Some results on Bell's degree variance of graphs

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Abstract

An irregularity measure (IM) of a (connected) graph G is a non-negative graph invariant satisfying the property: $IM(G) = 0$ if and only if G is regular. One of the most popular degree-based irregularity measures for a graph G with the degree sequence $(d_1, d_2, ..., d_n)$ and size m is Bell's degree-variance Var_B which is defined as $Var_B(G) = \frac{1}{n}\sum_{i=1}^{n}\left(d_i - \frac{2m}{n}\right)^2$. In this paper, we study the Bell's degree-variance for acyclic graphs, unicyclic graphs and some cactus graphs.

Keywords: Irregularity, Degree-variance, Tree, Unicyclic, Cactus graphs.

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1 Introduction

Let G be a connected simple graph for which $V(G)$ be the set of vertices with n vertices and $E(G)$ be the set of edges with m edges. The maximum degree and the minimum degree of vertices of G are denoted by $\Delta = \Delta(G)$ and $\delta = \delta(G)$, respectively. In a graph G, $d_G(u)$ shows the degree of vertex u and a vertex which is adjacent to all other vertices is called *universal vertex*. The number of vertices of degree i is denoted by N_i. If all vertices of a graph G have the same degree R, then G is called R-regular, otherwise, it is called an irregular graph.

If the degree sequence $D_s(G)$ of an irregular graph G contains exactly k different degrees, then G is said a k-degreed graph. Consequently, an irregular graph with exactly two different degrees is called a bidegreed graph and is denoted by $G(\Delta, \delta)$. According to Bell's definition [3], an irregularity measure (IM) of a (connected) graph G is a non-negative graph invariant satisfying the property: $IM(G) = 0$ if and only if

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G is regular. The oldest irregularity measure was proposed by Collatz and Sinogowitz [4] in 1957. Collatz-Sinogowitz index [3], Albertson index [1], Total irregularity [2] and Sigma index [7, 11] are some of the defined irregularity measures.

The most popular irregularity measures are *degree deviation* $S(G)$ and *degree variance* $Var(G)$. Nikiforov introduced the degree deviation [15], for a connected graph G with n vertices and m edges as

$$S(G) = \sum_{i=1}^{n} \left| d_i - \frac{2m}{n} \right|,$$

and Bell in [3] introduced degree variance as

$$Var_B(G) = \frac{1}{n} \sum_{i=1}^{n} \left(d_i - \frac{2m}{n} \right)^2.$$

Note that $\frac{2m}{n}$ is the average degree of the vertices of a graph G. We denote the $Var_B(G)$ simply by $Var(G)$. Additionally, we need two novel graph irregularity measures $IRD(G)$ and $IRR(G)$ defined as (see [8])

$$IRD(G) = \frac{2N_\Delta N_\delta}{N_\Delta + N_\delta}(\Delta - \delta),$$

and

$$IRR(G) = \frac{n}{2}(\Delta - \delta).$$

Cactus graphs were first known as Husimi trees; they appeared in the scientific literature some sixty years ago in papers by Husimi and Riddell concerned with cluster integrals in the theory of condensation in statistical mechanics [12, 14, 16]. These graphs have applications in chemistry [13, 18] and in electrical and communication networks [17]. We refer the reader to paper [6, 9, 10] for some aspects in cactus graphs.

A connected graph is a cactus graph, if no edge lies in more than one cycle. So, every block of a cactus graph is either an edge or a cycle. The cactus G is called m-uniform, if all of its blocks are cycles of the same size m.

In the next section, we study the degree deviation $S(G)$ and the degree variance $Var(G)$ of acyclic and unicyclic graphs. In Section 3, we consider some cactus chains and obtain the degree deviation $S(G)$ and the degree variance $Var(G)$ of these kind of graphs.

2 Results for acyclic and unicyclic graphs

In this section, we study the degree deviation and the degree variance of acyclic and unicyclic graphs. We start with the following lemma and theorems which will be used in the sequel.

Lemma 2.1 [8] *If G is an n-vertex connected bidegreed graph, then,* *(i)* $IRD(G) = S(G) \leq IRR(G) = \frac{n}{2}(\Delta - \delta)$ *with equality if and only if G is a bidegreed balanced graph.*

(ii) $Var(G) = \frac{1}{n^2} IRR(G) IRD(G) \leq \frac{1}{n^2}(IRR(G))^2 = \frac{(\Delta-\delta)^2}{4}$ *with equality if and only if G is a bidegreed balanced graph.*

(iii) $Var(G) = \frac{\Delta-\delta}{2n} S(G)$.

Theorem 2.2 [8] *If T is a tree with n vertices, then,*

(i) $S(T) = \frac{4(n-2)}{n} + \frac{2(n-2)}{n} \sum_{i=3}^{\Delta} N_i(i-2)$,

(ii) $Var(T) = \frac{2(n-2)}{n^2} + \frac{1}{n} \sum_{i=3}^{\Delta} N_i(i-1)(i-2)$.

By Theorem 2.2, since the path P_n and the star S_n have the minimum and maximum degree deviation and degree variance, respectively, so we have the following corollary:

Corollary 2.3 *(i) For any tree T of order n,* $\frac{4(n-2)}{n} \leq S(T) \leq \frac{2(n-1)(n-2)}{n}$.

(ii) For any tree T of order n, $\frac{2(n-2)}{n^2} \leq Var(T) \leq \frac{(n-1)(n-2)^2}{n^2}$.

Theorem 2.4 [8] *If G is an n-vertex connected unicyclic graph, then,*

(i) $S(G) = 2 \sum_{i=3}^{\Delta} N_i(i-2)$.

(ii) $Var(G) = \frac{1}{n} \sum_{i=3}^{\Delta} N_i(i-1)(i-2)$.

(iii) $nVar(G) - S(G) = \sum_{i=4}^{\Delta} N_i(i-2)(i-3)$.

An edge contraction is an operation that removes an edge from G while simultaneously merging the two vertices that it previously joined. The obtained graph is denoted as G/e. The following theorem compares the degree deviation of T and the degree deviation of T/e.

Theorem 2.5 *If T is a tree of order n and* $e \in E(T)$, *then* $S(T) > S(T/e)$.

Proof. By Theorem 2.2, the value $\sum_{i=3}^{\Delta} N_i(i-2)$ in $S(T)$ is greater than or equal to the value $\sum_{i=3}^{\Delta} N_i(i-2)$ in $S(T/e)$, and since $\frac{(n-2)}{n} > \frac{(n-3)}{(n-1)}$, so $S(T) > S(T/e)$. □

Theorem 2.6 *Let T be a tree of order n and $e = uv \in E(T)$.*

(i) If $\deg(u), \deg(v) \geq 2$ *or* $\deg(u), \deg(v) \leq 2$, *then* $Var(T) < Var(T/e)$.

(ii) If $\deg(u) \geq 3$ *and* $\deg(v) = 1$ *(or* $\deg(u) = 1$ *and* $\deg(v) \geq 3$*), then* $nVar(T) \geq (n-1)Var(T/e)$.

Proof. Suppose that $\deg(u) = s$ and $\deg(v) = t$.

(i) By Theorem 2.2, if $s, t \geq 2$ or $s, t \leq 2$, then we have

$$(s-1)(s-2) + (t-1)(t-2) < (s+t-3)(s+t-4).$$

So the value $\sum_{i=3}^{\Delta} N_i(i-1)(i-2)$ in $Var(T)$ is smaller than the value $\sum_{i=3}^{\Delta} N_i(i-1)(i-2)$ in $S(T/e)$, and since $\frac{(n-2)}{n^2} < \frac{(n-3)}{(n-1)^2}$, then $Var(T) < Var(T/e)$.

(ii) If $s \geq 3$ and $t = 1$, then the value $\sum_{i=3}^{\Delta} N_i(i-1)(i-2)$ in $Var(T)$ is greater than the value $\sum_{i=3}^{\Delta} N_i(i-1)(i-2)$ in $Var(T/e)$, and since $\frac{(n-2)}{n} > \frac{(n-3)}{(n-1)}$, so $nVar(T) \geq (n-1)Var(T/e)$. □

Here, we study the degree deviation and the degree variance of unicyclic connected graphs.

Definition 2.7 *Let C_n be a cycle with n-vertices. Unicyclic graph C_nT_k is obtained by identifying a vertex of a tree T_k of order k and a vertex of the cycle C_n.*

The following easy lemma can obtain by induction.

Theorem 2.8 *Let G be a connected unicyclic graph with the cycle C_n which obtained by identifying trees $T_{k_1}, T_{k_2}, \cdots, T_{k_r}$ to the vertices $u_1, u_2, \cdots, u_r (r \leq n)$ of the cycle C_n, respectively. Then*

$$S(G) = S(C_nT_{k_1}) + S(C_nT_{k_2}) + \cdots + S(C_nT_{k_r})$$

Let denote by $T_{m,n}$ (and call (m, n)-tadpole graph) the graph produced by joining a cycle C_m to a path P_n. We have shown the tadpole graphs $T_{3,1}$ and $T_{4,1}$ in Figure 1.

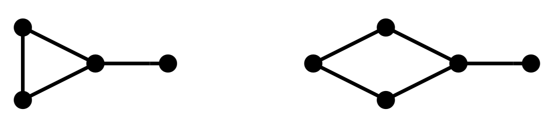

Figure 1: Tadpole graphs $T_{3,1}$ and $T_{4,1}$, respectively.

Lemma 2.9 *For the unicyclic graph C_nT_k, $S(C_nT_k) = 2t$, where t is the number of pendant vertices of C_nT_k.*

Proof. We prove the result by induction on t. If $t = 1$, then C_nT_k is a tadpole graph and it is clear that $S(C_nT_k) = 2$. Suppose that for any graph C_nT_k with t pendant vertices $S(C_nT_k) = 2t$. We show that for every graph C_nT_k with $t+1$ pendant vertices, $S(C_nT_k) = 2(t + 1)$. Consider the pendant vertex u_1 from the graph C_nT_k. If u_1 is adjacent to a vertex of degree 2, then clearly by removing u_1 the pendant vertices of $S(C_nT_k)$ and $S(C_nT_{k-1})$ are equal to $t + 1$. So $S(C_nT_k) = S(C_nT_{k-1})$. We continue this process until u_i is not adjacent to the vertex of degree 2. In this case, by removing the vertex u_i, the value of $S(C_nT_k)$ will decrease by 2, and the new graph $S(C_nT_{k-j})$ has t pendant vertices. Therefore $S(C_nT_k) = S(C_nT_{k-j} - \{u_i\}) + 2 = 2t + 2$. □

Note that the starlike tree is a tree with at most one vertex of degree ≥ 3. The path P_n and Star S_n are two special examples of starlike tree.

Corollary 2.10 *$S(C_nT_k) \geq 2(\Delta_{T_k} - 1)$ and $S(C_nT_k) = 2(\Delta_{T_k} - 1)$ if and only if T_k is starlike tree which connect to C_n by a pendant vertex.*

Using Theorem 2.8 and Lemma 2.9 we have the following theorem.

Theorem 2.11 *If G is a unicyclic graph with t pendant vertex, then $S(G) = 2t$.*

Corollary 2.12 *If G_n is a irregular unicycle graph of order n, then $S(G_n) \leq 2(n - 3)$ and the equality holds if and only if G_n has a universal vertex. Note that in the equality the cycle of G is a triangle.*

Using Cauchy-Schwarz inequality, it is not difficult to check that for every graph G with n vertices, $S(G) \leq n\sqrt{Var(G)}$. Also for each unicyclic graph G with n vertices and $\Delta_G \leq 3$, clearly $Var(G) \leq 1$ and equality holds if and only if G is a sun-graph ($C_n \circ K_1$, where $\circ$ is corona operation). In the following theorem we introduce another version of the above inequality in unicyclic graphs.

Theorem 2.13 *If G is a unicycle graph with n vertices, then $S(G) \leq nVar(G)$ and equality holds if and only if $\Delta_G \leq 3$.*

Proof. Since for each vertex v_i with $\deg(v_i) = d_i$, $|d_i - 2| \leq (d_i - 2)^2$, so we have the result. □

Using Theorems 2.11 and 2.13, we have

Corollary 2.14 *If G is a unicycle graph of order n and t pendant vertices, then $Var(G) \geq \frac{2t}{n}$ and equality holds if and only if $\Delta_G \leq 3$.*

Theorem 2.15 *If graph G is obtained by identifying the vertices $v_1, v_2, \cdots v_r$ of trees $T_{k_1}, T_{k_2}, \cdots, T_{k_r}$ to the different vertices $u_1, u_2, \cdots, u_r$ of cycle C_n, then*

$$Var(G) = \frac{\sum_{i=1}^{r} Var(C_nT_{k_i})(k_i + n - 1)}{n - r + \sum k_i}.$$

Proof. We know that the total number of vertices of graph G is $n-r+\sum_{i=1}^{r} k_i$. So

$$Var(G)=\frac{(\sum_{i=1}^{k_1}(d_{1i}-2)^2)+...+(\sum_{i=1}^{k_r}(d_{ri}-2)^2)}{n-r+\sum_{i=1}^{r}k_i}=\frac{\sum_{i=1}^{r}Var(C_nT_{k_i})(k_i+n-1)}{n-r+\sum k_i}.$$

□

Theorem 2.16 *If G_n is an irregular unicycle graph of order n, then $Var(G_n) \leq \frac{(n-2)(n-3)}{n}$. The equality holds if and only if G_n has a universal vertex.*

Proof. We prove the assertion by induction on n. If $n=4$, then $Var(G_4)=\frac{1}{2}$ and so $nVar(G_4)=2 \leq (n-2)(n-3)$. Suppose for the graph G_n, $nVar(G_n) \leq (n-2)(n-3)$ and consider an irregular unicycle graph G_{n+1} with $n+1$ vertices. Let u be a pendant vertex of graph G_{n+1}. By removing the vertex u, we have two cases.

Case 1) By removing u, the number of pendant vertices dose not change, then we have

$$\sum_{i=1}^{n+1}(d_i-2)^2=\sum_{i=1}^{n}(d_i-2)^2$$

and so

$$(n+1)Var(G_{n+1})=nVar(G_n) \leq (n-2)(n-3)<(n-1)(n-2).$$

Case 2) By removing the vertex u, the number of pendant vertices is reduced, then the maximum reduction value of $\sum_{i=1}^{n+1}(d_i-2)^2$ occurs, when u is a leaf adjacent to the universal vertex. The value of this reduction is equal to $(n-2)^2-(n-3)^2+1$, and so

$$\begin{aligned}(n+1)Var(G_{n+1}) &\leq nVar(G_n)+(n-2)^2-(n-3)^2+1\\ &\leq (n-2)(n-3)+(n-2)^2-(n-3)^2+1\\ &=(n-1)(n-2).\end{aligned}$$

For the equality, let $nVar(G_n)=(n-2)(n-3)$ and the graph G does not have a universal vertex. Suppose that the vertex v is a vertex of the cycle in the graph, whose degree is greater than 3. Since v is not universal vertex, there are some vertices that are not adjacent to v. Two cases can occur:

Case A) The graph G has a support vertex (a vertex adjucent to a pendant vertex) that is not on the cycle. Suppose u_0 is a support vertex on the tree T, where k pendant vertices are adjacent to u_0, and the tree T is connected to the cycle at the vertex v. We separate all k pendant vertices from u and connect them to v and consider the new graph as G'. We have

$$\begin{aligned}(n-2)(n-3) &= nVar(G)=(d_v-2)^2+(d-u_0-2)^2+k(1-2)^2+B\\ &= (d_v-2)^2+(k-1)^2+k+B,\end{aligned}$$

where B is the some of expressions in the form of $(d-2)^2$ on the rest of the vertices. On the other hand, we have

$$nVar(G') = (d_v + k - 2)^2 + (1-2)^2 + k(1-2)^2 + B.$$

It can be checked with a simple calculation

$$(n-2)(n-3) = nVar(G) \lneqq nVar(G') \le (n-2)(n-3),$$

which is a contradiction.

Case B) The graph G has a support vertex that is on the cycle. We consider two subcases:

Subcase 1) The cycle has length $n_1 \ge 4$. In this case, the graph G is obtained from the identifying the central vertices of $1 \le r \le n_1$ stars K_{1,k_i} and vertices of C_{n_1}. Then, $n = n_1 + k_1 + \cdots + k_r$ and

$$\begin{aligned} nVar(G) &= (k_1^2 + k_1) + \cdots (k_r^2 + k_r) \lneqq (k_1 + \cdots + k_r)^2 \\ &\lneqq (k_1 + \cdots + k_r + 2)(k_1 + \cdots + k_r + 1) \\ &\le (n_1 + k_1 + \cdots + k_r - 2)(n_1 + k_1 + \cdots + k_r - 3) = (n-2)(n-3). \end{aligned}$$

Subcase 2) The cycle has a length $n_1 = 3$. Since the graph G does not have a universal vertex, it is obtained from the connecting $2 \le r \le 3$ stars to C_3, where each star S_i has k_i pendant vertices and is connected to the cycle from the central vertex. Then, $n = 3 + k_1 + k_2 + k_3$ and the rest of the proof of this part is similar to the previous case. □

Theorem 2.17 *Let G be a unicyclic graph and $e = uv \in E(G)$ such that G/e be a unicyclic graph. Then $S(G/e) \le S(G)$, and equality holds if and only if* $\deg(u) \ge 2$ *and* $\deg(v) \ge 2$ *(or* $\deg(u) \le 2$ *and* $\deg(v) \le 2$*).*

Proof. Let $\deg(u) = s$ and $\deg(v) = t$. If $s, t \ge 2$ or $s, t \le 2$, then we have

$$|s-2| + |t-2| = |s+t-4|.$$

So $S(G) = \sum_{i=1}^{n} |d_i - 2| = \sum_{i=1 \& d_i \ne s,t}^{n} |d_i - 2| + |s+t-4| = S(G/e)$.
If $s < 2$ and $t > 2$ (or $s > 2$ and $t < 2$), then we have

$$|s-2| + |t-2| > |s+t-4|,$$

and so $S(G) > S(G/e)$. □

Theorem 2.18 *Let G be a unicyclic graph with n vertices and $e = uv \in E(G)$ such that G/e be a unicyclic graph. Then*

(i) If $\deg(u) \ge 2$ *and* $\deg(v) \ge 2$ *or* $\deg(u) \le 2$ *and* $\deg(v) \le 2$*, then $Var(G/e) \ge Var(G)$.*

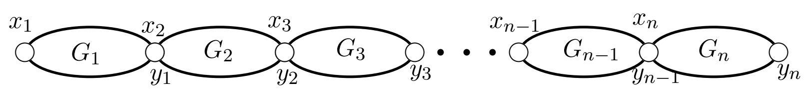

Figure 2: Chain of n graphs $G_1, G_2, \ldots, G_n$

(ii) If $\deg(u) > 2$ *and* $\deg(v) = 1$ *or vice versa, then* $(n-1)Var(G/e) < nVar(G)$*.*

Proof. Suppose that $\deg(u) = s$ and $\deg(v) = t$.

(i) If $s, t \geq 2$ or $s, t \leq 2$, then we have

$$(s-2)^2 + (t-2)^2 \leq (s+t-4)^2,$$

so

$$Var(G) = \frac{1}{n}[\sum_{i=1}^{n}(d_i-2)^2] \leq \frac{1}{n-1}[\sum_{i=1 \& d_i \neq s,t}^{n}(d_i-2)^2 + (s+t-4)^2] = Var(G/e).$$

(ii) If $s > 2$ and $t = 1$, then

$$(s-2)^2 + 1 > (s-3)^2,$$

so

$$nVar(G) = \sum_{i=1}^{n}(d_i-2)^2 > \sum_{i=1 \& d_i \neq s,t}^{n}(d_i-2)^2 + (s-3)^2 = (n-1)Var(G/e).$$

□

3 Results for some cactus graphs

In this section, we compute the degree deviation and the degree variance of some cactus graphs.

Let $G_1, G_2, \ldots, G_n$ be a finite sequence of pairwise disjoint connected graphs and let $x_i, y_i \in V(G_i)$. Suppose that $C(G_1, ..., G_n)$ is the chain of graphs $\{G_i\}_{i=1}^{n}$ with respect to the vertices $\{x_i, y_i\}_{i=1}^{k}$ which obtained by identifying the vertex y_i with the vertex x_{i+1} for $i = 1, 2, \ldots, n-1$ (Figure 2).

Proposition 3.1 *Suppose* $X_{n,m}$ *is a chain cactus graph which each block is a cycle of length* m *and* $n \geq 2$ *be the number of cycles. Then*

(i) $S(X_{n,m}) = \frac{4(m-2)n^2 - 4(m-4)n - 8}{(m-1)n+1}$.

(ii) $Var(X_{n,m}) = \frac{4(m-2)n^2 - 4(m-4)n - 8}{[(m-1)n+1]^2}$.

Proof.

(i) Since all of these chain cactus graphs are bidegree, by Lemma 2.1 we have

$$\begin{aligned} S(X_{n,m}) &= IRD(X_{n,m}) \\ &= \frac{2[n-1][n(m-2)+2]}{[n-1]+[n(m-2)+2]}(4-2) \\ &= \frac{4(m-2)n^2-4(m-4)n-8}{(m-1)n+1}. \end{aligned}$$

(ii) $Var(X_{n,m}) = \frac{(4-2)}{2[n(m-1)+1]} S(X_{n,m}) = \frac{4(m-2)n^2-4(m-4)n-8}{[(m-1)n+1]^2}$. □

Example 3.2 *(i) Let $T_n = X_{n,3} (n \geq 2)$ be a chain triangular cactus which is shown in Figure 3. By Proposition 3.1, we have*

$$S(T_n) = \frac{4(n-1)(n+2)}{2n+1},$$

and

$$Var(T_n) = \frac{4(n-1)(n+2)}{(2n+1)^2}.$$

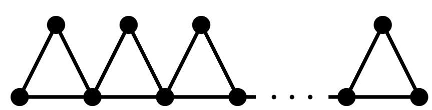

Figure 3: Chain triangular cactus T_n

(ii) If $Q_n = X_{n,4} (n \geq 2)$ is a para-chain square cactus shown in the Figure 4, then by Proposition 3.1,

$$S(Q_n) = \frac{8(n-1)(n+1)}{3n+1},$$

and

$$Var(Q_n) = \frac{8(n-1)(n+1)}{(3n+1)^2}.$$

Theorem 3.3 *For each $s, t \in \mathbb{N}$, and $3 \leq s < t$, we have*

$$S(X_{n,s}) < S(X_{n,t})$$

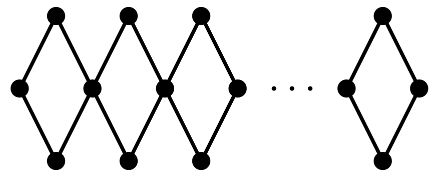

Figure 4: Para-chain square cactus Q_n

Proof. It is enough to prove it for the case $s = m$ and $t = m + 1$. By Proposition 3.1 we show that $S(X_{n,m+1}) - S(X_{n,m}) > 0$.

$$\begin{aligned} S(X_{n,m+1}) - S(X_{n,m}) &= \frac{4(m-1)n^2 - 4(m-3)n - 8}{mn+1} - \frac{4(m-2)n^2 - 4(m-4)n - 8}{(m-1)n+1} \\ &= \frac{2(n-2) + mn(m-2) + 3}{[mn+1][(m-1)n+1]} > 0. \end{aligned}$$

□

Corollary 3.4 *For each $m \geq 3$, $S(T_n) =\leq S(X_{n,m})$.*

Corollary 3.5 *For each $s, t \in \mathbb{N}$ and $3 \leq s < t$,*

$$[(s-1)n+1]Var(X_{n,s}) < [(t-1)n+1]Var(X_{n,t})$$

Proof. By lemma 2.1 and Theorem 3.3, the result follows. □

Theorem 3.6 *For any $m \geq 3$ and $n \geq 2$, $S(X_{n,m}) < S(X_{n+1,m})$.*

Proof. Using Proposition 3.1,

$$\begin{aligned} S(X_{n+1,m}) - S(X_{n,m}) &= \frac{4(m-2)(n+1)^2 - 4(m-4)(n+1) - 8}{(m-1)(n+1)+1} \\ &- \frac{4(m-2)n^2 - 4(m-4)n - 8}{(m-1)n+1} \\ &= \frac{4(m-2)^2n^2 + 4m(mn-2) + 8}{[(m-1)(n+1)+1][(m-1)n+1]} > 0. \end{aligned}$$

□

Definition 3.7 *The friendship (or Dutch windmill) graph F_n is a graph that can be constructed by coalescence n copies of the cycle graph C_3 with a common vertex. Clearly the friendship graph is a bidegree cactus graph. See Figure 5.*

Corollary 3.8 *Let F_n be a friendship graph. Then $S(F_n) = \frac{8n(n-1)}{2n+1}$ and $Var(F_n) = \frac{8n(n-1)^2}{(2n+1)^2}$.*

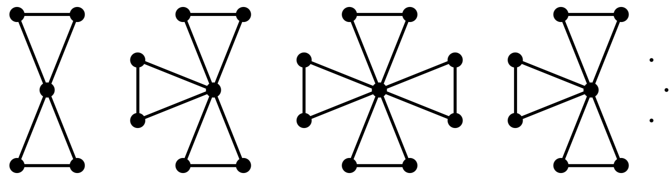

Figure 5: Friendship graph F_2, F_3, F_4 and F_n, respectively

Declaration of interests

The authors declare that they have no conflict of interest.

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