On Fuzzy n-Ary Profinite Groups

Mohsen Rostami

Abstract

The notion of an n-ary profinit group is a natural generalization of the notion of a n-ary group and ordinary group that has many applications in different branches. In this paper, We study the structure of fuzzy n-ary profinite group of an n-ary group is introduced and some related properties are investigated.

1 Introduction

A polyadic group is a natural generalization of the concept of group to the case where the binary operation of group is replaced with an n-ary associative operation, one variable linear equations in which have unique solutions. So, in this article, polyadic group means an n-ary group for a fixed natural number $n \geq 2$. These interesting algebraic objects are introduced by Kasner and Dörnte ([1] and [6]) and studied extensively by Emil Post during the first decades of the last century, [10]. During decades, many articles have been published regarding the structure of polyadic groups. In our previous work [12], we studied profinite polyadic groups: n-ary groups which are the inverse limits of finite n-ary groups. We proved that a polyadic topological group (G, f) is profinite if and only if it is compact, Hausdorf, and totally disconnected. Moreover, we showed that for a profinite polyadic group (G, f), its retract $(G, \bullet)$ as well as its Post cover G^* are profinite groups.In the latest paper we proved that the polyadic group (G, f) has a unique Haar measure m_p. In [?] We study the structure and properties of the nonabelian tensor products of projective limits of finite polyadic groups. B. Davvaz and W. A. Dudek in [?] they investigated the notion of (normal) fuzzy n-ary subgroup of an n-ary group is introduced and some related properties .

2 n-Ary Profinite Groups

A polyadic group is a pair (G, f) where G is a non-empty set and $f : G^n \longrightarrow G$ is an n-ary operation such that

i) the operation is associative, i.e.

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1})$$

for any $1 \leqslant i, j \leqslant n$ and for all $x_1, x_2, ..., x_{2n-1} \in G$, and

ii) for all $a_1, ..., a_n, b \in G$ and $1 \leqslant i \leqslant n$, there exists a unique element $x \in G$ such that

$$f(a_1^{i-1}, x, a_{i+1}^n) = b.$$

Note that, here we use the compact notation x_i^j for every sequence $x_i, x_{i+1}, ..., x_{j-1}, x_j$ of elements in G, and in the special case when all terms of this sequence are equal to a fixed x, we denote it by $\overset{(t)}{x}$, where t is the number of terms. Clearly, the case $n = 2$ is exactly the definition of ordinary groups. We assume that n is fixed. Note that an n-ary system (G, f) of the form $f(x_1^n) = x_1 \circ x_2 \circ ... \circ x_n \circ b$, where $(G, \circ)$ is a group and b a fixed element belonging to the center of $(G, \circ)$, is a polyadic group, which is called b-derived from the group $(G, \circ)$ and it is denoted by $Der_b^n(G, \circ)$. In the case when b is the identity of $(G, \circ)$, we say that such a polyadic group is reduced to the group $(G, \circ)$ or derived from $(G, \circ)$ and we use the notation $Der_b(G, \circ)$ for it.
For every $n > 2$, there are n-ary groups which are not derived from any group. A polyadic group (G, f) is derived from some group if and only if, it contains an element e (called an n-ary identity) such that

$$f(\overset{(n-2)}{e}, x, \overset{(n-i)}{e}) = x$$

holds for all $x \in G$ and for all $i = 1, ..., n$.
From the definition of an n-ary group (G, f), we can directly see that for every $x \in G$, there exists only one $y \in G$, satisfying the equation

$$f(\overset{(n-1)}{x}, y) = x.$$

This element is called skew to x and it is denoted by $\overline{x}$. As Dörnte [4] proved, the following identities hold for all $x, y \in G, 2 \leq i \leq n$,

$$f(\overset{(i-2)}{x}, \overline{x}, \overset{(n-i)}{x}, y) = f(y, \overset{(n-i)}{x}, \overline{x}, \overset{(i-2)}{x}) = y.$$

These identities together with the associativity identities, axiomatize the variety of polyadic groups in the algebraic language $(f,)$. Suppose (G, f) is a polyadic group and $a \in G$ is a fixed element. Define a binary operation

$$x \bullet y = f(x, \overset{(n-2)}{a}, y).$$

Then $(G, \bullet)$ is an ordinary group, called the retract of (G, f) over a. Such a retract will be denoted by $ret_a(G, f)$. All retracts of a polyadic group are isomorphic. The identity of the group $(G, \bullet)$ is a. One can verify that the inverse element to x has the form

$$y = f(\overline{a}, \overset{(n-3)}{x}, \overline{x}, \overline{a}).$$

One of the most fundamental theorems of polyadic group is the following, now known as Hosszú-Gloskin's theorem.

An n-ary subgroup H of an n-ary group (G,f) is called normal if

$$f(\overset{(n-3)}{a},\overline{a},h,a)\in H$$

for all $h\in H$ and $a\in G$. A normal subgroup $H\neq G$ containing at least two elements is called proper. If (G,f) has no any proper normal subgroup, then we say that it is simple. If $H=G$ is the only simple subgroup of G, then we say it is strongly simple.

Theorem 1. *Let (G,f) be a polyadic group. Then there exists an ordi-nary group $(G,\bullet)$, an automorphism θ of $(G,\bullet)$ and an element $b\in G$ such that*

i) $\theta(b)=b$,

ii) $\theta^{n-1}(x)=b\bullet x\bullet b^{-1}$, *for every* $x\in G$,

iii) $f(x_1^n)=x_1\bullet\theta(x_2)\bullet\theta^2(x_3)\bullet\ldots\bullet^{n-1}(x_n)\bullet b$, *for every* $x_1,\ldots,x_n\in G$.

According to this theorem, we use the notation $der_{\theta,b}(G,\bullet)$ for (G,f) and we say that (G,f) is (θ,b)-derived from the group $(G,\bullet)$.
Now, we can go back to profinite n-ary groups:A profinite polyadic group is the inverse limit of an inverse system of finite polyadic groups. More precisely, let $(I,\leq)$ be a directed set and suppose $\{(G_i,f_i),\varphi_{ij},I\}$ is an inverse system of finite polyadic groups. This means that for every pair (i,j) of elements of I with $j\leq i$, we are given a polyadic homomorphism $\varphi_{ij}:(G_i,f_i)\longrightarrow(G_j,f_j)$ such that the equality $\varphi_{jk}\varphi_{ij}=\varphi_{ik}$ holds for all $k\leq j\leq i$. Now, assume that

$$(G,f)=\varprojlim(G_i,f_i).$$

Then (G,f) is called a profinite polyadic group. From now on, we consider the pair (G,f) which is the above mentioned inverse limit. A realization of this pair can be given as follows: Let $\prod_i(G_i,f_i)$ be the direct product of the family $\{(G_i,f_i)\}_{i\in I}$. This is a polyadic group with the n-ary operation

$$(\prod f_i)((x_{i1}),(x_{i2}),\ldots,(x_{in}))=(f_i(x_{i1},x_{i2},\ldots,x_{in}))_{i\in I}.$$

Now, we have

$$(G,f)=\{(x_i)_{i\in I}:\forall j\leq i\varphi_{ij}(x_i)=x_j\},$$

and hence

$$f((x_{i1}),(x_{i2}),\ldots,(x_{in}))=(f_i(x_{i1},x_{i2},\ldots,x_{in}))_{i\in I}.$$

Note that, as each (G_i,f_i) is finite, being a closed subspace of the direct product of a family of finite sets, (G,f) is compact, Hausdorff, and totally disconnected topological polyadic group, of course, if it has been shown that G is non-empty. Indeed, using standard topological arguments, we can prove that $G\neq\varnothing$ as every G_i is compact.

Proposition 2.1. *Let $(G,f)=der_{\theta,b}(G,\bullet)$ be a profinite polyadic group. Then the retract group $(G,\bullet)$ is also profinite.*

Proposition 2.2. *Let $(G,f)=der_{\theta,b}(G,\bullet)$ be a polyadic group and $(G,f)=$*

$\varprojlim(G_i, f_i)$. *Then there exist elements* $a_i \in (G_i, f_i)$, *such that*

$$(G, f) = \varprojlim ret_{a_i}(G_i, f_i).$$

Theorem 2. *A topological polyadic group* (G, f) *is profinite, if and only if, it is compact, Hausdorff, and totally disconnected.*

Proposition 2.3. *A polyadic group* $(G, f) = der_{\theta,b}(G, \bullet)$ *is profinite, if and only if,* $(G, \bullet)$ *is profinite and* θ *is continuous.*

3 Fuzzy n-Ary Profinite Groups

Any function $\mu : G \longrightarrow [0, 1]$ is called a fuzzy subset of G. The set of all values of μ is denoted by $Im(\mu)$. If for every $S \subseteq G$, there exists $x_0 \in S$ such that $\mu(x_0) = sup\{\mu(x)|x \in S\}$ then we say that μ has sup-property. For usual groups A. Rosenfeld defined [20] fuzzy subgroups in the following way:

Definition 3.1. Definition 2.1. A fuzzy subset μ defined on a group $(G, .)$ is called a fuzzy subgroup if

1) $\mu(xy) \geq min\{\mu(x), \mu(y)\}$,
2) 2) $\mu(x^{-1}) \geq \mu(x)$

holds for all $x, y \in G$.
In fact we have $\mu(x^{-1}) = \mu(x)$ because $(x^{-1})^{-1} = x$ for every $x \in G$. Moreover, from the above definition we can deduce that $\mu(e) \geq \mu(x)$ for every $x \in G$.

Proposition 3.2. *Proposition 2.2. [20] A fuzzy subset* μ *on a group* $(G, .)$ *is a fuzzy subgroup if and only if each nonempty level subset* $\mu_t = \{x \in G|\mu(x) \geq t\}$ *is a subgroup of* $(G, .)$.

The above definition can be extended to n-ary case in the following way (cf. [8]):

Definition 3.3. Definition 2.3. Let (G, f) be an n-ary group. A fuzzy subset of G is called a fuzzy n-ary subgroup of (G, f) if the following axioms hold:

1) $\mu(f(x_1^n)) \geq min\{\mu(x_1), ..., \mu(x_n)\}$ for all $x_1^n \in G$,
2) $\mu(\overline{x}) \geq \mu(x)$ for all $x \in G$.

Note that for $n = 3$ the second condition of the Definition 2.3 can be replaced by the condition

3) $\mu(\overline{x}) = \mu(x)$ for all $x \in G$,

because in this case $n = 3$ we have $\overline{\overline{x}}$ for all $x \in G$ (cf. [2]). These two conditions are equivalent for all n-ary groups in which for every $x \in G$ there exists a natural number k such that $\overline{x}^{(k)} = x$, where $\overline{x}^{(k)}$ denotes the element skew to $\overline{x}^{(k-1)}$ and $\overline{x}^{(0)} = x$. But, as it was observed in [8], there are fuzzy n-ary subgroups in which $\mu(\overline{x}) > \mu(x)$ for some $x \in G$.

Proposition 3.4. *Proposition 2.5. Any* n*-ary subgroup of* (G, f) *can be realized as a level subset of some fuzzy* n*-ary subgroup of* G.

Corollary 3.5. *Corollary 2.6. The characteristic function of a nonempty subset A of an n-ary group (G, f) is a fuzzy n-ary subgroup of G if and only if A is an n-ary subgroup of G.*

Theorem 3. *Theorem 2.7. A fuzzy subset μ on an n-ary group (G, f) is a fuzzy n-ary subgroup if and only if each its nonempty level subset is an n-ary subgroup of $(G, .)$.*

Theorem 4. *Theorem 2.8. A fuzzy subset μ on an n-ary group (G, f) is a fuzzy n-ary subgroup if and only if for all $i = 1, 2, ..., n$ and all $x_1^n \in G$ it satisfies the following two conditions*

i) $\mu(f(x_1^n)) \geq min\{\mu(x_1), ..., \mu(x_n)\}$,

ii) $\mu(x_i) \geq min\{\mu(x_1), ..., \mu(x_{i-1}), \mu(f(x_1^n)), \mu(x_{i+1}), ..., \mu(x_n)\}$.

Corollary 3.6. *Corollary 2.9. A fuzzy subset μ defined on a group $(G, .)$ is a fuzzy subgroup if and only if*

1) $\mu(xy) \geq min\{\mu(x), \mu(y)\}$,

2) $\mu(x) \geq min\{\mu(y), \mu(xy)\}$,

3) $\mu(y) \geq min\{\mu(x), \mu(xy)\}$

holds for all $x, y \in G$.

Example 3.7. *Corollary 2.13. If (G, f) is a ternary group, then any fuzzy subgroup of $ret_a(G, f)$ is a fuzzy ternary subgroup of (G, f).*

Proof. Since a is $\overline{a}$ neutral element of a group $ret_a(G, f)$ then $\mu(a) \geq \mu(x)$ for all $x \in G$. Thus $\mu(\overline{a}) \geq \mu(x)$. But in ternary group $\overline{\overline{a}} = a$ for any $a \in G$, whence

$$\mu(a) = \mu(\overline{\overline{a}}) \geq \mu(\overline{a}) \geq \mu(a).$$

So, $\mu(a) = \mu(\overline{a}) \geq \mu(x)$ for all $x \in G$. This means that □

Theorem 5. *Theorem 2.15. Let $(G, f) = der_{\theta,b}(G, \bullet)$. Any fuzzy subgroup μ of $(G, \bullet)$ such that $\mu(b) \geq \mu(x)$ for every $x \in G$ is a fuzzy n-ary group of (G, f).*

Proof. The first condition of the Definition 2.3 is obvious. To prove the second, let $\theta = I$, then

$$\overline{x} = (x^{n-2} \bullet b)^{-1},$$

where x^{n-2} is the power of x in $(G, \bullet)$. Therefore

$$\mu(x) = \mu((x^{n-2} \bullet b)^{-1}) \geq \mu(x^{n-2} \bullet b) \geq min\{\mu(x), \mu(b)\} = \mu(x)$$

for all $x \in G$. The proof is complete. □

Corollary 3.8. *Corollary 2.16. Any fuzzy subgroup of a group $(G, \bullet)$ is a fuzzy n-ary subgroup of an n-ary group derived from $(G, \bullet)$.*

Proof. If an n-ary group (G, f) is derived from the group $(G, \bullet)$ then $b = e$ and $\mu(e) \geq \mu(x)$ for all $x \in G$. □

Theorem 6. *Theorem 2.10. Let μ be a fuzzy n-ary subgroup of (G, f). If there exists an element $a \in G$ such that $\mu(a) \geq \mu(x)$ for every $x \in G$, then μ is a fuzzy subgroup of a group $ret_a(G, f)$.*

Proof. Let $(G, \bullet) = ret_a(G, f)$, then

$$\mu(x \bullet y) = \mu(f(x, \overset{(n-2)}{a}, y) \geq min\{\mu(x), \mu(a), \mu(y)\} = min\{\mu(x), \mu(y)\}$$

and

$$\mu(x^{-1}) = \mu(f(\overline{a}, \overset{(n-3)}{x}, \overline{x}, \overline{a})) \geq min\{\mu(x), \mu(\overline{x}), \mu(a), \mu(\overline{a})\} = \mu(x),$$

which completes the proof. □

Recall that if $(I, \leq)$ be a directed set and suppose $\{(G_i, f_i), \varphi_{ij}, I\}$ is an inverse system of finite n-ary groups, and $(G, f) = \varprojlim_{i \in I}(G_i, f_i)$. Then according to Hossz´u -Gloskin's theorem, we have

$$(G_i, f_i) = der_{\theta_i, b_i}(G_i, \bullet_i),$$

for some ordinary group $(G_i, \bullet_i)$, an element $b_i \in G_i$, and an automorphism θ_i, satisfying the conclusions of Theorem 1. . In [] has proved that in some sense, there exists a binary operation $\bullet$ on G such that

$$(G, \bullet) = \varprojlim_{i \in I}(G_i, \bullet_i).$$

Proposition 3.9. *Proposition 2. Let $(G, f) = der_{\theta, b}(G, \bullet)$ be a polyadic group and $(G, f) = \varprojlim_{i \in I}(G_i, f_i)$. Then there exist elements $a_i \in G_i$, such that*

$$(G, \bullet) = \varprojlim_{i \in I} ret_{a_i}(G_i, f_i).$$

Theorem 7. *Theorem 2.12. Let (G_i, f_i) are be n-ary groups. If μ_i is a fuzzy subgroup of a group $ret_{a_i}(G_i, f_i)$ and $\mu_i(a_i) \geq \mu_i(x_i)$ for all $x_i \in G_i$, then μ_i is a fuzzy n-ary group of (G_i, f_i).*

Proof. Indeed, $(G_i, \bullet_i) = ret_{a_i}(G_i, f_i), \theta_i(x_i) = f_i(\overline{a_i}, x_i, \overset{(n-2)}{a_i})$ and $b_i = f_i(\overset{(n)}{\overline{a_i}})$. Obviously,

$$\mu_i(\theta_i(x_i)) = \mu_i(f_i(\overline{a_i}, x_i, \overset{(n-2)}{a_i})) \geq min\{\mu_i(\overline{a_i}), \mu_i(x_i), \mu_i(a_i)\} = \mu_i(x_i),$$

$$\mu_i(\theta_i^2(x_i)) = \mu_i(f_i(\overline{a_i}, \theta_i(x_i), \overset{(n-2)}{a_i})) \geq min\{\mu_i(\overline{a_i}), \mu_i(\theta_i(x_i)), \mu_i(a_i)\} = \mu_i(\theta_i(x_i)) \geq \mu_i(x_i).$$

Consequently, $\mu_i(\theta_i^k(x_i)) \geq \mu_i(x_i)$ for all $x_i \in G_i$ and $k \in \mathbb{N}$.
Similarly

$$\mu_i(b_i) = \mu_i(f_i(\overline{a_i}, \cdots, \overline{a_i})) \geq \mu_i(\overline{a_i}) \geq \mu_i(x_i)$$

for every $x_i \in G_i$.
Therefore

$$\begin{aligned} \mu_i(f_i(x_{i1}^{in})) &= \mu_i(x_{i1} \bullet_i \theta_i(x_{i2}) \bullet_i \theta_i^2(x_{i3}) \bullet_i \cdots \bullet_i \theta_i^{n-1}(x_{in}) \bullet_i b_i) \\ &\geq min\{\mu_i(x_{i1}), \mu_i(\theta_i(x_{i2})), \mu_i(\theta_i^2(x_{i3})), \cdots \mu_i(\theta_i^{n-1}(x_{in})), \mu_i(b_i)\} \\ &\geq min\{\mu_i(x_{i1}), \mu_i(x_{i2}), \mu_i(x_{i3}), \cdots \mu_i(x_{in}), \mu_i(b_i)\} \\ &\geq min\{\mu_i(x_{i1}), \mu_i(x_{i2}), \mu_i(x_{i3}), \cdots \mu_i(x_{in})\}, \end{aligned} \tag{1}$$

which proves that the first condition of the Definition 2.3 is satisfied. To prove the second condition observe that from (4) and (7) it follows

$$\overline{x_i} = (\theta_i(x_i) \bullet \theta_i^2(x_i) \bullet \cdots \bullet \theta_i^{n-2}(x_i) \bullet b_i)^{-1}.$$

Thus

$$\begin{aligned} \mu_i(\overline{x_i}) &= \mu_i((\theta_i(x_i) \bullet_i \theta_i^2(x_i) \bullet_i \cdots \bullet_i \theta_i^{n-2}(x_i) \bullet_i b_i)^{-1}) \\ &\geq \mu_i(\theta_i(x_i) \bullet_i \theta_i^2(x_i) \bullet_i \cdots \bullet_i \theta_i^{n-2}(x_i) \bullet_i b_i) \\ &\geq min\{\mu_i(\theta_i(x_i)), \mu_i(\theta_i^2(x_i)), \cdots \mu_i(\theta_i^{n-2}(x_i)), \mu_i(b_i)\} \\ &\geq min\{\mu_i(x_i), \mu_i(b_i) = \mu_i(x_i). \end{aligned} \tag{2}$$

which completes the proof. □

Proposition 3.10. *Let $\{(G_i, f_i), \varphi_{ij}, I\}$ be an inverse system of finite polyadic groups and for any i, suppose $(G_i, f_i) = der_{\theta_i, b_i}(G_i, \bullet_i)$. Then*

$$\varprojlim_{i \in I} der_{\theta_i, b_i}(G_i, \bullet_i) = der_{\hat{\theta}, \hat{b}} \varprojlim_{i \in I}(G_i, \bullet_i).$$

where

$$\hat{b} = (b_i)_i,$$

and

$$\hat{\theta}((x_i)_i) = (\theta_i(x_i))_i.$$

Corollary 3.11. *Let $(G, f) = \varprojlim_{i \in I}(G_i, f_i)$. If μ_i is a fuzzy n-ary group of (G_i, f_i). Then μ is a fuzzy n-ary group of (G, f) by*

$$\mu((x_i)_{i \in I}) = \prod_{i \in I} \mu_i((x_i)) = \bigwedge_{i \in I} \mu_i((x_i)).$$

Proof.

$$\begin{aligned}
\mu(f((x_{i1}^{in})_{i\in I})) &= (\mu_1 \times \cdots \times \mu_n)(f_i^n((x_{i1}^{in})_{i\in I})) \\
&= (\mu_1 \times \cdots \times \mu_n)(f_1(x_{i1}^{in}), f_2(x_{i1}^{in}), \cdots, f_n(x_{i1}^{in})) \\
&= min\{\mu_1(f(x_{i1}^{in})), \mu_2(f(x_{i1}^{in})), \cdots, \mu_n(f(x_{i1}^{in}))\} \\
&\geq min\{min\{\mu_1(x_{i1}), \mu_1(x_{i2}), \cdots, \mu_1(x_{in})\}, \cdots, min\{\mu_n(x_{i1}), \mu_n(x_{i2}), \cdots, \mu_n(x_{in})\}\} \\
&= min\{(\mu_1 \times \cdots \times \mu_n)(\overset{(n)}{x_{i1}}), \cdots, (\mu_1 \times \cdots \times \mu_n)(\overset{(n)}{x_{in}})\} \\
&= min\{(\mu_1 \times \cdots \times \mu_n)((x_{i1})), \cdots, (\mu_1 \times \cdots \times \mu_n)((x_{in}))\} \\
&= min\{\mu((x_{i1})), \cdots, \mu((x_{in}))\}
\end{aligned} \tag{3}$$

so

$$\mu(f((x_{i1}^{in}))) \geq min\{\mu((x_{i1})), \cdots, \mu((x_{in}))\}.$$

For second condition, we have

$$\begin{aligned}
\mu(\overline{(x_{i1}^{in})}) &= (\mu_1 \times \cdots \times \mu_n)(\overline{(x_{i1}^{in})}) \\
&= min\{\mu_1(\overline{(x_{i1}^{in})}), \cdots, \mu_n(\overline{(x_{i1}^{in})})\} \\
&\geq min\{\mu_1((x_{i1})), \cdots, \mu_n((x_{in}))\} \\
&= \mu((x_{i1}^{in})).
\end{aligned} \tag{4}$$

□

Lemma 3.12. *Let μ and λ be two fuzzy n-ary subgroups of (G, f) with the same family of levels. If $Im(\mu) = \{t_1, t_2, \cdots, t_m\}$ and $Im(\lambda) = \{s_1, s_2, \cdots, s_p\}$ where $t_1 > t_2 > \cdots > t_m$ and $s_1 > s_2 > \cdots > s_p$, then*

i) $m = p$

ii) $\mu_{t_i} = \lambda_{s_i}$, for $i = 1, \cdots, m$

iii) if $\mu(x) = t_i$, then $\lambda(x) = s_i$ for $x \in (G, f)$ and $i = 1, \cdots, m$

Proof. i) and ii) are obvious. To prove iii) consider $x \in (G, f)$ such that $\mu(x) = t_i$. If $\lambda(x) = s_i$ then $s_j \geq s_i$, i.e., $\lambda_{s_j} \subseteq \lambda_{s_i}$. Since $x \in \lambda_{s_j} = {}_{t_j}$, we obtain $t_i = (x) \geq t_j$, which gives $\mu_{t_i} \subseteq \mu_{t_j}$. Consequently, $\lambda_{s_i} = \mu_{t_i} \subseteq \mu_{t_j} = \lambda_{s_j}$. Thus $\lambda_{s_i} = \lambda_{s_j}$. Lemma 3.1 completes the proof. □

Theorem 8. *Let μ and λ be two fuzzy n-ary subgroups of (G, f) with the same family of levels. Then $\mu = \lambda$ if and only if $Im(\mu) = Im(\lambda)$.*

Proposition 3.13. *Let $(G, f) = \varprojlim_{i\in I}(G_i, f_i)$, and μ_i, λ_i be two fuzzy n-ary subgroups of (G_i, f_i) with the same family of levels, such that $Im(\mu_i) = \{t_{i1}^{im_i}\}$ and $Im(\lambda_i) = \{s_{i1}^{ip_i}\}$ where $t_{i1} > t_{i2} > \cdots > t_{im_i}$ and $s_{i1} > s_{i2} > \cdots > s_{ip_i}$. If μ, λ be two fuzzy n-ary subgroups* *of (G,f) with the same family of levels, such that $Im(\mu) = \{t_1^m\}$ and $Im(\lambda) = \{s_1^p\}$ where $t_1 > t_2 > \cdots > t_m$ and $s_1 > s_2 > \cdots > s_p$ then*

- *i)* $m = p$
- *ii)* $\mu_{t_i} = \lambda_{s_i}$, *for* $1 = 1, \cdots, m$
- *iii)* *if* $\mu((x_i)_i) = t_i$, *then* $\lambda((x_i)_i) = s_i$ *for* $(x_i)_i \in (G,f)$ *and* $i = 1, \cdots, m$
- *iv)* $t_i = min\{t_{ii}\}, s_i = min\{s_{ii}\}$

Proof. To prove iv), we have $\mu((x_i)_i) = min\{\mu_1(x_1), \cdots, \mu_n(x_n)\} = min\{t_{ii} : i \in I\}$ □

Definition 3.14. Definition 4.1. Let μ be a fuzzy set of G. An element $u \in G$ is called μ-maximal if $\mu(u) \geq \mu(x)$ for all $x \in G$. A fuzzy set with the property $\mu(u) = 1$ is called normal.

Any fuzzy set μ with finite image has a μ-maximal element. In n-ary groups derived from binary group $(G, \bullet)$ the identity of $(G, \bullet)$ is a μ-maximal element for any fuzzy subgroup of μ. In unipotent n-ary groups the element $u = f(x_1^n)$ is μ-maximal for all fuzzy n-ary subgroups. Thus a fuzzy n-ary subgroup μ of a unipotent n-ary group is normal if and only if $\mu(u) = 1$. Obviously a characteristic function χ_H of any n-ary subgroup H of G is normal.

Proposition 3.15. *Proposition 4.2. Let (u_i) be a μ-maximal element of a fuzzy n-ary subgroup of an n-ary profinite group $(G,f) = \varprojlim_{i \in I}(G_i, f_i)$. Then a fuzzy set μ^+ defined by $\mu^+((x_i)) = \mu((x_i)) + 1 - \mu((u_i))$ for all $(x_i) \in G$, is a normal fuzzy n-ary subgroup of (G,f) which contains μ.*

Proof. Indeed for every $x_i \in (G_i, f_i)$, with a μ_i-maximal element u_i, we have

$$\begin{aligned} \mu_i^+(f_i(x_{i1}^{in})) &= \mu_i(f_i(x_{i1}^{in})) + 1 - \mu_i(u_i) \\ &\geq min\{\mu_i(x_{i1}), \cdots, \mu_i(x_{in})\} + 1 - \mu_i(u_i) \\ &= min\{\mu_i(x_{i1}) + 1 - \mu_i(u_i), \cdots, \mu_i(x_{in}) + 1 - \mu_i(u_i)\} \\ &= min\{\mu_i^+(x_{i1}), \cdots, \mu_i^+(x_{in})\} \end{aligned} \tag{5}$$

and

$$\mu_i^+(\overline{x_i}) = \mu_i(\overline{x_i}) + 1 - \mu_i(u_i) \geq \mu_i(x) + 1 - \mu_i(u_i) = \mu_i^+(x_i)$$

. Clearly μ_i^+ is normal and $\mu_i \subseteq \mu_i^+$. Now for every $(x_i) \in (G,f) = \varprojlim_{i \in I}(G_i, f_i)$, with a μ-maximal element (u_i) and by Corollary 3.11, we have

$$\begin{aligned}\mu^+(f((x_{i1}^{in})_{i\in I})) &= \mu(f((x_{i1}^{in})_{i\in I})) + 1 - \mu((u_i)_{i\in I}) \\ &= (\mu_1 \times \cdots \times \mu_n)(f((x_{i1}^{in})_{i\in I})) + 1 - (\mu_1 \times \cdots \times \mu_n)((u_i)) \\ &\geq min\{(\mu_1 \times \cdots \times \mu_n)((x_{i1})), \cdots, (\mu_1 \times \cdots \times \mu_n)((x_{in})) + 1 - (\mu_1 \times \cdots \times \mu_n)((u_i))\} \\ &= min\{\mu((x_{i1})) + 1 - \mu((u_i)), \cdots, \mu((x_{in})) + 1 - \mu((u_i))\} \\ &= min\{\mu^+((x_{i1})), \cdots, \mu^+((x_{in}))\}\end{aligned} \tag{6}$$

and

$$\begin{aligned}\mu(\overline{(x_{i1}^{in})}) &= (\mu_1 \times \cdots \times \mu_n)(\overline{(x_{i1}^{in})}) \\ &= min\{\mu_1(\overline{(x_{i1}^{in})}), \cdots, \mu_n(\overline{(x_{i1}^{in})})\} \\ &\geq min\{\mu_1((x_{i1})), \cdots, \mu_n((x_{in}))\} \\ &= \mu((x_{i1}^{in})).\end{aligned} \tag{7}$$

$$\mu^+(\overline{(x_{i1}^{in})}) = \mu(\overline{(x_{i1}^{in})}) + 1 - \mu((u_i)) \geq \mu((x_{i1}^{in})) + 1 - \mu((u_i)) = \mu^+((x_{i1}^{in}))$$

. □

References

[1] Andreadakis, S. (1965), *On the automorphisms of free groups and free nilpotent groups*, Proc. Lond. Math. Soc.15(3), 239–268.

[2] Artamonov V., *Free n-groups*, Matematicheskie Zametki, 1970, 8, pp. 499-507.

[3] Dörnte, Wilhelm. (1929), *Untersuchungen über einen verallgemeinerten Gruppenbegriff*, Mathematische Zeitschrift 29, no. 1 : 1-19.

[4] Dörnte W., *Unterschungen uber einen verallgemeinerten Gruppenbegriff*, Math. Z., 1929, 29, pp. 1-19.

[5] Dudek W., *Remarks on n-groups*, Demonstratio Math., 1980, 13, pp. 65-181.

[6] Dudek W., *Around the Hosszu'-Gluskin Theorem for n-ary groups*, Glazek K , Discrete Math., 2008, 308, pp. 4861-4876.

[7] Dudek W. , Michalski J., *On retract of polyadic groups*, Demonstratio Math., 1984, 17, pp. 281-301.

[8] Dudek W., Shahryari M., *Representation theory of polyadic groups*, Algebras and Representation Theory, 2012, 15, pp. 29-51.

[9] Goldstein, D., Guralnick, R.M. (2006), *The direct product theorem for profinite groups*, J. Group Theory9, 317–322.

[10] Hossu M., *On the explicit form of n-groups*, Publ. Math., 1963, 10, pp. 88-92.

[11] Kasner, E. (1904), *An extension of the group concept*, Bull. Amer. Math. Soc, 10, 290-291.

[12] Khodabandeh H., *On free polyadic groups*, 2019, arXiv:1905-1182.

[13] Khodabandeh H., Shahryari M., *Simple polyadic groups*, Siberian Math. Journal, 2014, 55, pp. 734-744.

[14] Mel'nikov, O.V. (1980), *Projective limits of free profinite groups*, Dokl. Akad. Nauk BSSR 24, 968–970.

[15] Pickel, P.F. (1976), *Fitting subgroups and profinite completions of polycyclic groups*, J. Algebra42, 41–45.

[16] Post E. L. (1940), *Polyadic groups*, Transactions of the American Mathematical Society 48, no. 2 : 208-350.

[17] Ribes, L. (1971), *On amalgamated products of profinite groups*, Math. Z.123, 357–364.

[18] Ribes L. and Zalesskii P, *Profinite Groups*, Springer Heidelberg Dordecht London New York 2010.

[19] Segal, D. (2000), *Closed subgroups of profinite groups*, Proc. Lond. Math. Soc. (3)81(1), 29–54.

[20] Shahryari, M. (2012), *Representations of finite polyadic groups. Communications in Algebra*, 40(5), 1625-1631.

[21] Khodabandeh, H., Shahryari, M. (2012), *On the representations and automorphisms of polyadic groups*, Commun. Algebra, 40, 2199-2212.

[22] Shahryari, M., Rostami, M., *ON PROFINITE POLYADIC GROUPS*, Siberian Electronic Mathematical Reports,20 ,pp. 814-823.(2023)

[23] Sokolov E., *On the Gluskin-Hosszu theorem for Dornte n-groups*, Mat. Issled., 1976, 39, pp. 187-189.