}(\xi)(i\xi)^{\alpha}D_{\varepsilon}(\xi,\lambda),$$

where

$$D_{\varepsilon}(\xi,\lambda)=\left[\widehat{A}(\xi)+L_{\varepsilon}(\xi)+\lambda\right]^{-1}.$$

In our old work [?] we proved the following lemma.

Assume that Condition 1 is satisfied. If the operator functions $\sigma_{i\varepsilon}(\xi,\lambda)$ for $\lambda\in S_{\varphi_2},\ \varphi_1\in[0,\pi)$ and the operators $|\xi|^{|\beta|}D_{\xi}^{\beta}\sigma_{i\varepsilon}(\xi,\lambda),\ i=0,1,2$ are uniformly bounded, then the following sets

$$S_{i\varepsilon}(\xi,\lambda)=\left\{|\xi|^{|\beta|}D_{\xi}^{\beta}\sigma_{i\varepsilon}(\xi,\lambda);\xi\in R^{n}\backslash\left\{0\right\}\right\},\ i=0,1,2$$

are uniformly $R-$ bounded for $\beta_k\in\{0,1\}$ , $0\leq|\beta|\leq n$.

Here, $E$ is a Banach space satisfying the multiplier condition with respect to weighted function $\gamma$ and $p\in(1,\infty)$ and $\widehat{A}(\xi)$ is a uniformly $R-$ sectorial operator in $E$ with $\varphi\in[0,\pi)$.

Now, consider the following nonlocal differential operator equation

$$\sum_{|\alpha|\leq l}\varepsilon_{\alpha}a_{\alpha}*D^{\alpha}u+(A+\lambda)*u=f, \tag{3}$$

where $\varepsilon,\varepsilon_{\alpha},\lambda$ are parameters, $a_{\alpha}$ are complex $-$ valued functions defined in (1.1) and $A$ is a linear operator in a Banach space $E$.

Assume that Condition 1 holds and $E$ is a Banach space satisfying the multiplier condition with respect to weighted function $\gamma$ and $p\in(1,\infty)$. Let $\widehat{A}(\xi)$ be a uniformly $R$-sectorial operator in $E$ with $\varphi\in[0,\pi)$. $\lambda\in S_{\varphi_2}$ and $0\leq\varphi+\varphi_1+\varphi_2<\pi$. Then, problem (3.3) has a unique solution $u$ and the coercive uniform estimate holds

$$\sum_{|\alpha|\leq l}\varepsilon_{\alpha}|\lambda|^{1-\frac{|\alpha|}{l}}\left\|a_{\alpha}*D^{\alpha}u\right\|_{X}+\left\|A*u\right\|_{X}+|\lambda|\left\|u\right\|_{X}\leq C\left\|f\right\|_{X} \tag{4}$$

for all $f\in X$ and $\lambda\in S_{\varphi}$.

Here,

$$X=L_{p,\gamma}\left(R^{n};E\right),\ Y=W_{p,\gamma}^{l}\left(R^{n};E(A),E\right),\ p\in(1,\infty).$$

Let $\Phi_{\varepsilon}$ be an operator in $X$ generated by problem (3.3) for $\lambda=0$, i.e.

$$D\left(\Phi_{\varepsilon}\ \right)\subset Y,\ \Phi_{\varepsilon}u=\sum_{|\alpha|\leq l}\varepsilon_{\alpha}a_{\alpha}*D^{\alpha}u+A*u.$$

Assume that Theorem 1 holds and the following conditions are satisfied:

1) $C_1 \left\| \widehat{A}(\xi_0) u \right\|_E \leq \|A(x)u\|_E \leq C_2 \left\| \widehat{A}(\xi_0) u \right\|_E, \xi_0 \in R^n, \ u \in D(A), x \in R^n$;

2) for $\alpha(l,k) = (0,0,\ldots,l,0,0,\ldots,0)$, *i.e.* $\alpha_i = 0, \quad i \neq k, \ \alpha_k = l$,

$$C_1 \sum_{k=1}^{n} \varepsilon_k \left| \widehat{a}_{\alpha(l,k)} \right| |\xi_k|^l \leq |L_\varepsilon(\xi)| \leq C_2 \sum_{k=1}^{n} \varepsilon_k \left| \widehat{a}_{\alpha(l,k)} \right| |\xi_k|^l, \ \xi \in R^n,$$

and there exists $x_0 \in R^n$ such that

$$\widehat{A}(\xi) A^{-1}(x_0) \in L_\infty(R^n; B(E)), \ \xi, x_0 \in R^n,$$

$$C_1 \|A(x_0) u\| \leq \|A(x)u\| \leq C_2 \|A(x_0) u\|, \ u \in D(A), x \in R^n$$

where $C_1, \ C_2$ are positive constants.

Then for $u \in Y$ there are positive constants $M_1, \ M_2$ such that

$$M_1 \|u\|_Y \leq \|\Phi_\varepsilon u\|_X \leq M_2 \|u\|_Y .$$

From Theorem 1 we have:

**Result 1.** Assume that the all conditions of Theorem 1 are satisfied. Then, for all $\lambda \in S_{\varphi_2}$ the following uniform coercive estimate holds

$$\sum_{|\alpha| \leq l} \varepsilon_\alpha |\lambda|^{1-\frac{|\alpha|}{l}} \left\| a_\alpha * D^\alpha (\Phi_\varepsilon + \lambda)^{-1} \right\|_{B(X)} + \left\| A * (\Phi_\varepsilon + \lambda)^{-1} \right\|_{B(X)} + \left\| \lambda (\Phi_\varepsilon + \lambda)^{-1} \right\|_{B(X)} \leq C.$$

## 4.The Cauchy problem for parabolic nonlocal differential operators equations

In this section, we shall consider the Cauchy problem for the nonlocal parabolic equation

$$\frac{\partial u}{\partial t} + \sum_{|\alpha| \leq l} \varepsilon_\alpha a_\alpha * D^\alpha u + A * u = f(t,x), \qquad (5)$$

$$u(0,x) = 0, \quad t \in (0,T), \ x \in R^n, \ T < \infty$$

where $\varepsilon_\alpha$ is defines as in (1.1), $a_\alpha$ are complex valued functions defined as in (1.1) and $A$ is a linear operator in a Banach space $E$.

By using the definition of the norm of the function for the space $L_{\mathbf{p}}(R^n)$, $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ which the norm is

$$\|f\|_{\mathbf{p},R^n} = \|f\|_{(p_1,p_2,\ldots,p_n),R^n} = \left\{ \int_{R^n} \left[ \cdots \left\{ \int_{R^2} \left( \int_{R^1} |f(x)|^{p_1} dx_1 \right)^{\frac{p_2}{p_1}} dx_2 \right\}^{\frac{p_3}{p_2}} \cdots \right]^{\frac{p_n}{p_{n-1}}} dx_n \right\}^{\frac{1}{p_n}},$$

we can be denoted the space of all $\mathbf{p}$-summable $E$-valued functions for $R_T^{n+1} = (0,T)\times R^n$, $\mathbf{p}=(p,p_1)$, $Z=L_{\mathbf{p},\gamma}\left(R_T^{n+1};E\right)$ with mixed norm, which is

$$\|f\|_Z = \left(\int_{R^n}\left(\int_0^T \|f(x,t)\|_E^p \gamma(x)dx\right)^{\frac{p_1}{p}} dt\right)^{\frac{1}{p_1}} < \infty.$$

Here, the space of all measurable $E$-valued functions $f$ defined on $R_T^{n+1}$.

Let $E_0$ and $E$ be two Banach spaces, where $E_0$ is continuously and densely embedded into $E$. Suppose $l$ is an integer and $Z_0 = W_{\mathbf{p},\gamma}^{1,l}\left(R_T^{n+1};E_0,E\right)$ denotes the space of all functions $u \in Z$ such that the generalized derivatives $D_t u,\ D_k^l u \in Z$, with the norm

$$\|u\|_{Z_0} = \|u\|_{Z(E_0)} + \|D_t u\|_Z + \sum_{k=1}^{n}\left\|D_k^l u\right\|_Z,$$

where

$$Z(E_0) = L_{\mathbf{p},\gamma}\left(R_T^{n+1};\ E_0\right)$$

Applying Theorem 1 we establish the maximal regularity of (4.5) in $Z$. For this purpose, we need the following result:

Assume that the all conditions of Theorem 1 are satisfied. Then operator $\Phi_\varepsilon$ is uniformly $R$ sectorial in $X$.

The Result 1 implies that $\Phi_\varepsilon$ is a sectorial operator in $X$. We have to prove the $R$ boundedness of the set

$$\sigma_\varepsilon(\xi,\lambda) = \left\{\lambda\left(\Phi_\varepsilon+\lambda\right)^{-1} : \lambda\in S_\varphi\right\}.$$

Indeed, from the proof of Theorem 1 we have

$$\lambda\left(\Phi_\varepsilon+\lambda\right)^{-1} f = F^{-1}\sigma_{0\varepsilon}(\xi,\lambda)\hat{f},\ f\in X,$$

where

$$\sigma_{0\varepsilon}(\xi,\lambda) = \lambda\left[\hat{A}(\xi)+L_\varepsilon(\xi)+\lambda\right]^{-1}.$$

By using Lemma 1 and definition of $R$ boundedness, it is enough to show that the operator function $\sigma_{0\varepsilon}(\xi,\lambda)$ (depended on variable $\lambda$ and parameter $\xi$) is a multiplier in $X$. Then, we have

$$\int_0^1\left\|\sum_{j=1}^m \mu_j\lambda_j\left(\Phi_\varepsilon+\lambda_j\right)^{-1} f_j\right\|_X dy = \int_0^1\left\|\sum_{j=1}^m \mu_j F^{-1}\sigma_{0\varepsilon}\left(\xi,\lambda_j\right)\hat{f}_j\right\|_X dy =$$

$$= \int_0^1\left\|F^{-1}\sum_{j=1}^m \mu_j\sigma_{0\varepsilon}\left(\xi,\lambda_j\right)\hat{f}_j\right\|_X dy \le C\int_0^1\left\|\sum_{j=1}^m \mu_j f_j\right\|_X dy$$

for all $\xi\in R^n$, $\lambda_1,\lambda_2,\dots,\lambda_m\in S_\varphi$, $f_1,f_2,\dots,f_m\in X$, $m\in N$ where $\{\mu_j\}$ is a sequence of independent symmetric $\{-1;1\}$ valued random variables on $[0;1]$.

Hence, the set $\sigma_\varepsilon(\xi,\lambda)$ is uniformly $R$ bounded.

Now, we are ready to state the main result of this section.

Assume that all the conditions of Theorem 1 are satisfied for $\varphi \in \left(\frac{\pi}{2},\pi\right)$. Then the equation (4.1) has a unique solution $u \in W^{1,l}_{\mathbf{p},\gamma}\left(R^{n+1}_T; E(A), E\right)$. Moreover, the following coercive uniform estimate holds

$$\left\|\frac{\partial u}{\partial t}\right\|_Z + \sum_{|\alpha|\le l} \varepsilon_\alpha \left\|a_\alpha * D^\alpha u\right\|_Z + \left\|A * u\right\|_Z \le C \left\|f\right\|_Z . \tag{6}$$

By Fubini's theorem we have $Z = L_{p1}(0,\ T;X)$. Moreover, by definition of spaces $Y,\ Z_0$ for $E_0 = E(A)$ and by Theorem 2 we obtain

$$\|u\|_{Z_0} = \|u\|_{Z(A)} + \left\|\frac{du}{dt}\right\|_{L_{p1}(0,\ T;X)} + \|\Phi_\varepsilon u\|_{L_{p1}(0,\ T;X)} \simeq \left\|\frac{\partial u}{\partial t}\right\|_Z + \|\Phi_\varepsilon u\|_Z \simeq$$

$$\simeq \|Au\|_Z + \left\|\frac{\partial u}{\partial t}\right\|_Z + \sum_{k=1}^{n} \left\|D^l_k u\right\|_Z \simeq \|u\|_{Z_0}$$

where

$$Z(A) = L_{p1}\left(0,\ T; X(A)\right),\ X(A) = L_{\mathbf{p},\gamma}\left(R^n_T; E(A)\right),\ X = L_{p,\gamma}\left(R^n; E\right).$$

Hence, we get

$$Z_0 = W^1_{p1}\left(0,T; D\left(\Phi_\varepsilon\right), X\right),\ for\ E_0 = E(A).$$

Therefore, the problem (4.5) can be expressed as

$$\frac{du}{dt} + \Phi_\varepsilon u(t) = f(t),\ u(0) = 0,\ t \in R_+. \tag{7}$$

By virtue of [2, Theorem 4.5.2] and [6], $X$ is a Banach space satisfying the multiplier condition with respect to $p \in (1,\infty)$. Then due to $R$ sectoriality of $\Phi_\varepsilon$ with $\varphi \in \left(\frac{\pi}{2},\pi\right)$, by virtue of [12, *Theorem* 4.2] , for $f \in L_{p1}(0,\ T;X)$ the problem

$$\left\|\frac{du}{dt}\right\|_{L_{p1}(0,\ T;X)} + \|\Phi_\varepsilon u\|_{L_{p1}(0,\ T;X)} \le C \|f\|_{L_{p1}(0,\ T;X)} .$$

In view of Results 1 and from the above estimate, we get (4.6).

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