

ASCENTS AND DESCENTS S-SPECTRUM IN QUATERNIONIC FRAMEWORK

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Abstract

In this paper, we present and study the ascent S-spectrum (A.S.S) and descent S-spectrum (D.S.S) of a bounded right linear operator \mathbf{T} defined on a two-sided quaternionic Banach space \mathbf{X} , as well as, the essential ascent S-spectrum (E.A.S.S) and essential descent S-spectrum (E.D.S.S). On the one hand, we establish a connection between A.S.S, D.S.S, E.A.S.S, and E.D.S.S with the usual ascent, descent, essential ascent, and essential descent spectra, respectively. Furthermore, we show that, under certain conditions, the spectral mapping theorem holds for these spectra for any intrinsic slice hyperholomorphic function \mathbf{f} defined on $\sigma_{\mathbf{S}}(\mathbf{T})$. On the other hand, we prove some analogous properties known in the complex framework for their quaternionic counterparts. In particular, if \mathbf{s} is an arbitrary element of the essential ascent S-spectrum (respectively, the essential descent S-spectrum), then the pseudo-resolvent $\mathbf{Q}_{\mathbf{q}}(\mathbf{T})$ of \mathbf{T} is semi-regular and upper semi-Fredholm (respectively, semi-regular and lower semi-Fredholm) for all \mathbf{q} in $\mathcal{V} \setminus [\mathbf{s}]$, where $[\mathbf{s}]$ is the the 2-dimensional sphere associated to \mathbf{s} and \mathcal{V} is a neighborhood of \mathbf{s} . As a result, the A.S.S, D.S.S, E.A.S.S, and E.D.S.S are compact subsets of the S-spectrum. Moreover, this work introduces the concept of spherical poles of the S-resolvent, the quaternionic analog of the resolvent poles in the complex frame. We investigate the relationships between these poles and the ascent/descent S-spectra (A.S.S./D.S.S.) and their essential counterparts

(E.A.S.S./E.D.S.S.). Finally, we establish that the A.S.S., D.S.S., and E.A.S.S. of the product operator TS (excluding zero) coincide with the corresponding spectra of ST (excluding zero).

Keywords: Quaternions, ascent S-spectrum, essential ascent S-spectrum, descent S-spectrum, essential descent S-spectrum.

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1 Introduction

Quaternion numbers are a type of hypercomplex number system that extends the complex numbers to a four-dimensional space. They were first introduced by the Irish mathematician William Rowan Hamilton in 1843. Functional analysis in the quaternion field, which generalizes classical functional analysis into the quaternionic framework, is a new field of research that specializes in studying the properties of quaternionic Banach and Hilbert spaces, quaternionic linear operators, and quaternionic versions of various concepts, such as linear operators, S-resolvents, S-spectra, S-functional calculus, and quaternionic spectral theory.

One of the key differences between quaternionic functional analysis and classical functional analysis is the non-commutativity of quaternion multiplication. Indeed, since the standard basis of quaternion numbers, that is the set $\{1, i, j, k\}$, is formed by the three non-commutative imaginary units i , j and k , then the quaternion multiplication is consequently non-commutative. Owing to that and depending on how the scalar multiplication is defined, the structure of quaternionic Banach spaces can be classified into three types: right, left, and two-sided. In that respect, the linearity of an operator T defined on a two-sided quaternionic Banach space X can be classified as either right-linear or left-linear. Moreover, in contrast to the complex framework, the new spectrum concept unique to the quaternionic framework gives it an additional layer of complexity not found in the complex counterpart. These distinctions present significant challenges for functional analysis and spectral theory in the quaternionic framework, where numerous major properties of various spectra in the complex framework do not hold necessarily in the quaternionic setting.

We denote by \mathbb{S} the unit sphere of purely imaginary quaternions. If we consider X as a right (respectively, left) quaternionic Banach space, then the right (respectively, left) multiplication of X restricted to the complex plane $\mathbb{C}_j := \{u + jv : u, v \in \mathbb{R}\}$, where j is an arbitrary element of \mathbb{S} , forms a Banach space over the field \mathbb{C}_j , and X becomes a complex Banach space denoted by X_j . Note that in the case where X is two-sided, we can restrict only the right or the left multiplication. Hence, it is possible to move from the quaternionic framework to the complex framework. In fact, Colombo and Sabadini in [17] clarified an important property: if we restrict the S-spectrum of a closed right linear operator T to \mathbb{C}_j , we obtain the spectrum of T considered as a closed complex linear operator on X_j . For any $j \in \mathbb{S}$, the spectra of T as a right linear operator on X are related to the spectra of T considered as \mathbb{C}_j -linear on X_j .

Precisely, if the operator T is defined on the complex Banach space X_j , its spectrum is determined using the operator $(\lambda I_j - T)$, where $\lambda \in \mathbb{C}_j$ and I_j is the identity on X_j ; however, if T is defined on the quaternionic Banach space X , its spectrum is determined using the pseudo-resolvent $Q_s(T) = T^2 - 2\operatorname{Re}(s)T + |s|^2$, where $s \in \mathbb{H}$, which is equal to $(\bar{s}I_j - T)(sI_j - T)$ when T is considered as \mathbb{C}_j -linear. This explains the deep relation between the spectrum of T as a right linear operator on X and the spectrum of T as a \mathbb{C}_j -linear operator defined on X_j .

In spectral theory, many researchers have given special attention to the concepts of ascent $p(T)$ and descent $q(T)$ of a bounded right linear operator T , which are defined as the minimum values of integer $n \geq 0$ such that $\mathcal{N}(T^n) = \mathcal{N}(T^{n+1})$ and $\mathcal{R}(T^n) = \mathcal{R}(T^{n+1})$, respectively. In the complex case, Fredj, Burgos, Oudghiri, and Olfa Bel Hadj studied the spectra of ascent, descent, essential ascent, and essential descent in their publications [7] and [8]. The study of these spectra in the quaternionic framework plays a central role in advancing the quaternionic functional analysis and the quaternionic spectral theory, and it provides a necessary foundation for advancing the development of mathematics in the quaternion field in the future.

Quaternion numbers find widespread applications in various fields of science and engineering. These include quantum mechanics, robotics and control systems, computer graphics and animation, aerospace and navigation, and signal processing, along with molecular dynamics simulations [see, [20], [21], [22], [23]]. In particular, quaternionic functional analysis and spectral theory have significant applications in physics, notably in the study of quantum mechanics and relativity theory, as well as in engineering and signal processing. They, also, have links to other branches of mathematics, such as algebraic geometry and representation theory.

To the best of our knowledge, the concepts of ascent, essential ascent, descent and essential descent S-spectrum have not yet been studied in the quaternionic framework. The objective of this paper is to introduce for the first time these concepts in the quaternionic framework, then, compare them to the corresponding spectra in the complex setting, and lastly, render the most well-known results in the complex framework into their quaternionic equivalents, which require fundamentally different approaches and ideas to prove.

In this paper, we define and study the ascent and essential ascent spectra, as well as the descent and essential descent spectra of a bounded right linear operator T that is defined on a two-sided quaternionic Banach space, denoted by X . We will refer to these four spectra, in the quaternionic setting, as the ascent and essential ascent spherical spectra (A.S.S and E.A.S.S, respectively) and the descent and essential descent spherical spectra (D.S.S and E.D.S.S, respectively). In particular, we demonstrate that the last four spectra are axially symmetric, which is a crucial result for the rest of this paper. We also establish a connection between these spectra and the classical spectra in the complex setting, enabling us to prove the spectral mapping theorem for A.S.S, E.A.S.S, D.S.S, and E.D.S.S for any intrinsic slice hyperholomorphic function under certain conditions. Moreover, we prove that if s is an element of the essential ascent (respectively, essential descent) of T , then the pseudo-resolvent $Q_q(T)$ is semi-regular and upper semi-Fredholm (respectively, semi-regular and lower semi-Fredholm) for all $q \in \mathcal{V} \setminus [s]$, where \mathcal{V} is an open neighborhood of s . As a consequence, the fact that

A.S.S, E.A.S.S, D.S.S, and E.D.S.S are axially symmetric implies that they are compact subsets of $\sigma_S(T)$. Finally, we demonstrate that the A.S.S, E.A.S.S, and D.S.S of the product TS minus $\{0\}$, equals the A.S.S, E.A.S.S, and D.S.S of the product ST minus $\{0\}$, respectively, where T and S are two bounded right linear operators. We note that the majority of the properties presented in this paper for the right linear operators also hold for their left linear counterparts.

To make the paper self-contained, we recall some definitions and properties of quaternion numbers. For more details, we refer the reader to ([16], [17], [19], [18]). Let \mathbb{H} denote the field of all quaternion numbers, and $\mathbb{H}^* = \mathbb{H} \setminus \{0\}$ the group of all invertible quaternions. An arbitrary quaternion number s can be written as $s = s_0 + s_1i + s_2j + s_3k$, $s_l \in \mathbb{R}$, $l = 0, 1, 2, 3$, where i, j, k are three quaternion imaginary units, satisfying $i^2 = j^2 = k^2 = -1$ and $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$, the number s_0 also denoted by $Re(s)$ is called the real part of s , while $s_1i + s_2j + s_3k$ is called vector part or the imaginary part of s denoted by $Im(s)$. The quaternion conjugate of s is $\bar{s} = s_0 - s_1i - s_2j - s_3k$, while $(\bar{s}s)^{1/2}$ denotes the usual norm of s ; moreover, if s is the non-zero element, it has an inverse $s^{-1} = \bar{s}/|s|^2$. One of the most important sets in \mathbb{H} is the sphere of purely imaginary quaternion of modulus 1, denote by \mathbb{S} , i.e.,

$$\mathbb{S} = \{I = s_1i + s_2j + s_3k : s_1, s_2, s_3 \in \mathbb{R} \text{ and } s_1^2 + s_2^2 + s_3^2 = 1\}.$$

If $I \in \mathbb{S}$, then we have $I^2 = -1$. Additionally, every non-real quaternion $s = s_0 + s_1i + s_2j + s_3k$ is uniquely determining an element $I_s = (s_1i + s_2j + s_3k)/|s_1i + s_2j + s_3k| \in \mathbb{S}$, such that $s = s_0 + I_s|s_1i + s_2j + s_3k| = s_0 + |s_1i + s_2j + s_3k|I_s$ (note that $rs = sr$ for all $r \in \mathbb{R}$ and $s \in \mathbb{H}$). If s is real, then $s = s + I0$ for all $I \in \mathbb{S}$. On the one hand, for all $s \in \mathbb{H}$ we can associate to s the 2-dimensional sphere

$$[s] = \{s_0 + I|Im(s)| : I \in \mathbb{S}\},$$

which is centered at the real point $s_0 = Re(s)$ and has radius $|Im(s)|$. On the other hand, if $I \in \mathbb{S}$, then the set $\mathbb{C}_I = \{s_1 + Is_2 : s_1, s_2 \in \mathbb{R}\}$ is an isomorphic copy of the complex numbers

This paper is organized as follows. In Section 2, we recall the definition of quaternionic Banach space, closed right linear operators, and some important results that will be used in the rest of this work. In Section 3, firstly, we define and investigate the ascent and essential ascent S-spectra for closed right linear operators, and prove their connection with the usual ascent and essential ascent spectra known for the complex frame. As a result, the spectral mapping theorem holds for those S-spectra for any intrinsic slice hyperholomorphic function under certain conditions. Moreover, we prove that if s is an arbitrary element of the ascent (respectively, essential ascent) S-resolvent of a bounded right linear operator T , then there exists a neighborhood \mathcal{V} of s such that $Q_q(T)$ is semi-regular and upper semi-Fredholm for any element $q \in \mathcal{V} \setminus [s]$; and in particular, we show that in the case where s is an element of the ascent S-resolvent of T , then $Q_q(T)$ is bounded below for every $q \in \mathcal{V} \setminus [s]$. As a consequence, we prove that the ascent and essential ascent S-spectra of T are compact subsets of $\sigma_S(T)$. Secondly, we investigate the relation between the poles of the S-resolvent $\Pi_S(T)$, the

ascent S-resolvent, and the essential ascent S-resolvent; specifically, $\Pi_S(T)$ equals the intersection of the ascent S-resolvent (respectively, essential ascent S-resolvent) and the boundary of the S-resolvent of T . Finally, we prove that the ascent and essential ascent of the product TS minus the singleton $\{0\}$ and ST minus the singleton $\{0\}$ are equal, where S is a bounded right linear operator that do not necessarily commute with T . In Section 4, we present and study almost the same properties as in Section 3 for the descent and essential descent S-spectra of a bounded right linear operator T . In particular, we show the connection between these spectra, the descent spectrum, and the essential descent spectrum in the complex plane. We also prove that the spectral mapping theorem holds for descent and essential descent S-spectra for any intrinsic slice hyperholomorphic function under certain conditions. Furthermore, we prove that if s is an arbitrary element of the essential descent (respectively, descent) S-spectrum, then there exists a neighborhood \mathcal{U} of s such that $Q_q(T)$ is semi-regular and lower semi-Fredholm (respectively, surjective) for any $q \in \mathcal{U} \setminus [s]$. This leads us to conclude that the descent and essential descent S-spectra are compact subsets of $\sigma_S(T)$. Finally, we end this section by proving that the descent S-spectrum of the product TS minus the singleton $\{0\}$ equals the descent S-spectrum of the product ST minus the singleton $\{0\}$.

2 PRELIMINARIES

This section covers the fundamental results pertaining to the structure of quaternionic Banach spaces and basic properties of the quaternionic field. For further details, we refer readers to [17] and [16]. Additionally, we present some results in the context of two-sided quaternionic Banach spaces, which can be proven similarly to their complex counterparts.

2.1 Quaternionic Banach space

In the beginning, we start by presenting the primary explanations of various structures of the quaternionic vector Banach spaces. It is worth mentioning that some of the definitions mentioned in this section can be found in [17].

Definition 1 ([17]). • *A quaternionic right vector space is an additive group $(X_R, +)$ endowed with a quaternionic scalar multiplication from the right such that for all $x, y \in X_R$ and all $a, b \in \mathbb{H}$, we have*

$$(x + y)a = xa + ya, \quad x(a + b) = xa + xb, \quad y(ab) = (ya)b, \quad y1 = y. \quad (1)$$

• *A quaternionic left vector space is an additive group $(X_L, +)$ endowed with a quaternionic scalar multiplication from the left such that for all $x, y \in X_L$ and all $a, b \in \mathbb{H}$, we have*

$$a(x + y) = ax + ay, \quad (a + b)y = ay + by, \quad (ab)y = a(by), \quad 1y = y. \quad (2)$$

• *Finally, a two-sided quaternionic vector space is an additive group $(X, +)$ endowed with a quaternionic scalar multiplication from the right and a quaternionic scalar*

multiplication from the left that satisfy (1) and (2) such that in addition $ay = ya$ for all $y \in X$ and all $a \in \mathbb{R}$.

- Remark 1.** 1. Any quaternion right (resp, left) vector space X_R (resp, X_L) also carry the structure of real vector space if we simply restrict the quaternion scalar multiplication to \mathbb{R} , in this case we obtain a real vector space. Similarly, if we choose some $j \in \mathbb{S}$ and identify \mathbb{C}_j with the field of complex numbers, then X_R (resp, X_L) also carries the structure of a complex vector space over \mathbb{C}_j . Again, we obtain this structure by restricting the quaternionic scalar multiplication to \mathbb{C}_j .
2. If we consider a two-sided quaternion vector space X , then the left and the right scalar multiplications of the real numbers coincide, so we can constrain them to \mathbb{R} to get a real vector space again. This is; however, not true for the multiplication with scalars in one complex plane \mathbb{C}_j , for some $j \in \mathbb{S}$. Indeed, in general, $sy \neq ys$ for $s \in \mathbb{C}_j$ and $y \in X$; hence, we can only restrict either the left or the right multiplication to \mathbb{C}_j to consider X as a \mathbb{C}_j -vector space, but not both simultaneously.

Definition 2 ([17]). A norm on a right, left, or two-sided quaternionic vector space X is a norm in the sense of real vector spaces that is compatible with the quaternionic right, left, or two-sided scalar multiplication. Precisely, this means that $\|ya\| = \|y\|\|a\|$, $\|ay\| = \|a\|\|y\|$, or $\|ay\| = \|a\|\|y\| = \|ya\|$ for all $a \in \mathbb{H}$ and all $y \in X$. A quaternionic right, left, or two-sided Banach space is a quaternionic right, left, or two-sided vector space that is endowed with a norm $\|\cdot\|$ and is complete with respect to the topology induced by this norm.

2.2 Right linear operator in quaternionic frame

Let X be a two-sided quaternionic Banach space. A mapping $T : \mathcal{D}(T) \subseteq X \rightarrow X$ is said to be right \mathbb{H} -linear (or right linear) operator if

$$T(xa + yb) = (Tx)a + (Ty)b \text{ for any } x, y \in \mathcal{D}(T) \text{ and } a, b \in \mathbb{H},$$

while T is said to be left \mathbb{H} -linear (or left linear) operator if

$$T(ax + by) = a(Tx) + b(Ty) \text{ for any } x, y \in \mathcal{D}(T) \text{ and } a, b \in \mathbb{H},$$

where $\mathcal{D}(T)$ stands for the domain of T .

The right and the left multiplication of T are defined and given by

$$(sT)(x) = s(Tx) \text{ and } (Ts)(x) = T(sx) \text{ for all } s \in \mathbb{H},$$

in this case, sT and Ts are two right linear operators. Moreover, the range and the kernel of T will be

$$\begin{aligned} \mathcal{R}(T) &= \{y \in X : Tx = y \text{ for } x \in X\}, \\ \mathcal{N}(T) &= \{x \in \mathcal{D}(T) : Tx = 0\}. \end{aligned}$$

Furthermore, the nullity of T , denoted by $\alpha(T)$, is the dimension of $\mathcal{N}(T)$. While the dimension of the quotient space $X/\mathcal{R}(T)$, denoted by $\beta(T)$, is the defect of T .

For a right linear operator T with domain $\mathcal{D}(T) \subseteq X$ and range $\mathcal{R}(T) \subseteq X$, the right subspace $\mathcal{G}(T) = \{(x, Tx) : x \in \mathcal{D}(T)\}$ of the product space $X \times X$ is called the

graph of T . Then, T is said to be closed whenever $\mathcal{G}(T)$ is a closed subspace of $X \times X$; equivalently, T is closed whenever it follows from $x_n \in \mathcal{D}(T)$ for $n = 1, 2, \dots, x_n \rightarrow x$ and $Tx_n \rightarrow y$ that $x \in \mathcal{D}(T)$ and $Tx = y$. The set of all closed right linear operators with domain in X into X will be denoted by $\mathcal{K}_R(X)$, and the identity linear operator on X will be denoted by \mathcal{I} .

By induction we define the iterates T^n for $n \geq 1$, where T^n is the linear operator with domain

$$\mathcal{D}(T^n) = \{x : x, Tx, \dots, T^{n-1}x \text{ are in } \mathcal{D}(T)\},$$

and for each x in $\mathcal{D}(T^n)$, we have $T^n x = T(T^{n-1}x)$. Also, we define $T^0 = I$ the identity operator from X into X . Let n and m be non-negative integers, then $x \in \mathcal{D}(T^{n+m})$ if and only if $T^n x \in \mathcal{D}(T^m)$, and in this case

$$T^m(T^n x) = T^{n+m}x.$$

Note that we have always the following properties

$$\mathcal{N}(T^n) \subseteq \mathcal{N}(T^{n+1}), \mathcal{D}(T^{n+1}) \subseteq \mathcal{D}(T^n), \text{ and } \mathcal{R}(T^{n+1}) \subseteq \mathcal{R}(T^n) \text{ for all } n \in \mathbb{N}.$$

Moreover, we define the hyper range and the hyper kernel of T by

$$\mathcal{N}^\infty(T) := \bigcup_{n \in \mathbb{N}} \mathcal{N}(T^n) \quad \text{and} \quad \mathcal{R}^\infty(T) := \bigcap_{n \in \mathbb{N}} \mathcal{R}(T^n).$$

Furthermore, if there is an integer $n \geq 0$ such that $\mathcal{N}(T^n) = \mathcal{N}(T^{n+1})$ (resp. $\mathcal{R}(T^n) = \mathcal{R}(T^{n+1})$), then the smallest such integer that verifies the last property is called the ascent of T , denoted by $p(T)$ (respectively, the descent of T , denoted by $q(T)$). Moreover, the sequence $c_n(T) := \dim(\mathcal{R}(T^n)/\mathcal{R}(T^{n+1}))$, $n \in \mathbb{N}$ is decreasing (the proof follows exactly as the complex case, see [12]). Furthermore, following the definitions of the complex frame (Mbekhta and Muller in [15]), the essential descent of T is defined as $q_e(T) := \inf\{n \geq 0 : c_n(T) < \infty\}$, where the infimum over the empty set is taken to be infinite. If $q_e(T)$ is finite, let $d(T)$ be the smallest positive integer k with $c_n(T) = c_k(T)$ for all $n \geq k$, trivially $q_e(T) \leq d(T)$ and if $q(T)$ is finite, then $q(T) = d(T)$. Similarly, the essential ascent of T is defined as $p_e(T) := \inf\{n \geq 0 : b_n(T) < \infty\}$, where $b_n(T) := \dim(\mathcal{N}(T^{n+1})/\mathcal{N}(T^n))$ and the infimum over the empty set is taken to be infinite. If $p_e(T)$ is finite, let $a(T)$ be the smallest positive integer k with $b_n(T) = b_k(T)$ for all $n \geq k$. In this case, $p_e(T) \leq a(T)$ and if $p(T)$ is finite, then $p(T) = a(T)$.

Suppose that T is closed right linear operator such that $\mathcal{D}(T) = X$ (i.e., its domain is the entire space X). Then, by the closed graph theorem in quaternionic frame ([17], Theorem 12.0.5), T is bounded right linear operator. This means that $\|T\| = \sup_{\|x\|=1} \|Tx\| < \infty$ [see [17], Definition 2.2.5 for the definition of norm]. The set of all bounded right linear will be denoted by $\mathcal{B}_R(X)$.

2.3 Basic results

To conclude this section, we provide next some fundamental properties that will be essential for the rest of this paper. Let us first begin by presenting the following lemma.

Lemma 1. *Let $T \in \mathcal{K}_R(X)$ and j be an arbitrary element of \mathbb{S} and let P and Q be two relatively prime polynomials with coefficients in \mathbb{C}_j . If $A = P(T)$ and $B = Q(T)$, then we have the following properties:*

- (i) $\mathcal{R}(A^n B^n) = \mathcal{R}(A^n) \cap \mathcal{R}(B^n)$ for all $n \in \mathbb{N}$;
- (ii) $\mathcal{N}(A^n B^n) = \mathcal{N}(A^n) + \mathcal{N}(B^n)$ for all $n \in \mathbb{N}$;
- (iii) $\mathcal{N}^\infty(A) \subseteq \mathcal{R}^\infty(B)$ and $\mathcal{N}^\infty(B) \subseteq \mathcal{R}^\infty(A)$;
- (iv) $\max\{p_e(A), p_e(B)\} \leq p_e(AB) \leq p_e(A) + p_e(B)$ and $p(AB) = \max\{p(A), p(B)\}$;
- (v) $\max\{q_e(A), q_e(B)\} \leq q_e(AB) \leq q_e(A) + q_e(B)$ and $q(AB) = \max\{q(A), q(B)\}$.

Proof. Due to the similarity of the proof to its complex counterpart, the details are omitted (see, [4], Lemma 4.4). \square

Let X be two-sided quaternionic Banach space. The right Banach dual space of X will be denoted by X^* , which represents the set of all bounded right linear functional h defined on X onto \mathbb{H} . Let T be a right linear operator with a dense domain $\mathcal{D}(T)$ in X , then we will define the adjoint operator T^* with domain $\mathcal{D}(T^*)$ consisting of all $h \in X^*$ such that $h \circ T$ is a bounded right linear functional on $\mathcal{D}(T)$. Thus, for any $f \in \mathcal{D}(T^*)$ and since $\mathcal{D}(T)$ is dense in X , there exists a unique bounded right linear function $h' \in X^*$ such that $h(Tx) = h'(x)$ for all $x \in \mathcal{D}(T)$. Hence, the range of T^* is now defined by $T^*h = h'$, i.e.,

$$(T^*h)(x) = h(Tx); h \in \mathcal{D}(T^*), x \in \mathcal{D}(T).$$

The operator T^* is called the adjoint operator of T . Furthermore, if T is closed, T^* is a closed left linear operator. Indeed, if the sequence $\{f_n; n = 1, 2, \dots\}$ in $\mathcal{D}(T^*)$ has the property that $f_n \rightarrow f$ in X^* and $T^*f_n \rightarrow g$ in X^* , then $f(Tx) = \lim f_n(Tx) = \lim(T^*f_n)(x) = g(x)$ for all $x \in \mathcal{D}(T)$, so $f \in \mathcal{D}(T^*)$ and $g = T^*f$.

Following the definitions in the complex framework (see [10]), for a non-empty subset $S \subset X$, we define the annihilator S^\perp of S as the set of all $h \in X^*$ such that $h(x) = 0$ for all $x \in S$. Similarly, if we choose a non-empty $H \subseteq X^*$, then we define the inverse annihilator ${}^\perp H$ as the set of all $x \in X$ such that $h(x) = 0$ for all $h \in H$. We now introduce the following lemma, which will be useful throughout the paper.

Lemma 2. *Let $T : \mathcal{D}(T) \subseteq X \rightarrow \mathbb{H}$ be a right linear operator with domain $\mathcal{D}(T)$ dense in X . Then, we have*

- (i) $\mathcal{R}(T)^\perp = \mathcal{N}(T^*)$ and $\overline{\mathcal{R}(T)} = {}^\perp \mathcal{N}(T^*)$;
- (ii) If T is closed, then $\mathcal{N}(T) = {}^\perp \mathcal{R}(T^*)$ and $\mathcal{R}(T^*) \subseteq \mathcal{N}(T)^\perp$. Furthermore, if $\mathcal{R}(T)$ is closed we get $\mathcal{R}(T^*) = \mathcal{N}(T)^\perp$;
- (iii) $\mathcal{R}(T)$ is closed if and only if $\mathcal{R}(T^*)$ is closed.

Proof. The proof is similar to the complex counterpart (see, [[10], Theorem 7.4, Theorem 2.4, and lemma 7.1]). \square

The following definitions have been established by Muraleetharan and Thirulogasanthar (see, [5] and [6]). In what follows, we generalize those notions of Fredholm operators and semi-regular operators in a two-sided quaternionic Banach space.

Definition 3. 1. A bounded right linear operator $T \in \mathcal{B}_R(X)$ is said to be Fredholm operator (respectively, lower semi-Fredholm, upper semi-Fredholm) if $\alpha(T)$ and $\beta(T)$ are both finite (respectively, $\beta(T)$ is finite, $\alpha(T)$ is finite and $\mathcal{R}(T)$ is closed). While T is said to be semi-regular if the range of T is closed and T verifies one of the equivalent conditions of Theorem 5.6 in [5].

2. If T is a semi-Fredholm operator, then the index of T is defined by $\text{ind}(T) = \alpha(T) - \beta(T)$. It is clear that $\text{ind}(T)$ is an integer or $\pm\infty$.

The next results establishes similar properties for the quaternionic framework, essential for our paper, as those seen in the complex frame.

Proposition 1. Let $T \in \mathcal{B}_R(X)$ and T^* its adjoint. Then, the following statements hold:

- (i) T is semi-regular if and only if T^* is semi-regular;
- (ii) T is upper semi-Fredholm if and only if T^* is lower semi-Fredholm;
- (iii) T is lower semi-Fredholm if and only if T^* is upper semi-Fredholm;
- (iv) $\alpha(T) = \beta(T^*)$ and $\alpha(T^*) = \beta(T)$.

Proof. Due to the similarity of the proof to its complex counterpart, the details are omitted (see, [2], Theorem 1.19 and [11], 1.3, Proposition 8). \square

Theorem 2. Let $T \in \mathcal{B}_R(X)$ such that T is upper semi-Fredholm (respectively, lower semi-Fredholm), then there exists an element $\epsilon := \epsilon(T)$ such that $S \in \mathcal{B}_R(X)$ and $\|S\| < \epsilon$ implies that $T+S$ is upper semi-Fredholm (respectively, lower semi-Fredholm). Moreover, $\text{ind}(T+S) = \text{ind}(T)$ in this case.

Proof. Due to the similarity of the proof to its complex counterpart, the details are omitted (see, [13], Theorem 5.23 and Theorem 5.29). \square

Finally, we present the definition of the Drazin inverse in the quaternionic setting which is almost identical to the complex counterpart.

Definition 4. Let $T \in \mathcal{B}_R(X)$. An element $B \in \mathcal{B}_R(X)$ is a Drazin inverse of T , written $B = T^d$, if it satisfies the following conditions

$$\begin{cases} TB = BT \text{ on } X, & BTB = B \text{ on } X, \\ T(I - TB) \text{ is nilpotent, that is, } T^k(I - TB) = 0 \text{ for some non-negative integer } k. \end{cases}$$

Proposition 3. Let $T \in \mathcal{B}_R(X)$, then the following assertions are equivalent:

- (i) T is Drazin invertible;
- (iv) $p = p(T) < \infty$ and $q = q(T) < \infty$.

Proof. The proof is easily obtained by following the same steps as in Theorem 1.132 of [1]. \square

3 ASCENT AND ESSENTIAL ASCENT S-SPECTRUM

This section introduces the concepts of ascent spectrum and essential ascent S-spectrum for bounded right linear operators acting on a two-sided quaternionic Banach space X . We then establish quaternionic analogs of important results and properties known in the complex setting. We begin with the following definition.

Definition 5. For a given right linear operator $T : \mathcal{D}(T) \subseteq X \rightarrow X$ and $s \in \mathbb{H}$, we introduce the pseudo-resolvent operator of T by

$$Q_s(T) = T^2 - 2\operatorname{Re}(s)T + |s|^2I,$$

such that $\mathcal{D}(Q_s(T)) = \mathcal{D}(T^2)$. Then, we define the ascent S-resolvent set $\rho_{asc}^S(T)$ (respectively, the essential ascent S-resolvent set $\rho_{asc}^{Se}(T)$) of T as the set of all $s \in \mathbb{H}$ such that $Q_s(T)$ has finite ascent (respectively, essential ascent); otherwise, the ascent S-spectrum $\sigma_{asc}^S(T)$ (respectively, essential ascent S-spectrum $\sigma_{asc}^{Se}(T)$) of T is defined by setting $\sigma_{asc}^S(T) := \mathbb{H} \setminus \rho_{asc}^S(T)$ (respectively, $\sigma_{asc}^{Se}(T) := \mathbb{H} \setminus \rho_{asc}^{Se}(T)$), i.e.,

$$\rho_{asc}^S(T) = \{s \in \mathbb{H} \mid Q_s(T) \text{ has a finite ascent}\} \text{ and } \sigma_{asc}^S(T) = \mathbb{H} \setminus \rho_{asc}^S(T).$$

(respectively, $\rho_{asc}^{Se}(T) = \{s \in \mathbb{H} \mid Q_s(T) \text{ has a finite essential ascent}\}$ and $\sigma_{asc}^{Se}(T) = \mathbb{H} \setminus \rho_{asc}^{Se}(T)$.)

Recall that $\rho_S(T)$ and $\sigma_S(T)$ are the spherical resolvent and the spherical spectrum of T respectively (see Definition 3.1.1 of [17]). Further, it is clear that

$$\sigma_{asc}^S(T) \subseteq \sigma_S(T) \text{ and } \sigma_{desc}^S(T) \subseteq \sigma_S(T).$$

This proposition demonstrates that both the ascent S-spectrum and the essential ascent S-spectrum exhibit axial symmetry, a property that plays a vital role in the remainder of this paper.

Proposition 4. Let $T \in \mathcal{K}_R(X)$. Then, the sets

$$\rho_{asc}^S(T) \text{ and } \sigma_{asc}^S(T) \text{ (respectively, } \rho_{asc}^{Se}(T) \text{ and } \sigma_{asc}^{Se}(T))$$

are axially symmetric.

Proof. If $s = u + jv \in \mathbb{H}$ for some $u, v \in \mathbb{R}$ and $j \in \mathbb{S}$, then for every $i \in \mathbb{S}$ and $\tilde{s} = u + iv \in [s]$, we have

$$Q_{\tilde{s}}(T) = T^2 - 2uT + (u^2 + v^2)I = Q_s(T),$$

which asserts that $Q_{\tilde{s}}(T)$ has a finite ascent if and only if $Q_s(T)$ has a finite ascent, and so $s \in \rho_{asc}^S(T)$ if and only if $\tilde{s} \in \rho_{asc}^S(T)$. Therefore, $\rho_{asc}^S(T)$ and $\sigma_{asc}^S(T)$ are axially symmetric. Similarly, we get that $\rho_{asc}^{Se}(T)$ and $\sigma_{asc}^{Se}(T)$ are axially symmetric. \square

Let X be a two-sided quaternionic Banach space, $j \in \mathbb{S}$, and T be a bounded right linear operator defined on X . If we restrict the right multiplication to the complex

plane \mathbb{C}_j , the quaternionic Banach space carries the structure of a Banach space on the field \mathbb{C}_j , we denote this \mathbb{C}_j -complex Banach space by X_j . In this case, T is also a \mathbb{C}_j -linear operator on X_j .

Remark 2. 1. The \mathbb{C}_j -complex multiples of the identity \mathcal{I}_{X_j} on X_j act as $(\lambda\mathcal{I}_{X_j})y = y\lambda$ for any $\lambda \in \mathbb{C}_j$ and $y \in X_j$.

2. Let $T \in \mathcal{B}_R(T)$ and $j \in \mathbb{S}$. Then, for any $s \in \mathbb{C}_j$, the pseudo-resolvent $Q_s(T)$ can be written as the following

$$Q_s(T) = (s\mathcal{I}_{X_j} - T)(\bar{s}\mathcal{I}_{X_j} - T).$$

Indeed, for any $x \in X$ we have $(s\mathcal{I}_{X_j} - T)(\bar{s}\mathcal{I}_{X_j} - T)x = (s\mathcal{I}_{X_j} - T)(x\bar{s} - Tx) = x|s|^2 - (Tx)s - T(x\bar{s}) + T^2x = |s|^2x - 2\text{Re}(s)Tx + T^2x = Q_s(T)x$. Hence, the result holds.

The next theorem clarifies that for any $j \in \mathbb{S}$, the intersection of the ascent S-spectrum of T with the complex plane \mathbb{C}_j is the union of the ascent spectrum of T considered as \mathbb{C}_j -linear operator on X_j with its conjugate. We denote the ascent resolvent, essential ascent resolvent, ascent spectrum, and essential ascent spectrum of T as a \mathbb{C}_j -linear operator on X_j by $\rho_{asc}^{\mathbb{C}_j}(T)$, $\rho_{asc}^{\mathbb{C}_j^e}(T)$, $\sigma_d^{\mathbb{C}_j}(T)$, and $\sigma_{asc}^{\mathbb{C}_j^e}(T)$, respectively.

Theorem 5. Let $T \in \mathcal{K}_R(X)$ and let j be an arbitrary element $\in \mathbb{S}$. Then, the union of the ascent (respectively, essential ascent) spectrum of T considered as a closed complex linear operator on X_j and its conjugate, is equal to the intersection of the ascent (respectively, essential ascent) S-spectrum with \mathbb{C}_j , i.e.,

$$\sigma_{asc}^S(T) \cap \mathbb{C}_j = \sigma_{asc}^{\mathbb{C}_j}(T) \cup \overline{\sigma_{asc}^{\mathbb{C}_j}(T)} \text{ and } \sigma_{asc}^{Se}(T) \cap \mathbb{C}_j = \sigma_{asc}^{\mathbb{C}_j^e}(T) \cup \overline{\sigma_{asc}^{\mathbb{C}_j^e}(T)}.$$

Proof. Let $j \in \mathbb{S}$ and $s \in \rho_{asc}^S(T) \cap \mathbb{C}_j$ (respectively, $s \in \rho_{asc}^{Se}(T) \cap \mathbb{C}_j$), then $Q_s(T)$ has a finite ascent (respectively, essential ascent), since the pseudo-resolvent of T can be written as $Q_s(T) = (s\mathcal{I}_j - T)(\bar{s}\mathcal{I}_j - T)$ (last remark). Then, following property (iv) of Lemma 1, we get that $p(s\mathcal{I}_j - T)$ and $p(\bar{s}\mathcal{I}_j - T)$ are both finite (respectively, $p_e(s\mathcal{I}_j - T)$ and $p_e(\bar{s}\mathcal{I}_j - T)$ are both finite), where T considered as a \mathbb{C}_j -linear on X_j ; hence, $s \in \rho_{asc}^{\mathbb{C}_j}(T) \cap \overline{\rho_{asc}^{\mathbb{C}_j}(T)}$ (respectively, $s \in \rho_{asc}^{\mathbb{C}_j^e}(T) \cap \overline{\rho_{asc}^{\mathbb{C}_j^e}(T)}$).

Conversely, suppose that $s \in \rho_{asc}^{\mathbb{C}_j}(T) \cap \overline{\rho_{asc}^{\mathbb{C}_j}(T)}$ (respectively, $s \in \rho_{asc}^{\mathbb{C}_j^e}(T) \cap \overline{\rho_{asc}^{\mathbb{C}_j^e}(T)}$), then both $(s\mathcal{I}_j - T)$ and $(\bar{s}\mathcal{I}_j - T)$ have a finite ascent (respectively, essential ascent); hence, by property (iv) of Lemma 1, we obtain that $Q_s(T) = (s\mathcal{I}_j - T)(\bar{s}\mathcal{I}_j - T)$ has a finite ascent (respectively, essential ascent); thus, $s \in \rho_{asc}^S(T) \cap \mathbb{C}_j$. \square

For $T \in \mathcal{B}_R(X)$ and any intrinsic slice hyperholomorphic function f on $\sigma_S(T)$, the spectral mapping theorem holds for the ascent and essential ascent S-spectra under certain conditions. The following theorem clarifies this result in more detail. Note that the set of intrinsic slice hyperholomorphic functions on $\sigma_S(T)$ denoted by $\mathcal{N}(\sigma_S(T))$, for more details we refer the readers to Definition 2.1.2 of [17].

Theorem 6. (Spectral Mapping Theorem). Let $T \in \mathcal{B}_R(X)$. If $f \in \mathcal{N}(\sigma_S(T))$ is an intrinsic slice hyperholomorphic function on $\sigma_S(T)$ such that for any $j \in \mathbb{S}$, $f_j = f|_{\mathbb{C}_j}$ is not identically constant in any connected component of its domain $\mathcal{D}(f_j) = \mathcal{D}(f) \cap$

\mathbb{C}_j , then we have

$$\sigma_{asc}^S(f(T)) = f(\sigma_{asc}^S(T)) \text{ (respectively, } \sigma_{asc}^{Se}(f(T)) = f(\sigma_{asc}^{Se}(T))).$$

Proof. Let $T \in \mathcal{B}_R(X)$. By the equality $\mathbb{H} = \cup_{j \in \mathbb{S}} \mathbb{C}_j$ and Theorem 5, we get that $\sigma_{asc}^S(f(T)) = \cup_{j \in \mathbb{S}} \mathbb{C}_j \cap \sigma_{asc}^S(f(T)) = \cup_{j \in \mathbb{S}} [\sigma_{asc}^{\mathbb{C}_j}(f(T)) \cup \overline{\sigma_{asc}^{\mathbb{C}_j}(f(T))}]$ (respectively, $\sigma_{asc}^{Se}(f(T)) = \cup_{j \in \mathbb{S}} \mathbb{C}_j \cap \sigma_{asc}^{Se}(f(T)) = \cup_{j \in \mathbb{S}} [\sigma_{asc}^{\mathbb{C}_j^{e}}(f(T)) \cup \overline{\sigma_{asc}^{\mathbb{C}_j^{e}}(f(T))}]$). Furthermore, since $f|_{\mathbb{C}_j}$ is an holomorphic function from \mathbb{C}_j to \mathbb{C}_j and $f\left(\overline{\sigma_{asc}^{\mathbb{C}_j}(T)}\right) = \overline{f(\sigma_{asc}^{\mathbb{C}_j}(T))}$ (respectively, $f\left(\overline{\sigma_{asc}^{\mathbb{C}_j^e}(T)}\right) = \overline{f(\sigma_{asc}^{\mathbb{C}_j^e}(T))}$), then by the spectral mapping theorem for the ascent and essential ascent spectrum in the complex frame [[4], Corollary 4.3], we deduce the following

$$\begin{aligned} \sigma_{asc}^S(f(T)) &= \cup_{j \in \mathbb{S}} f\left(\sigma_{asc}^{\mathbb{C}_j}(T) \cup \overline{\sigma_{asc}^{\mathbb{C}_j}(T)}\right) \\ &= f\left(\cup_{j \in \mathbb{S}} \left[\sigma_{asc}^{\mathbb{C}_j}(T) \cup \overline{\sigma_{asc}^{\mathbb{C}_j}(T)}\right]\right) \\ &= f(\sigma_{asc}^S(T)) \text{ (respectively, } \sigma_{asc}^{Se}(f(T)) = f(\sigma_{asc}^{Se}(T))). \end{aligned}$$

□

One of the main results of this paper is the following theorem, which states that if s is an element of the ascent S-resolvent, then there exists an open neighborhood \mathcal{V} of s such that $Q_q(T)$ is an upper semi-Fredholm and semi-regular operator for any $q \in \mathcal{V} \setminus [s]$. Now, we present the following lemmas and propositions which play crucial roles in the proof. In what follows, we shall denote by T_∞ the restriction of T to the right invariant subspace $\mathcal{R}^\infty(T)$.

Lemma 3. *If T is an upper semi-Fredholm operator defined on a two-sided quaternionic Banach space X , then T_∞ is a Fredholm operator.*

Proof. The proof is similar to the complex framework (see, [[2], Lemma 1.57]). □

The following lemma is a generalization of Theorem 5.4 in [5].

Lemma 4. *Let $T \in \mathcal{B}_R(X)$ and $s \in \mathbb{H}$. Then, we have*

- (i) $Q_s(T)(\mathcal{N}^\infty(Q_q(T))) = \mathcal{N}^\infty(Q_q(T))$ for every $q \in \mathbb{H} \setminus [s]$;
- (ii) $\mathcal{N}^\infty(Q_s(T)) \subseteq \mathcal{R}^\infty(Q_q(T))$ for every $q \in \mathbb{H} \setminus [s]$.

Proof. Let $T \in \mathcal{B}_R(X)$, $s \in \mathbb{H}$, and $q \in \mathbb{H} \setminus [s]$.

- (i) To prove the first assertion, it suffice to show that $Q_s(T)(\mathcal{N}(Q_q(T)^n)) = \mathcal{N}(Q_q(T)^n)$ for every $q \in \mathbb{H} \setminus [s]$ and $n \in \mathbb{N}$. On the one hand, it is clear that $Q_s(T)(\mathcal{N}(Q_q(T)^n)) \subseteq \mathcal{N}(Q_q(T)^n)$. On the other hand, since $q \in \mathbb{H} \setminus [s]$, then $[q] \cap [s] = \emptyset$; hence, from Theorem 2.3 of [14], Q_s and Q_q^n are two relatively prime

polynomials with real coefficients. Further, by Lemma 5.3 of [5], there exist two polynomials P_1 and P_2 with real coefficients such that

$$Q_s(T)P_1(T) + Q_q^n(T)P_2(T) = \mathcal{I}.$$

If $x \in \mathcal{N}(Q_q(T)^n)$, then $Q_s(T)P_1(T)x = x$; further, since $Q_q^n(T)$ and $P_1(T)$ are commute we have $P_1(T)x \in \mathcal{N}(Q_q^n(T))$. Therefore, $x \in Q_s(T)(\mathcal{N}^n(Q_q(T)))$ and hence $Q_s(T)(\mathcal{N}^n(Q_q(T))) = \mathcal{N}^n(Q_q(T))$ for any $n \in \mathbb{N}$.

- (ii) By property (i), we have $\mathcal{N}^\infty(Q_s(T)) = Q_q(T)(\mathcal{N}^\infty(Q_s(T)))$. Moreover, it is easy to see from the last equality that $\mathcal{N}^\infty(Q_s(T)) = Q_q(T)^n(\mathcal{N}^\infty(Q_s(T)))$ for all $n \in \mathbb{N}$; hence, $\mathcal{N}^\infty(Q_s(T)) \subseteq \mathcal{R}^\infty(Q_q(T))$. □

Lemma 5. *Let T be a bounded right linear operator and $s \in \mathbb{H}$, if $Q_s(T)$ is upper semi-Fredholm (respectively, lower semi-Fredholm), then there exists an element $\epsilon > 0$ such that $Q_q(T)$ is a upper semi-Fredholm (respectively, lower semi-Fredholm) and $\text{ind}(Q_q(T)) = \text{ind}(Q_s(T))$ for all $q \in \mathbb{H}$ such that $|q - s| < \epsilon$.*

Proof. Let $T \in \mathcal{B}_R(X)$ and $s \in \mathbb{H}$, suppose that $Q_s(T)$ is upper semi-Fredholm operator (respectively, lower semi-Fredholm), then by Theorem 2, there exists a constant c such that for every bounded right linear operator $S \in \mathcal{B}_R(X)$ with $\|S\| < c$, we have $Q_s(T) + S$ is an upper semi-Fredholm operator (respectively, lower semi-Fredholm) and $\text{ind}(Q_s(T) + S) = \text{ind}(Q_s(T))$. If we set $\epsilon = \frac{c}{(1+2\|T\|+2|s|)}$, then for all $q \in \mathbb{H}$ such that $|s - q| < \min(1, \epsilon)$, we get

$$\begin{aligned} \|2(\text{Re}(s) - \text{Re}(q))T + |q|^2 - |s|^2\| &\leq 2|\text{Re}(s) - \text{Re}(q)|\|T\| + (|s - q|)(|s| + |q|) \\ &\leq 2|s - q|\|T\| + (|s - q|)(2|s| + 1) \\ &\leq |s - q|(2\|T\| + 2|s| + 1) < c. \end{aligned}$$

Hence, $Q_s(T) + (2(\text{Re}(s) - \text{Re}(q))T + |q|^2 - |s|^2) = Q_q(T)$ is an upper semi-Fredholm operator (respectively, lower semi-Fredholm) and $\text{ind}(Q_q(T)) = \text{ind}(Q_s(T))$ for all $q \in \mathbb{H}$ such that $|s - q| < \min(1, \epsilon)$. □

From Lemma 4 and Lemma 3, we derive the following proposition.

Proposition 7. *If $T \in \mathcal{B}_R(X)$ and $s \in \mathbb{H}$ such that $Q_s(T)$ is upper semi-Fredholm and semi-regular, then there exists an element $\epsilon > 0$ such that $\alpha(Q_s(T)) = \alpha(Q_q(T))$ for all $q \in \mathbb{H}$ with $|s - q| < \epsilon$.*

Proof. Let $T \in \mathcal{B}_R(X)$ and $s \in \mathbb{H}$, suppose that $Q_s(T)$ is upper semi-Fredholm and semi-regular. It is easy to see that for all $q \in \mathbb{H}$ we have $Q_q(T/\mathcal{R}^\infty(Q_s(T))) = Q_q(T)/\mathcal{R}^\infty(Q_s(T))$, where $T/\mathcal{R}^\infty(Q_s(T))$ denotes the restriction of T to $\mathcal{R}^\infty(Q_s(T))$ and $Q_q(T)/\mathcal{R}^\infty(Q_s(T))$ the restriction of $Q_q(T)$ to $\mathcal{R}^\infty(Q_s(T))$.

On the one hand, we have

$$\alpha(Q_q(T)) = \alpha(Q_q(T/\mathcal{R}^\infty(Q_s(T)))) \text{ for all } q \in \mathbb{H}.$$

Indeed, since $\mathcal{N}(Q_s(T)) \subseteq \mathcal{N}^\infty(Q_s(T)) \subseteq \mathcal{R}^\infty(Q_s(T))$, then $\mathcal{N}(Q_q(T)) = \mathcal{N}(Q_s(T)) = \mathcal{N}(Q_s(T/\mathcal{R}^\infty(Q_s(T)))) = \mathcal{N}(Q_q(T/\mathcal{R}^\infty(Q_s(T))))$ for all $q \in [s]$. Otherwise, for all $q \notin [s]$, we have $\mathcal{N}(Q_q(T)) \subseteq \mathcal{N}^\infty(Q_q(T)) \subseteq \mathcal{R}^\infty(Q_s(T))$ by property (ii) of Lemma 4; hence, $\mathcal{N}(Q_q(T)) = \mathcal{N}(Q_q(T/\mathcal{R}^\infty(Q_s(T))))$ and so $\alpha(Q_q(T)) = \alpha(Q_q(T/\mathcal{R}^\infty(Q_s(T))))$ for all $q \in \mathbb{H}$.

On the other hand, since $Q_s(T/\mathcal{R}^\infty(Q_s(T)))$ verifies the conditions of Theorem 5.12 of [5], then we have $Q_s(T/\mathcal{R}^\infty(Q_s(T)))(\mathcal{R}^\infty(Q_s(T))) = \mathcal{R}^\infty(Q_s(T))$; hence, $Q_s(T/\mathcal{R}^\infty(Q_s(T)))$ is surjective and so $\beta(Q_s(T/\mathcal{R}^\infty(Q_s(T)))) = 0$, thus from Proposition 4.11 of [5], there exists an element $\epsilon_1 > 0$, such that $Q_q(T/\mathcal{R}^\infty(Q_s(T)))$ is surjective (which means that $\beta(Q_q(T/\mathcal{R}^\infty(Q_s(T)))) = 0$) for any $q \in \mathbb{H}$ such that $|s - q| < \epsilon_1$. Moreover, Lemma 3 implies that $Q_s(T/\mathcal{R}^\infty(Q_s(T))) = Q_s(T/\mathcal{R}^\infty(Q_s(T)))$ is Fredholm, hence by the last lemma, there exists an element ϵ_2 such that

$$\text{ind}(Q_q(T/\mathcal{R}^\infty(Q_s(T)))) = \text{ind}(Q_s(T/\mathcal{R}^\infty(Q_s(T)))) \quad \text{for all } q \in \mathbb{H} \text{ such that } |s - q| < \epsilon_2.$$

Therefore, $\alpha(Q_q(T/\mathcal{R}^\infty(Q_s(T)))) = \alpha(Q_s(T/\mathcal{R}^\infty(Q_s(T))))$ and hence $\alpha(Q_q(T)) = \alpha(Q_s(T))$ for all $q \in \mathbb{H}$ such that $|s - q| < \epsilon = \min(\epsilon_1, \epsilon_2)$. \square

Lemma 6. *Let T be a bounded right linear operator defined on X such that $p_e(T)$ is finite and $\mathcal{R}(T^{p_e(T)+1})$ is closed, then the operator induced by T on $X/\mathcal{N}(T^{a(T)})$ is both semi-regular and upper semi-Fredholm.*

Proof. Due to the similarity of the proof to its complex counterpart, the details are omitted (see, [7], Lemma 2.1). \square

Lemma 7. *If $T \in \mathcal{B}_R(X)$ is a semi-regular operator such that $\alpha(T)$ is finite, then we have $\dim(\mathcal{N}(T^n)) = n \dim(\mathcal{N}(T))$ for all $n \in \mathbb{N}$.*

Proof. Due to the similarity of the proof to its complex counterpart, the details are omitted (see, [7], Lemma 2.2). \square

We build upon the previous lemmas and proposition to prove the next theorem, which is one of the most important results in this paper.

Theorem 8. *Let $T \in \mathcal{B}_R(X)$ and $s \in \mathbb{H}$ be such that $p_e(Q_s(T))$ is finite and $\mathcal{R}(Q_s(T)^{p_e(Q_s(T))+1})$ is closed. Then, there exists an element $\epsilon > 0$ such that for every $n \in \mathbb{N}$ and $q \in \mathbb{H}$ such that $0 < |s - q| < \epsilon$ and $q \notin [s]$, the following statements hold:*

- (i) $Q_q(T)$ is semi-regular and upper semi-Fredholm;
- (ii) $\dim(\mathcal{N}(Q_q(T)^n)) = n \dim(\mathcal{N}(Q_s(T)^{a(Q_s(T))+1})/\mathcal{N}(Q_s(T)^{a(Q_s(T))}))$;
- (iii) $\text{codim}(\mathcal{R}(Q_q(T)^n)) = n \dim(\mathcal{R}(Q_s(T)^{a(Q_s(T))})/\mathcal{R}(Q_s(T)^{a(Q_s(T))+1}))$.

Proof. Let $a := a(Q_s(T))$ and let \tilde{T} be the operator induced by T on $X/\mathcal{N}(Q_s(T)^a)$.

On the one hand, from Lemma 6, the operator induced by $Q_s(T)$ on $X/\mathcal{N}(Q_s(T)^a)$ which equals $Q_s(\tilde{T})$ is both semi-regular and upper semi-Fredholm. Hence, by Proposition 7.10 of [6] and Theorem 7.6 of [5], there exists an element $\epsilon > 0$ such that $Q_q(\tilde{T})$ is semi-regular and upper semi-Fredholm for any $q \in \mathbb{H}$ such that $|s - q| < \epsilon$. Furthermore, from Proposition 7 and Lemma 5, we can take ϵ such that $\alpha(Q_q(\tilde{T})) = \alpha(Q_s(\tilde{T}))$ and $\text{ind}(Q_q(\tilde{T})) = \text{ind}(Q_s(\tilde{T}))$.

On the other hand, let $q \in \mathbb{H}$ such that $0 < |s - q| < \epsilon$ and $q \notin [s]$. For any such q and for any $n \in \mathbb{N}$, we have

$$\begin{aligned} \mathcal{N}(Q_q(\tilde{T})^n) &= \mathcal{N}(Q_q(T)^n Q_s(T)^a) / \mathcal{N}(Q_s(T)^a) \\ &= (\mathcal{N}(Q_q(T)^n) \oplus \mathcal{N}(Q_s(T)^a)) / \mathcal{N}(Q_s(T)^a); \end{aligned} \quad (3)$$

moreover, property (ii) of Lemma 4 implies that $\mathcal{N}(Q_s(T)^a) \subseteq \mathcal{R}(Q_q(T))$, hence

$$\begin{aligned} \mathcal{R}(Q_q(\tilde{T})) &= (\mathcal{R}(Q_q(T)) + \mathcal{N}(Q_s(T)^a)) / \mathcal{N}(Q_s(T)^a) \\ &= \mathcal{R}(Q_q(T)) / \mathcal{N}(Q_s(T)^a). \end{aligned} \quad (4)$$

Hence, $\mathcal{R}(Q_q(T))$ is closed and $\mathcal{N}(Q_q(T)^n) \subseteq \mathcal{R}(Q_q(T))$ for all $n \in \mathbb{N}$. Thus, $Q_q(T)$ is semi-regular and upper semi-Fredholm.

Finally, by 3 and Lemma 7, we get

$$\begin{aligned} \dim(\mathcal{N}(Q_q(T)^n)) &= \dim(\mathcal{N}(Q_q(\tilde{T})^n)) \\ &= n \dim(\mathcal{N}(Q_q(\tilde{T}))) \\ &= n \dim(\mathcal{N}(Q_s(\tilde{T}))) = n \dim(\mathcal{N}(Q_s(T)^{a+1}) / \mathcal{N}(Q_s(T)^a)). \end{aligned}$$

Furthermore, by 4 and Lemma 5, we get

$$\begin{aligned} \operatorname{codim}(\mathcal{R}(Q_q(T))^n) &= \operatorname{codim}(\mathcal{R}(Q_q(\tilde{T}))^n) \\ &= \dim(\mathcal{N}(Q_q(\tilde{T})^n)) - \operatorname{ind}(Q_q(\tilde{T})^n) \\ &= n \dim(\mathcal{N}(Q_s(\tilde{T}))) - n \operatorname{ind}(Q_s(\tilde{T})) \\ &= n \operatorname{codim}(\mathcal{R}(Q_s(\tilde{T}))) \\ &= n \operatorname{codim}((\mathcal{R}(Q_q(T)) + \mathcal{N}(Q_s(T)^a)) / \mathcal{N}(Q_s(T)^a)) \\ &= n \dim(X / (\mathcal{R}(Q_s(T)) + \mathcal{N}(Q_s(T)^a))) \\ &= n \dim(\mathcal{R}(Q_s(T)^a) / \mathcal{R}(Q_s(T)^{a+1})). \end{aligned}$$

□

Theorem 8 implies the following corollary, which holds when $p(Q_s(T))$ is finite and $\mathcal{R}(Q_s(T)^{p+1})$ is closed. This corollary is a special case of Theorem 8 for such a bounded right linear operator T .

Corollary 1. *If $T \in \mathcal{B}_R(X)$ and $s \in \mathbb{H}$ such that $p(Q_s(T))$ is finite and $\mathcal{R}(Q_s(T)^{p+1})$ is closed. Then, there exists an element $\epsilon > 0$ such that for all $n \in \mathbb{N}$ and for all $q \in \mathbb{H}$ with $0 < |s - q| < \epsilon$ and $q \notin [s]$, the following statements hold:*

- (i) $Q_q(T)$ is bounded below;
- (ii) $\operatorname{codim}(\mathcal{R}(Q_q(T)^n)) = n \dim(\mathcal{R}(Q_s(T)^{p(Q_s(T))}) / \mathcal{R}(Q_s(T)^{p(Q_s(T))+1}))$.

Theorem 8 also shows that the ascent and essential ascent S-spectra of a bounded right linear operator T are compact subsets of its S-spectrum as the following corollary shows.

Corollary 2. *If $T \in \mathcal{B}_R(X)$, then $\sigma_{asc}^S(T)$ and $\sigma_{asc}^{Se}(T)$ are compact subsets of $\sigma_S(T)$; furthermore, $\sigma_{asc}^S(T) \setminus \sigma_{asc}^{Se}(T)$ is an open set.*

Proof. Let $T \in \mathcal{B}_R(X)$ and $s \in \rho_{asc}^{Se}(T)$ (respectively, $s \in \rho_{asc}^S(T)$), then from Theorem 8, there exists an element $\epsilon > 0$ such that $Q_q(T)$ is semi-regular and semi-Fredholm (respectively, bounded below) for any $q \in \mathbb{H}$ such that $0 < |s - q| < \epsilon$ and $q \notin [s]$. In particular, $q \in \rho_{asc}^{Se}(T)$ (respectively, $q \in \rho_{asc}^S(T)$) for all $q \in \mathbb{B}(s, \epsilon) \setminus [s]$. We now need to prove that $[s] \subseteq \rho_{asc}^{Se}(T)$ (respectively, $[s] \subseteq \rho_{asc}^S(T)$). To see this, since $\rho_{asc}^{Se}(T)$ (respectively, $\rho_{asc}^S(T)$) is axially symmetric and $s \in \rho_{asc}^{Se}(T)$ (respectively, $s \in \rho_{asc}^S(T)$), then $[s] \subseteq \rho_{asc}^{Se}(T)$ (respectively, $[s] \subseteq \rho_{asc}^S(T)$) and so $\mathbb{B}(s, \epsilon) \subseteq \rho_{asc}^{Se}(T)$ (respectively, $\mathbb{B}(s, \epsilon) \subseteq \rho_{asc}^S(T)$); thus, $\sigma_{asc}^{Se}(T)$ and $\sigma_{asc}^S(T)$ are closed subsets.

To prove the second statement let $s \in \sigma_{asc}^S(T) \setminus \sigma_{asc}^{Se}(T)$, then by property (ii) of Theorem 8, there exists an open neighborhood \mathcal{V} of s such that $\mathcal{V} \setminus [s] \subseteq \rho_{asc}^{Se}(T)$ and

$$\dim(\mathcal{N}(Q_q(T)^n)) = n \dim(\mathcal{N}(Q_s(T)^{a(Q_s(T))+1}) / \mathcal{N}(Q_s(T)^{a(Q_s(T))})),$$

for all $n \in \mathbb{N}$ and $q \in \mathcal{V} \setminus [s]$. Furthermore, since the ascent of $Q_s(T)$ is infinite, then the sequence $\{\dim(\mathcal{N}(Q_q(T)^n))\}_n$ is strictly increasing for each $q \in \mathcal{V} \setminus [s]$; hence, $\mathcal{V} \setminus [s] \subseteq \sigma_{asc}^S(T)$, the fact that $\sigma_{asc}^S(T) \setminus \sigma_{asc}^{Se}(T)$ is axially symmetric implies that $\mathcal{V} \subseteq \sigma_{asc}^S(T) \setminus \sigma_{asc}^{Se}(T)$ that is $\sigma_{asc}^S(T) \setminus \sigma_{asc}^{Se}(T)$ is an open set. \square

In the following, we define the spherical poles of the S-resolvent and prove some basic results concerning them, as well as the ascent S-resolvent and essential ascent S-resolvent. For a bounded right linear operator T , the Drazin S-resolvent and Drazin S-spectrum are defined by: $\rho_d^S(T) = \{s \in \mathbb{H} \mid Q_s(T) \text{ is Drazin invertible } \mathcal{B}_R(X)\}$ and $\sigma_d^S(T) = \mathbb{H} \setminus \rho_d^S(T)$. Furthermore, by Proposition 3, we have $\rho_d^S(T) \subseteq \rho_{asc}^S(T) \subseteq \rho_{asc}^{Se}(T)$.

Definition 6. *For a bounded right linear operator T , we define the spherical poles of the S-resolvent of T as the intersection of the boundary of the S-spectrum, denoted by $\partial\sigma_S(T)$, and the Drazin S-resolvent $\rho_d^S(T)$, i.e.,*

$$\Pi_S(T) := \partial\sigma_S(T) \cap \rho_d^S(T),$$

Theorem 9. *Let $T \in \mathcal{B}_R(X)$, then we have*

$$\rho_{asc}^{Se}(T) \cap \partial\sigma_S(T) = \rho_{asc}^S(T) \cap \partial\sigma_S(T) = \Pi_S(T).$$

Moreover, the following assertions are equivalent:

- (i) $\sigma_{asc}^S(T) = \emptyset$;
- (ii) $\sigma_{asc}^{Se}(T) = \emptyset$;
- (iii) $\partial\sigma_S(T) \subseteq \rho_{asc}^S(T)$;
- (iv) $\partial\sigma_S(T) \subseteq \rho_{asc}^{Se}(T)$;
- (v) $\partial\sigma_S(T) = \Pi_S(T)$.

Proof. Firstly, from the above definition, we have $\Pi_S(T) = \partial\sigma_S(T) \cap \rho_d^S(T) \subseteq \partial\sigma_S(T) \cap \rho_{asc}^S(T) \subseteq \partial\sigma_S(T) \cap \rho_{asc}^{Se}(T)$. Secondly, if $s \in \partial\sigma_S(T) \cap \rho_{asc}^{Se}(T)$, then Theorem 8 implies the existence of an open neighborhood \mathcal{V} of s

such that $\dim(\mathcal{N}(Q_q(T))) = \dim(\mathcal{N}(Q_s(T)^{a(Q_s(T))+1})/\mathcal{N}(Q_s(T)^{a(Q_s(T))}))$ and $\text{codim}(\mathcal{R}(Q_q(T))) = \dim(\mathcal{R}(Q_s(T)^{a(Q_s(T))})/\mathcal{R}(Q_s(T)^{a(Q_s(T))+1}))$ for all $q \in \mathcal{V} \setminus [s]$. Since $\mathcal{V} \setminus \sigma_S(T)$ is non-empty, then we get

$$\begin{aligned} 0 &= \dim(\mathcal{R}(Q_s(T)^{a(Q_s(T))})/\mathcal{R}(Q_s(T)^{a(Q_s(T))+1})) \\ &= \dim(\mathcal{N}(Q_s(T)^{a(Q_s(T))+1})/\mathcal{N}(Q_s(T)^{a(Q_s(T))})); \end{aligned}$$

hence, $Q_s(T)$ has finite ascent and descent, which in turn implies that s is a spherical pole of the S-resolvent of T .

To prove the last equivalences, it is easy to see that properties (iii), (iv), and (v) are equivalent. Moreover, by Corollary 2, properties (i) and (ii) are also equivalent. To complete the proof of the equivalences, it suffices to prove that properties (ii) and (v) are equivalent. Indeed, suppose that $\partial\sigma_S(T) = \Pi_S(T)$, then $\sigma_S(T) \setminus \partial\sigma_S(T)$ is a closed and open set, which implies that $\sigma_S(T) = \partial\sigma_S(T)$; hence, $\sigma_S(T) = \partial\sigma_S(T) = \Pi_S(T) \subseteq \rho_{asc}^{Se}(T)$. Moreover, $\mathbb{H} = \rho_S(T) \cup \sigma_S(T) \subseteq \rho_{asc}^{Se}(T)$, which means that $\mathbb{H} = \rho_{asc}^{Se}(T)$; therefore, $\sigma_{asc}^{Se}(T) = \emptyset$. Conversely, it is easy to see that property (ii) implies (iv), which in turn implies (v). Hence, the equivalences hold. \square

Remark 3. From Theorem 9, for a bounded right linear operator T , the following property holds: $\partial\sigma_S(T) \subseteq \sigma_{asc}^{Se}(T) \cup \Pi_S(T)$.

We now turn to studying the ascent and essential ascent S-spectra of TS and ST (where T and S are bounded right linear operators) by first considering the following lemma.

Lemma 8. Let $T, S \in \mathcal{B}_R(T)$. Then, the following assertions hold:

- (i) For every $\lambda \in \mathbb{R}^*$, $\lambda\mathcal{I} + TS$ has closed range if and only if $\lambda\mathcal{I} + ST$ has closed range;
- (ii) For any $s \in \mathbb{H}^*$ and $n \in \mathbb{N}$, $\mathcal{R}(Q_s(TS)^n)$ is closed if and only if $\mathcal{R}(Q_s(ST)^n)$ is closed;
- (iii) For every $s \in \mathbb{H}^*$, $p(Q_s(TS)) = p(Q_s(ST))$ and $p_e(Q_s(TS)) = p_e(Q_s(ST))$.

Proof. (i) The proof is easily achievable similarly to Theorem 2.149 of [1].

(ii) Let $s \in \mathbb{H}^*$ and $n \in \mathbb{N}$, if we set $s_0 = \text{Re}(s)$, then we have

$$\begin{aligned} Q_s(TS)^n &= ((TS)^2 - 2s_0TS + |s|^2I)^n \\ &= \sum_{j=0}^n C_n^j ((TS)^2 - 2s_0TS)^j (|s|^2I)^{n-j} \\ &= |s|^{2n}I + \sum_{j=1}^n C_n^j (T(STS - 2s_0S))^j (|s|^2I)^{n-j} \\ &= |s|^{2n}I + \sum_{j=1}^n C_n^j T((STS - 2s_0S)T)^{j-1} (STS - 2s_0S) (|s|^2I)^{n-j} \\ &= |s|^{2n}I + T \left(\sum_{j=1}^n C_n^j ((STS - 2s_0S)T)^{j-1} (STS - 2s_0S) (|s|^2I)^{n-j} \right). \end{aligned}$$

Moreover,

$$\begin{aligned}
Q_s(ST)^n &= ((ST)^2 - 2s_0ST + |s|^2I)^n \\
&= \sum_{j=0}^n C_n^j ((ST)^2 - 2s_0ST)^j (|s|^2I)^{n-j} \\
&= |s|^{2n}I + \sum_{j=1}^n C_n^j ((STS - 2s_0S)T)^j (|s|^2I)^{n-j} \\
&= |s|^{2n}I + \sum_{j=1}^n C_n^j ((STS - 2s_0S)T)^{j-1} (STS - 2s_0S)T (|s|^2I)^{n-j} \\
&= |s|^{2n}I + \left(\sum_{j=1}^n C_n^j ((STS - 2s_0S)T)^{j-1} (STS - 2s_0S) (|s|^2I)^{n-j} \right) T.
\end{aligned}$$

Hence, for any $s \neq 0$, $Q_s(TS)^n = |s|^{2n}I + TU_n$ and $Q_s(ST)^n = |s|^{2n}I + U_nT$, where

$$U_n = \sum_{j=1}^n C_n^j ((STS - 2s_0S)T)^{j-1} (STS - 2s_0S) (|s|^2I)^{n-j}.$$

Then, from property (i) we have $Q_s(TS)^n = |s|^{2n}I + TU_n$ has closed range if and only if $|s|^{2n}I + U_nT = Q_s(ST)^n$ has closed range; thus, the result holds.

- (iii) Suppose that $p(Q_s(ST)) < \infty$. Let $n > p(Q_s(ST))$ and $x \in \mathcal{N}((Q_s(ST))^n)$ such that $x \notin \mathcal{N}((Q_s(ST))^{n-1})$. On the one hand, we have $(Q_s(ST))^n(x) = 0 = T(Q_s(ST))^{n-1}(x) = (Q_s(TS))^n T(x)$; hence,

$$Tx \in \mathcal{N}((Q_s(TS))^n). \quad (5)$$

On the other hand, if $Tx \in \mathcal{N}((Q_s(TS))^{n-1})$, then we have

$$(Q_s(TS))^{n-1}Tx = 0 = T(Q_s(ST))^{n-1}x. \quad (6)$$

It is clear that (6) leads to

$$\begin{aligned}
0 &= (Q_s(ST))^n x \\
&= Q_s(ST)(Q_s(ST))^{n-1}x \\
&= (ST)^2(Q_s(ST))^{n-1}x - 2\operatorname{Re}(s)ST(Q_s(ST))^{n-1}x + |s|^2(Q_s(ST))^{n-1}x \\
&= |s|^2(Q_s(ST))^{n-1}x,
\end{aligned}$$

since $s \neq 0$, this contradicts the fact that $(Q_s(ST))^{n-1}x \neq 0$; therefore,

$$Tx \notin \mathcal{N}((Q_s(TS))^{n-1}). \quad (7)$$

Finally, from (5) and (7) we conclude that $n > p(Q_s(TS))$; thus, $p(Q_s(ST)) \geq p(Q_s(TS))$. By applying the same steps with S replacing T and T replacing S , we can obtain the reverse inequality. Hence, the first equality holds.

In order to prove the second equality, it suffices to show that for any $s \in \mathbb{H}^*$ and any $n \in \mathbb{N}$, the right spaces $\mathcal{N}(Q_s(ST)^{n+1})/\mathcal{N}(Q_s(ST)^n)$ and $\mathcal{N}(Q_s(TS)^{n+1})/\mathcal{N}(Q_s(TS)^n)$ are isomorphic.

From the proof of property (ii), we have $Q_s(TS)^n = |s|^{2n}I + TU_n$ and $Q_s(ST)^n = |s|^{2n}I + U_nT$, where $U_n = \sum_{j=1}^n C^j n((STS - 2s_0S)T)^{j-1}(STS - 2s_0S)(|s|^2I)^{n-j}$. Then, $T(\mathcal{N}(Q_s(ST)^n)) = \mathcal{N}(Q_s(TS)^n)$; hence, the operator \tilde{T} induced by T from $\mathcal{N}(Q_s(ST)^{n+1})/\mathcal{N}(Q_s(ST)^n)$ to $\mathcal{N}(Q_s(TS)^{n+1})/\mathcal{N}(Q_s(TS)^n)$ can be defined. Furthermore, \tilde{T} is an isomorphism. Indeed, let $\bar{x} \in \mathcal{N}(Q_s(TS)^{n+1})/\mathcal{N}(Q_s(TS)^n)$, where x is an arbitrary element of $\mathcal{N}(Q_s(TS)^{n+1})$, i.e., $Q_s(TS)^{n+1}x = (I + TU_{n+1})x = 0$. If we set $y = -U_{n+1}x$, then $y \in \mathcal{N}(Q_s(ST)^{n+1})$ and $\tilde{T}(\bar{y}) = \overline{T y} = \bar{x}$, which implies that \tilde{T} is surjective. Moreover, the condition $\mathcal{N}(T) \cap \mathcal{N}(Q_s(ST)) = 0$ implies that \tilde{T} is injective. Therefore, \tilde{T} is an isomorphism. \square

Building upon the results of the previous lemma, we now prove the following theorem.

Theorem 10. *For any $T, S \in \mathcal{B}_R(X)$ the following properties hold:*

$$\sigma_{asc}^S(TS) \setminus \{0\} = \sigma_{asc}^S(ST) \setminus \{0\} \text{ and } \sigma_{asc}^{Se}(TS) \setminus \{0\} = \sigma_{asc}^{Se}(ST) \setminus \{0\}.$$

Proof. By properties (ii) and (iii) of Lemma 8, we have that $s \in \rho_{asc}^{Se}(TS) \setminus \{0\}$ (respectively, $s \in \rho_{asc}^S(TS) \setminus \{0\}$) if and only if $p_e(Q_s(TS))$ is finite and $\mathcal{R}(Q_s(TS)^{p_e(Q_s(TS))})$ is closed (respectively, $p(Q_s(TS))$ is finite and $\mathcal{R}(Q_s(TS)^{p(Q_s(TS))})$ is closed), and also if and only if $p_e(Q_s(ST))$ is finite and $\mathcal{R}(Q_s(ST)^{p_e(Q_s(ST))})$ is closed (respectively, $p(Q_s(ST))$ is finite and $\mathcal{R}(Q_s(ST)^{p(Q_s(ST))})$ is closed), and if and only if $s \in \rho_{asc}^{Se}(ST)$ (respectively, $s \in \rho_{asc}^S(ST)$). Thus, the results hold. \square

4 DESCENT AND ESSENTIAL DESCENT S-SPECTRUM

This section delves into the concepts of descent and essential descent S-spectra within the framework of quaternionic spectral theory. We formally define these spectra and explore their key properties. Our analysis builds upon the fundamental principles of quaternionic spectral theory. To begin this exploration, we present the following definition

Definition 7. *For a given right linear operator $T : \mathcal{D}(T) \subseteq X \rightarrow X$, we define the descent S-resolvent set $\rho_{desc}^S(T)$ of T (respectively, essential descent S-resolvent $\rho_{asc}^{Se}(T)$) as the set of all $s \in \mathbb{H}$ such that $Q_s(T)$ has a finite descent (respectively, essential descent); otherwise, the descent S-spectrum $\sigma_{desc}^S(T)$ of T (respectively, essential descent S-spectrum $\sigma_{desc}^{Se}(T)$) is defined by setting $\sigma_{desc}^S(T) := \mathbb{H} \setminus \rho_{desc}^S(T)$*

(respectively, $\sigma_{desc}^{Se}(T) := \mathbb{H} \setminus \rho_{desc}^{Se}(T)$), i.e.,

$$\rho_{desc}^S(T) = \{s \in \mathbb{H} \mid Q_s(T) \text{ has a finite descent}\} \text{ and } \sigma_{desc}^S(T) = \mathbb{H} \setminus \rho_{desc}^S(T).$$

(respectively, $\rho_{desc}^{Se}(T) = \{s \in \mathbb{H} \mid Q_s(T) \text{ has a finite essential descent}\}$ and $\sigma_{desc}^{Se}(T) = \mathbb{H} \setminus \rho_{desc}^{Se}(T)$). It is clear that

$$\sigma_{asc}^{Se}(T) \subseteq \sigma_{asc}^S(T) \subseteq \sigma_S(T) \text{ and } \sigma_{desc}^{Se}(T) \subseteq \sigma_{desc}^S(T) \subseteq \sigma_S(T).$$

In the following proposition, we present a highly useful result that will be referenced throughout the remainder of this section. This result demonstrates that the descent and essential descent S-spectra are axially symmetric, and its proof follows a similar approach as in Proposition 4.

Proposition 11. *Let $T \in \mathcal{K}_R(X)$. Then, the sets*

$$\rho_{desc}^S(T) \text{ and } \sigma_{desc}^S(T) \text{ (respectively, } \rho_{desc}^{Se}(T) \text{ and } \sigma_{desc}^{Se}(T))$$

are axially symmetric.

Proof. The proof can be established in a manner analogous to Proposition 4. \square

For any $j \in \mathbb{S}$, we denote the descent resolvent, essential descent resolvent, descent spectrum, and essential descent spectrum of T as complex linear operators on X_j by $\rho_{desc}^{\mathbb{C}_j}(T)$, $\rho_{desc}^{\mathbb{C}_j^e}(T)$, $\sigma_{desc}^{\mathbb{C}_j}(T)$, and $\sigma_{desc}^{\mathbb{C}_j^e}(T)$, respectively. The following theorem clarifies the relation between the last two spectra, the descent S-spectrum, and the essential descent S-spectrum.

Theorem 12. *Let $T \in \mathcal{K}_R(X)$ and let $j \in \mathbb{S}$ be an arbitrary element. Then, the union of the descent (respectively, essential descent) spectrum and its conjugate of T considered as a closed complex linear operator on X_j is equal to the intersection of the descent (respectively, essential descent) S-spectrum with \mathbb{C}_j , i.e.,*

$$\sigma_{desc}^S(T) \cap \mathbb{C}_j = \sigma_{desc}^{\mathbb{C}_j}(T) \cup \overline{\sigma_{desc}^{\mathbb{C}_j}(T)} \text{ and } \sigma_{desc}^{Se}(T) \cap \mathbb{C}_j = \sigma_{desc}^{\mathbb{C}_j^e}(T) \cup \overline{\sigma_{desc}^{\mathbb{C}_j^e}(T)}.$$

Proof. Let j an arbitrary element of \mathbb{S} and let $s \in \rho_{desc}^S(T) \cap \mathbb{C}_j$, then $Q_s(T)$ has a finite descent (respectively, essential descent). Consider T as a \mathbb{C}_j -linear on X_j , since $Q_s(T) = (s\mathcal{L}_j - T)(\overline{s}\mathcal{L}_j - T)$, then by property (v) of Lemma 1, we have $\delta(s\mathcal{L}_j - T)$ and $\delta(\overline{s}\mathcal{L}_j - T)$ are both finite (respectively, $\delta_e(s\mathcal{L}_j - T)$ and $\delta_e(\overline{s}\mathcal{L}_j - T)$ are both finite); hence, $s \in \rho_{desc}^{\mathbb{C}_j}(T) \cap \overline{\rho_{desc}^{\mathbb{C}_j}(T)}$ (respectively, $s \in \rho_{desc}^{\mathbb{C}_j^e}(T) \cap \overline{\rho_{desc}^{\mathbb{C}_j^e}(T)}$).

Conversely, suppose that $s \in \overline{\rho_{desc}^{\mathbb{C}_j}(T)} \cap \rho_{desc}^{\mathbb{C}_j}(T)$ (respectively, $s \in \overline{\rho_{desc}^{\mathbb{C}_j^e}(T)} \cap \rho_{desc}^{\mathbb{C}_j^e}(T)$), then both $(s\mathcal{L}_j - T)$ and $(\overline{s}\mathcal{L}_j - T)$ have a finite descent (respectively, essential descent); hence, property (v) of Lemma 1 implies that $Q_s(T) = (s\mathcal{L}_j - T)(\overline{s}\mathcal{L}_j - T)$ has a finite descent (respectively, essential descent), so $s \in \rho_{desc}^S(T) \cap \mathbb{C}_j$. \square

For any bounded right linear operator T , the spectral mapping theorem holds for any intrinsic slice hyperholomorphic function on $\sigma_S(T)$, under certain conditions. This result is a consequence of the previous theorem, as described in the following corollary.

Corollary 3. (Spectral Mapping Theorem). Let $T \in \mathcal{B}_R(X)$. If $f \in \mathcal{N}(\sigma_S(T))$ is an intrinsic slice hyperholomorphic function on $\sigma_S(T)$ such that for any $j \in \mathbb{S}$, $f_j = f|_{\mathbb{C}_j}$ is not identically constant in any connected component of its domain $\mathcal{D}(f_j) = \mathcal{D}(f) \cap \mathbb{C}_j$, then we have

$$\sigma_{desc}^S(f(T)) = f(\sigma_{desc}^S(T)) \text{ (respectively, } \sigma_{desc}^{Se}(f(T)) = f(\sigma_{desc}^{Se}(T))).$$

Proof. By Theorem 12, the proof is easily achievable similarly to Theorem 6. \square

The following theorem is one of the main results of this paper, which clarifies that if s belongs to the descent S -resolvent, then there exists an open neighborhood \mathcal{V} of s such that $Q_q(T)$ is a lower semi-Fredholm and semi-regular operator for any q in $\mathcal{V} \setminus [s]$. We begin by introducing the following lemmas.

Lemma 9. Let $s \in \mathbb{H}$ and $T \in \mathcal{B}_R(X)$, if $Q_s(T)$ is lower semi-Fredholm and semi-regular operator, then there exists an element $\epsilon > 0$ such that $\beta(Q_q(T)) = \beta(Q_s(T))$ for all $q \in \mathbb{H}$ with $|q - s| < \epsilon$.

Proof. It is easy to see that $Q_s(T^*) = Q_s(T)^*$. Furthermore, if we suppose that $Q_s(T)$ is lower semi-Fredholm and semi-regular, then from property (i) and (ii) of Proposition 1, $Q_s(T)^* = Q_s(T^*)$ is upper semi-Fredholm and semi-regular. Thus, by Proposition 7, there exists an element $\epsilon > 0$ such that $\alpha(Q_q(T^*)) = \alpha(Q_s(T^*))$ for any $q \in \mathbb{H}$ such that $|s - q| < \epsilon$. Further, from property (iii) of Proposition 1, we get

$$\beta(Q_q(T)) = \alpha(Q_q(T^*)) = \alpha(Q_s(T^*)) = \beta(Q_s(T)),$$

for all $q \in \mathbb{H}$ such that $|s - q| < \epsilon$. \square

Lemma 10. If $T \in \mathcal{B}_R(X)$ is a semi-regular operator such that $\beta(T)$ is finite, then for all $n \in \mathbb{N}$ we have $\text{codim}(\mathcal{R}(T^n)) = n \text{codim}(\mathcal{R}(T))$.

Proof. Due to the similarity of the proof to its complex counterpart, the details are omitted (see, [8], Lemma 2.2). \square

Lemma 11. Let $T \in \mathcal{B}_R(X)$, then for all $m, n \in \mathbb{N}$, $s \in \mathbb{H}$, and $q \in \mathbb{H}$ such that $q \notin [s]$, we have $X = \mathcal{R}(Q_s(T)^n) + \mathcal{R}(Q_q(T)^m)$.

Proof. Let $m, n \in \mathbb{N}$, $s \in \mathbb{H}$, and $q \in \mathbb{H}$ such that $q \notin [s]$, then $[q] \cap [s] = \emptyset$; hence, from Theorem 2.3 of [14], we have Q_s^n and Q_q^m are relatively prime polynomials with real coefficients, then by Lemma 5.3 in [5], there exist two polynomials P_1 and P_2 with real coefficients such that

$$Q_s(T)^m P_1(T) + Q_q^n(T) P_2(T) = \mathcal{I}.$$

Thus, $X = \mathcal{R}(Q_s(T)^n) + \mathcal{R}(Q_q(T)^m)$. \square

Theorem 13. Let $T \in \mathcal{B}_R(X)$ and $s \in \mathbb{H}$ be such that $q_e(Q_s(T))$ is finite. Then, there exists an element $\epsilon > 0$ such that for every $n \in \mathbb{N}$ and $q \in \mathbb{H}$ such that $0 < |s - q| < \epsilon$ and $q \notin [s]$ (i.e., $q \in \mathbb{B}(s, \epsilon) \setminus [s]$), the following statements hold:

- (i) $Q_q(T)$ is semi-regular and lower semi-Fredholm;
- (ii) $\dim(\mathcal{N}(Q_q(T)^n)) = n \dim(\mathcal{N}(Q_s(T)^{d(Q_s(T))+1})/\mathcal{N}(Q_s(T)^{d(Q_s(T))}))$;
- (iii) $\text{codim}(\mathcal{R}(Q_q(T)^n)) = n \dim(\mathcal{R}(Q_s(T)^{d(Q_s(T))})/\mathcal{R}(Q_s(T)^{d(Q_s(T))+1}))$.

Proof. Let $d = d(Q_s(T))$. On the one hand, we denote by $Q_s(T)_0$ the restriction of $Q_s(T)$ to $\mathcal{R}(Q_s(T)^d)$ which also equals to $Q_s(T_0)$ (where T_0 is the restriction of T to $\mathcal{R}(Q_s(T)^d)$), note that $\mathcal{R}(Q_s(T)^d)$ is a right quaternionic Banach space equipped with the following norm

$$\|y\|' = \|y\| + \inf\{\|x\| : x \in X \text{ and } y = Q_s(T)^d x\}, \text{ for all } y \in \mathcal{R}(Q_s(T)^d).$$

On the other hand, we have $Q_s(T_0)$ is both semi-regular and lower semi-Fredholm. Indeed, since $\mathcal{R}(Q_s(T_0)) = \mathcal{R}(Q_s(T)^{d+1})$ and $\dim(\mathcal{R}(Q_s(T)^d)/\mathcal{R}(Q_s(T)^{d+1}))$ is finite, then $\mathcal{R}(Q_s(T_0))$ is lower semi-Fredholm. Moreover, by Theorem 3.1 of [12], we have $\mathcal{N}(Q_s(T)) \cap \mathcal{R}(Q_s(T)^d) = \mathcal{N}(Q_s(T)) \cap \mathcal{R}(Q_s(T)^{d+n})$ for all $n \in \mathbb{N}$; hence,

$$\begin{aligned} \mathcal{N}(Q_s(T_0)) &= \mathcal{N}(Q_s(T)) \cap \mathcal{R}(Q_s(T)^d) \\ &= \mathcal{N}(Q_s(T)) \cap \mathcal{R}(Q_s(T)^{d+n}) \subseteq \mathcal{R}(Q_s(T)^{d+n}) = \mathcal{R}(Q_s(T_0)^n). \end{aligned}$$

Thus, $Q_s(T_0)$ is semi-regular and lower semi-Fredholm.

Finally, by Proposition 7.10 of [6] and Theorem 7.6 of [5], there exists an element $\epsilon > 0$ such that $Q_q(T_0)$ is semi-regular and lower semi-Fredholm for any $q \in \mathbb{H}$ such that $|s - q| < \epsilon$. Consequently, by the last lemma, Lemma 9, and Lemma 10, we have the following equalities hold for every $n \in \mathbb{N}$ and $q \in \mathbb{H}$ such that $|s - q| < \epsilon$ and $q \notin [s]$ (i.e., $q \in \mathbb{B}(s, \epsilon) \setminus [s]$):

$$\begin{aligned} \text{codim}(\mathcal{R}(Q_q(T)^n)) &= \dim(X/(\mathcal{R}(Q_q(T)^n))) \\ &= \dim((\mathcal{R}(Q_s(T)^d) + \mathcal{R}(Q_q(T)^n))/Q_q(T)^n) \text{ (last lemma)} \\ &= \dim(\mathcal{R}(Q_s(T)^d)/(\mathcal{R}(Q_s(T)^d) \cap \mathcal{R}(Q_q(T)^n))) \\ &= \text{codim}(\mathcal{R}(Q_q(T_0)^n)) = n \text{codim}(\mathcal{R}(Q_q(T_0)^n)) \text{ (Lemma 10)} \\ &\text{ (Lemma 9)} = n \text{codim}(\mathcal{R}(Q_s(T_0))) = n \dim(\mathcal{R}(Q_s(T)^d)/\mathcal{R}(Q_s(T)^{d+1})). \end{aligned}$$

In particular, $Q_q(T)$ is semi-Fredholm. Furthermore, by property (ii) of Lemma 4, we have $\mathcal{N}(Q_q(T)) = \mathcal{R}(Q_s(T)^d) \cap \mathcal{N}(Q_q(T)) = \mathcal{N}(Q_q(T_0)) \subseteq \mathcal{R}(Q_q(T_0)^k) \subseteq \mathcal{R}(Q_q(T)^k)$ for every $k \in \mathbb{N}$. Hence, $Q_q(T)$ is both semi-regular and upper semi-Fredholm. Otherwise, by Lemma 5 and Lemma 10 we get

$$\begin{aligned} \dim(\mathcal{N}(Q_q(T)^n)) &= \dim(\mathcal{N}(Q_q(T_0)^n)) = \text{ind}(Q_q(T_0)^n) + \text{codim}(\mathcal{R}(Q_q(T_0)^n)) \\ &= n \text{ind}(Q_s(T_0)) + n \text{codim}(\mathcal{R}(Q_s(T_0))) \\ &= n \dim(\mathcal{N}(Q_s(T_0))) \\ &= n \dim(\mathcal{R}(Q_s(T)^d) \cap \mathcal{N}(Q_s(T))) \\ &= n \dim(\mathcal{N}(Q_s(T)^{d+1})/\mathcal{N}(Q_s(T)^d)). \end{aligned}$$

This completes the proof. □

Theorem 13 yields the following corollary for the special case when the descent of $Q_s(T)$ is finite.

Corollary 4. *Let $T \in \mathcal{B}_R(X)$ and $s \in \mathbb{H}$ be such that $q(Q_s(T))$ is finite. Then, there exists an element $\epsilon > 0$ such that for every $n \in \mathbb{N}$ and $q \in \mathbb{B}(s, \epsilon) \setminus [s]$ (i.e., $q \in \mathbb{H}$ such that $0 < |s - q| < \epsilon$ and $q \notin [s]$), the following statements hold:*

- (i) $Q_q(T)$ is surjective;
- (ii) $\dim(\mathcal{N}(Q_q(T)^n)) = n \dim(\mathcal{N}(Q_s(T)^{q(Q_s(T))+1})/\mathcal{N}(Q_s(T)^{q(Q_s(T))}))$.

Another consequence of Theorem 13 is the following corollary, which states that the descent and essential descent S-spectrum are compact subsets.

Corollary 5. *If $T \in \mathcal{B}_R(X)$, then $\sigma_{desc}^S(T)$ and $\sigma_{desc}^{Se}(T)$ are compact subsets of $\sigma_S(T)$; furthermore, $\sigma_{desc}^S(T) \setminus \sigma_{desc}^{Se}(T)$ is an open set.*

Proof. For a bounded right linear operator T , let $s \in \rho_{desc}^{Se}(T)$ (respectively, $s \in \rho_{desc}^S(T)$), then by property (i) of Theorem 13, there exists an element $\epsilon > 0$ such that $Q_q(T)$ is semi-regular and lower semi-Fredholm (respectively, surjective) for any $q \in \mathbb{B}(s, \epsilon) \setminus [s]$, since $\rho_{asc}^{Se}(T)$ and $\rho_{asc}^S(T)$ are axially symmetric, then $\mathbb{B}(s, \epsilon) \subseteq \rho_{asc}^{Se}(T)$ (respectively, $\mathbb{B}(s, \epsilon) \subseteq \rho_{asc}^S(T)$) which implies that $\sigma_{desc}^S(T)$ and $\sigma_{desc}^{Se}(T)$ are closed subsets. Furthermore, if $s \in \sigma_{desc}^S(T) \setminus \sigma_{desc}^{Se}(T)$, then by Property (ii) of Theorem 13, there exists an open neighborhood \mathcal{V} of s such that $\mathcal{V} \subseteq \rho_{asc}^{Se}(T)$ and the following equality is satisfied

$$\text{codim}(\mathcal{R}(Q_q(T)^n)) = n \dim(\mathcal{R}(Q_s(T)^{d(Q_s(T))})/\dim(\mathcal{R}(Q_s(T)^{d(Q_s(T))+1})),$$

for all $n \in \mathbb{N}$ and $q \in \mathcal{V} \setminus [s]$. Further, since $Q_s(T)$ has infinite descent, then the sequence $\{\text{codim}(\mathcal{R}(Q_q(T)^n))\}_n$ is strictly increasing for each $q \in \mathcal{V} \setminus [s]$; thus, $\mathcal{V} \setminus [s] \subseteq \sigma_{desc}^S(T)$. Since $\sigma_{desc}^S(T)$ is axially symmetric and $s \in \sigma_{desc}^S(T)$, then $\mathcal{V} \subseteq \sigma_{desc}^S(T)$, which implies that $\sigma_{asc}^S(T) \setminus \sigma_{desc}^{Se}(T)$ is an open set. \square

Building on Theorem 9, the following theorem shows that similar properties exist between the spherical poles of the S-resolvent and the descent/essential descent S-spectra, reflecting the connection between ascent/essential ascent spectra established previously.

Theorem 14. *Let $T \in \mathcal{B}_R(X)$, then we have*

$$\rho_{desc}^{Se}(T) \cap \partial\sigma_S(T) = \rho_{desc}^S(T) \cap \partial\sigma_S(T) = \Pi_S(T).$$

Moreover, the following assertions are equivalent:

- (i) $\sigma_{desc}^S(T) = \emptyset$;
- (ii) $\sigma_{desc}^{Se}(T) = \emptyset$;
- (iii) $\partial\sigma_S(T) \subseteq \rho_{desc}^S(T)$;
- (iv) $\partial\sigma_S(T) \subseteq \rho_{desc}^{Se}(T)$;
- (v) $\partial\sigma_S(T) = \Pi_S(T)$.

Proof. On the one hand, by Definition 6 we have $\Pi_S(T) = \partial\sigma_S(T) \cap \rho_d^S(T) \subseteq \partial\sigma_S(T) \cap \rho_{desc}^S(T) \subseteq \partial\sigma_S(T) \cap \rho_{desc}^{Se}(T)$. On the other hand, if $s \in \partial\sigma_S(T) \cap \rho_{asc}^{Se}(T)$,

then property (ii) and (iii) of Theorem 13 imply the existence of an open neighborhood \mathcal{V} of s such that $\dim(\mathcal{N}(Q_q(T))) = \dim(\mathcal{N}(Q_s^{d(Q_s(T))+1})/\mathcal{N}(Q_s^{d(Q_s(T))}))$ and $\text{codim}(\mathcal{R}(Q_q(T))) = \dim(\mathcal{R}(Q_s(T)^{d(Q_s(T))})/\mathcal{R}(Q_s(T)^{d(Q_s(T))+1}))$ for all $q \in \mathcal{V} \setminus [s]$. Since $(\mathcal{V} \setminus [s]) \cap \rho_S(T) \neq \emptyset$, then we get

$$\dim(\mathcal{N}(Q_s(T)^d)/\mathcal{N}(Q_s(T)^{d+1})) = \dim(\mathcal{R}(Q_s(T)^d)/\mathcal{R}(Q_s(T)^{d+1})) = 0.$$

Further, $Q_s(T)$ has finite ascent and descent; hence, by Proposition 3, $s \in \rho_d^S(T)$ and so $s \in \Pi_S(T)$.

To establish the equivalences between the final assertions, we first observe that properties (iii), (iv), and (v) are equivalent. Furthermore, Corollary 4 implies that properties (i) and (ii) are also equivalent. The remaining step is to demonstrate that properties (ii) and (v) are equivalent, which completes the proof. Indeed, suppose that $\partial\sigma_S(T) = \Pi_S(T)$, then $\sigma_S(T) \setminus \partial\sigma_S(T)$ is both closed and open, which implies that it must be empty. Hence, $\sigma_S(T) = \partial\sigma_S(T) = \Pi_S(T) \subseteq \rho_{desc}^{Se}(T)$. Moreover, $\mathbb{H} = \rho_S(T) \cup \sigma_S(T) \subseteq \rho_{desc}^{Se}(T)$, that is $\mathbb{H} = \rho_{desc}^{Se}(T)$, which means that $\sigma_{desc}^{Se}(T) = \emptyset$. Conversely, we can readily demonstrate that property (ii) necessitates property (iv), which itself implies property (v). Therefore, the equivalences are established. \square

Remark 4. 1. For a bounded right linear operator T , the following property holds:

$$\partial\sigma_S(T) \subseteq \sigma_{desc}^{Se}(T) \cup \Pi_S(T).$$

Furthermore, by Remark 3, we get $\partial\sigma_S(T) \subseteq (\sigma_{desc}^{Se}(T) \cap \sigma_{asc}^{Se}(T)) \cup \Pi_S(T)$.

2. By Theorem 14 and Theorem 9, we get

$$\begin{aligned} \rho_{asc}^{Se}(T) \cap \partial\sigma_S(T) &= \rho_{asc}^S(T) \cap \partial\sigma_S(T) \\ &= \rho_{desc}^{Se}(T) \cap \partial\sigma_S(T) = \rho_{desc}^S(T) \cap \partial\sigma_S(T) = \Pi_S(T). \end{aligned}$$

Furthermore, the properties (i), (ii), (iii), (iv), and (v) of Theorem 14 are equivalent to the properties (i), (ii), (iii), (iv), and (v) of Theorem 9. In particular

$$\sigma_{desc}^S(T) = \emptyset \Leftrightarrow \sigma_{desc}^{Se}(T) = \emptyset \Leftrightarrow \sigma_{asc}^{Se}(T) = \emptyset \Leftrightarrow \sigma_{asc}^S(T) = \emptyset.$$

We conclude this section by proving that for bounded right linear operators T and S defined on X , the descent S-spectra of TS and ST (excluding $\{0\}$ in both cases) coincide. We begin with the following lemma

Lemma 12. Let $T, S \in \mathcal{B}_R(T)$. Then, the following assertion holds:

$$q(Q_s(TS)) = q(Q_s(ST)).$$

Proof. To prove the first inequality, let $n \geq q(Q_s(TS))$ and let $x \in \mathcal{R}((Q_s(TS))^{n-1})$ such that $x \notin \mathcal{R}((Q_s(TS))^n)$, then there exists an element $y \in X$ such that $x = (Q_s(TS))^{n-1}y$; hence, $Sx = (Q_s(ST))^{n-1}Sy$. Consequently, we get

$$Sx \in \mathcal{R}((Q_s(ST))^{n-1}). \quad (8)$$

Assume that $Sx \in \mathcal{R}((Q_s(ST))^n)$, then there exists an element $z \in X$ such that $Sx = (Q_s(ST))^n z$, so

$$Q_s(TS)x = (TST - 2Re(s)T)(Q_s(ST))^n z + |s|^2 x.$$

Since $x \in \mathcal{R}((Q_s(TS))^{n-1})$, we get $Q_s(TS)x \in \mathcal{R}((Q_s(TS))^n)$; hence,

$$x = 1/|s|^2 [Q_s(TS)x - (Q_s(ST))^n (TST - 2Re(s)T)z] \in \mathcal{R}((Q_s(TS))^n).$$

This contradicts the fact that $x \notin \mathcal{R}((Q_s(TS))^n)$; thus, $Sx \notin \mathcal{R}((Q_s(ST))^n)$, which implies that $q(Q_s(ST)) \leq q(Q_s(TS))$. Similarly, the reverse inequality holds; hence, (iii) holds. \square

Theorem 15. *For any $T, S \in \mathcal{B}_R(X)$ we have*

$$\sigma_{asc}^{Se}(TS) \setminus \{0\} = \sigma_{asc}^{Se}(ST) \setminus \{0\}.$$

Proof. By Lemma 12, we have $s \in \rho_{desc}^{Se}(TS) \setminus \{0\}$ if and only if $q_e(Q_s(TS))$ is finite, and if and only if $q_e(Q_s(ST))$ is finite, and also if and only if $s \in \rho_{desc}^{Se}(ST)$. Thus, the result holds. \square

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