On J-Polyadic Groups

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Abstract

Assume that (G, f) be an n-ary group. In this paper we study the groups $(G, \bullet) = ret_a(G, f)$ with the curious property that there exists an element $g \in G$ and a function

$$\begin{cases} F : (G, \bullet) \longrightarrow (G, \bullet) \\ F(x \bullet g) = x \bullet F(x) \end{cases}$$

holds for all $x \in (G, \bullet)$.
This property arose from the study of near-rings and input-output automata on groups. We call a group with this property a J-group. Finite J-groups must have odd order, and hence are solvable. We prove that every finite nilpotent group of odd order is a J-group if its nilpotency class C satisfies some properties. If $(G, \bullet)$ is a finite p-group, with $p > 2$.

1 Introduction

A polyadic group is a natural generalization of the concept of group to the case where the binary operation of group is replaced with an n-ary associative operation, one variable linear equations in which have unique solutions. So, in this article, polyadic group means an n-ary group for a fixed natural number $n \geq 2$. These interesting algebraic objects are introduced by Kasner and Dörnte ([1] and [6]) and studied extensively by Emil Post during the first decades of the last century, [10]. During decades, many articles have been published regarding the structure of polyadic groups. In our previous work [12], we studied profinite polyadic groups: n-ary groups which are the inverse limits of finite n-ary groups. We proved that a polyadic topological group (G, f) is profinite if and only if it is compact, Hausdorf, and totally disconnected. Moreover, we showed that for a profinite polyadic group (G, f), its retract $(G, \bullet)$ as well as its Post cover G^* are profinite groups.In the latest paper we proved that the polyadic group (G, f) has a unique Haar measure m_p. We study the structure and properties of the nonabelian tensor products of projective limits of finite polyadic groups.

2 n-Ary Groups

A polyadic group is a pair (G, f) where G is a non-empty set and $f : G^n \longrightarrow G$ is an n-ary operation such that

i) the operation is associative, i.e.

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1})$$

for any $1 \leqslant i, j \leqslant n$ and for all $x_1, x_2, ..., x_{2n-1} \in G$, and

ii) for all $a_1, ..., a_n, b \in G$ and $1 \leqslant i \leqslant n$, there exists a unique element $x \in G$ such that

$$f(a_1^{i-1}, x, a_{i+1}^n) = b.$$

Here we use the compact notation x_i^j for every sequence $x_i, x_{i+1}, ..., x_{j-1}, x_j$ of elements in G, and in the special case when all terms of this sequence are equal to a fixed x, we denote it by $\overset{(t)}{x}$, where t is the number of terms. Clearly, the case $n = 2$ is exactly the definition of ordinary groups. We assume that n is fixed. An n-ary group may be considered as an algebra (G, f, f_1, f_n) with one associative n-ary operation f and two n-ary operations f_1, f_n such that

$$f(f_1(a_2^n, b), a_2^n) = f(a_2^n, f_n(a_2^n, b)) = b$$

for all $a_2^n, b \in G$.
The solution of the equation $f(x, \overset{(n-2)}{a}, f(\overset{(n)}{a})) = a$ is denoted by $a^{[-2]}$. An n-ary semigroup (G, f) with an unary operation $^{-2} : G \longrightarrow G$ satisfying some natural identities is an n-ary group (cf. [16]). The map is called an j-th n-ary translation determined by a_1^n. In an ngroup each n-ary translation is a bijection. In an n-ary group (G, f) for any sequence a_1^{n-2} there exists only one $a \in G$ such that

$$f(x, a_1^{n-2}, a) = f(a_1^{n-2}, a, x) = f(a, a_1^{n-2}, x) = f(x, a, a_1^{n-2}) = x$$

for all $x \in G$ (cf. [17]). An element a is called inverse for a_1^{n-2}. In the binary case, i.e. in the case $n = 2$, when the sequence a_1^{n-2} is empty by the inverse we mean the neutral element of a group (G, f). A sequence a_2^n is called a left (right) neutral sequence if $f(a_2^n, x) = x$ (respectively $f(x, a_2^n) = x$) holds for all $x \in G$. A left and right neutral sequence is called a neutral sequence. In an n-ary group for every sequence a_1^{n-2} may be extended to a neutral sequence, but there are n-ary semigroups without left (right) neutral sequences.
Note that an n-ary system (G, f) of the form $f(x_1^n) = x_1 \circ x_2 \circ ... \circ x_n \circ b$, where $(G, \circ)$ is a group and b a fixed element belonging to the center of $(G, \circ)$, is a polyadic group, which is called b-derived from the group $(G, \circ)$ and it is denoted by $Der_b^n(G, \circ)$. In the case when b is the identity of $(G, \circ)$, we say that such a polyadic group is reduced to the group $(G, \circ)$ or derived from $(G, \circ)$ and we use the notation $Der_b(G, \circ)$ for it.
For every $n > 2$, there are n-ary groups which are not derived from any group. A polyadic group (G, f) is derived from some group if and only if, it contains an element e (called an n-ary identity) such that

$$f(\overset{(n-2)}{e}, x, \overset{(n-i)}{e}) = x$$

holds for all $x \in G$ and for all $i = 1, ..., n$.
From the definition of an n-ary group (G, f), we can directly see that for every $x \in G$, there exists only one $y \in G$, satisfying the equation

$$f(\overset{(n-1)}{x}, y) = x.$$

This element is called skew to x and it is denoted by $\overline{x}$. As Dörnte [?] proved, the following identities hold for all $x, y \in G, 2 \leq i \leq n$,

$$f(\overset{(i-2)}{x}, \overline{x}, \overset{(n-i)}{x}, y) = f(y, \overset{(n-i)}{x}, \overline{x}, \overset{(i-2)}{x}) = y.$$

These identities together with the associativity identities, axiomatize the variety of polyadic groups in the algebraic language $(f,)$. Suppose (G, f) is a polyadic group and $a \in G$ is a fixed element. Define a binary operation

$$x \bullet y = f(x, \overset{(n-2)}{a}, y).$$

Then $(G, \bullet)$ is an ordinary group, called the retract of (G, f) over a. Such a retract will be denoted by $ret_a(G, f)$. All retracts of a polyadic group are isomorphic. The identity of the group $(G, \bullet)$ is a. One can verify that the inverse element to x has the form

$$y = f(\overline{a}, \overset{(n-3)}{x}, \overline{x}, \overline{a}).$$

3 J-Polyadic Groups

Recall that if (G, f) is an n-ary group and $a \in (G, f)$ is a fixed element. Then define the binary operation

$$x \bullet y = f(x, \overset{(n-2)}{a}, y)$$

Then $(G, \bullet)$ is an ordinary group called the retract of (G, f) over a. Such a retract will be denoted by $ret_a(G, f)$. All retracts of an n-ary group are isomorphic [10]. The identity of $(G, \bullet)$ is $\overline{a}$. We can verify that the inverse element to x has the form $y = f(\overline{a}, \overset{(n-3)}{x}, \overline{x}, \overline{a})$.

Definition 3.1. A group $(G, \bullet) = ret_a(G, f)$ is a J-group if there exists an element $g \in G$ and a function $F : (G, \bullet) \longrightarrow (G, \bullet)$ satisfying

$$F(x \bullet g) = F(f(x, \overset{(n-2)}{a}, g)) = f(x, \overset{(n-2)}{a}, F(x)) = x \bullet F(x)$$

for all $x \in (G, f)$.

Lemma 1. *Lemma 2.1. Let $(G, \bullet)$ be a J-group with function F, and $g \in G$ (g is called a witness). The following equations hold for any positive integer m and any $x \in (G, f)$:*

i) $F(x \bullet g^m) = x \bullet g^{m-1} \bullet x \bullet g^{m-2} \bullet x ... x \bullet g^2 \bullet x \bullet g \bullet x \bullet F(x)$

ii) $F(x \bullet g^{-m}) = g^m \bullet x^{-1} \bullet g^{m-1} \bullet x^{-1} \bullet g^{m-2} \bullet x^{-1} ... x^{-1} \bullet g^2 \bullet x^{-1} \bullet g \bullet x^{-1} \bullet F(x)$

Proof. To prove i), let $m = 2$. Then

$$\begin{aligned} F(x \bullet g^2) &= F(f(f(x, \overset{(n-2)}{a}, g), \overset{(n-2)}{a}, g)) \\ &= f(f(x, \overset{(n-2)}{a}, g), \overset{(n-2)}{a}, f(x, \overset{(n-2)}{a}, F(x))) \\ &= f(f(x, \overset{(n-2)}{a}, g), \overset{(n-2)}{a}, F(x \bullet g)) \\ &= f(f(x, \overset{(n-2)}{a}, g), \overset{(n-2)}{a}, x \bullet F(x))) \\ &= x \bullet g \bullet x \bullet F(x). \end{aligned} \tag{1}$$

Now assume that $m = 3$. Then we have

$$\begin{aligned} F(x \bullet g^3) &= F(f(f(f(x, \overset{(n-2)}{a}, g), \overset{(n-2)}{a}, g), \overset{(n-2)}{a}, g)) \\ &= f(f(f(x, \overset{(n-2)}{a}, g), \overset{(n-2)}{a}, f(f(x, \overset{(n-2)}{a}, g), \overset{(n-2)}{a}, f(x, \overset{(n-2)}{a}, F(x)))) \\ &= f(f(x, \overset{(n-2)}{a}, g), \overset{(n-2)}{a}, F(x \bullet g^2)) \\ &= f(f(x, \overset{(n-2)}{a}, g), \overset{(n-2)}{a}, x \bullet g \bullet x \bullet F(x)) \\ &= x \bullet g^2 \bullet x \bullet g \bullet x \bullet F(x). \end{aligned} \tag{2}$$

Similarly by induction

$$\begin{aligned} F(x \bullet g^m) &= F(\overbrace{f(f(...f(f}^{m-times}(x, \overset{(n-2)}{a}, g), \overset{(n-2)}{a}, g), \overset{(n-2)}{a}, g))...)) \\ &= ... \\ &= f(f(...f(f(x, \overset{(n-2)}{a}, g))...)), \overset{(n-2)}{a}, F(x \bullet g^{m-1})) \\ &= f(f(...f(f(x, \overset{(n-2)}{a}, g))...)), \overset{(n-2)}{a}, x \bullet g^{m-2} \bullet x...x \bullet g \bullet x \bullet F(x)) \\ &= x \bullet g^{m-1} \bullet x \bullet g^{m-2} \bullet x...x \bullet g^2 \bullet x \bullet g \bullet x \bullet F(x). \end{aligned} \tag{3}$$

To prove ii), observe that

$$F(x) = F(x \bullet g^{-1} \bullet g) = x \bullet g^{-1} \bullet F(x \bullet g^{-1}),$$

and so

$$F(x \bullet g^{-1}) = g \bullet x^{-1} \bullet F(x).$$

Suppose $m > 1$. by part part i) , we have

$$F(x \bullet g^{-m} \bullet g) = x \bullet g^{-m} \bullet F(x \bullet g^{-m}),$$

and by induction,

$$F(x \bullet g^{-m}) = g^m \bullet x^{-1} \bullet g^{m-1} \bullet x^{-1} \bullet g^{m-2} \bullet x^{-1}...x^{-1} \bullet g^2 \bullet x^{-1} \bullet g \bullet x^{-1} \bullet F(x).$$

□

Lemma 2. *Let* $(G, \bullet) = ret_a(G, f)$ *be a group with an element* g *of finite order.*

Then $(G,\bullet)$ *is a J- group with constant g if and only if for all* $x \in (G,\bullet)$

$$\prod_{i=1}^{ord(g)} x \bullet g^{ord(g)-i} = 1.$$

Proof. Let $ord(g) = k$. Assume first that $(G,\bullet) = ret_a(G,f)$ is a J-group with constant g and function F satisfying (1.1). Setting $m = k$ in Lemma 2.1 shows

$$\begin{aligned} F(x) &= F(x \bullet g^k) \qquad (4)\\ &= F(\overbrace{f(f(...f(f}^{k-times}(x, \overset{(n-2)}{a}, g), \overset{(n-2)}{a}, g), \overset{(n-2)}{a}, g))...)) \\ &= x \bullet g^{k-1} \bullet x \bullet ... \bullet x \bullet g \bullet x \bullet F(x) = (\prod_{i=1}^{k} x \bullet g^{k-i})F(x) \end{aligned}$$

Hence $\prod_{i=1}^{k} x \bullet g^{k-i} = 1$ for all $x \in G$. This proves the forward implication. Assume now that $\prod_{i=1}^{k} x \bullet g^{k-i} = 1$. Let us define a function F satisfying (1.1). Choose a left transversal T of $\langle g \rangle$ in $(G,\bullet)$, that is, $(G,\bullet)$ is a disjoint union . (If $(G,\bullet)$ is infinite, the existence of T requires the Axiom of Choice.) Since each element in $(G,\bullet)$ can be written uniquely as $t \bullet g^m$, where $t \in T$ and $0 \le m < k$, we define

$$F(t \bullet g^m) := t \bullet g^{m-1} \bullet t \bullet g^{m-2} \bullet ... \bullet g^2 \bullet t \bullet g \bullet t = (\prod_{i=1}^{m} t \bullet g^{m-i}).$$

In particular, we have $F(t) = 1$ for all $t \in T$. We now prove this function satisfies

$$F(f(x, \overset{(n-2)}{a}, g)) = f(x, \overset{(n-2)}{a}, F(x))$$

for all $x \in ret_a(G,f)$.
Let $x = t \bullet g^m$. Observe that $x \bullet g = t \bullet g^{m+1}$ and $1 \le m+1 < k+1$. Assume first that $m+1 < k$. Then

$$\begin{aligned} F(f(x, \overset{(n-2)}{a}, g)) &= F(t \bullet g^{m+1}) \qquad (5)\\ &= \prod_{i=1}^{m+1} t \bullet g^{m+1-i} \\ &= t \bullet g^m \bullet \prod_{i=2}^{m+1} t \bullet g^{m+1-i} \\ &= t \bullet g^m \bullet \prod_{i=1}^{m} t \bullet g^{m-i} \\ &= f(x, \overset{(n-2)}{a}, F(x)). \end{aligned}$$

Remark 3.2. Let $(G,\bullet) = ret_a(G,f)$ be an J group. Then we sometimes suppress the role of f and call g a witness if it satisfies lemma 2., yielding a "function-free" (equivalent) definition.

i) An element $g \in ret_a(G,f)$ with finite order $ord(g)$ is called a witness if $\prod_{i=1}^{ord(g)} x \bullet g^{ord(g)-i} = 1$ for all $x \in ret_a(G,f)$.

ii) If g is a witness for $(G,\bullet) = ret_a(G,f)$, we say that $(ret_a(G,f), g)$ is a

J-group.

iii) If we want to emphasise the choice of function F , we say that $(ret_a(G,f),F,g)$ is a J-group.em

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