S©MR

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ
Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru
ISSN 1813-3304

Том 20, № 2, стр. 144–145 (2023) УДК 517.984.5, 532.135, 537.84
https://doi.org/10.33048/semi.2023.20.??? MSC 35B35,
76A05, 76A10,76W05

SPECTRUM OF A LINEAR PROBLEM ABOUT THE MHD FLOWS OF A POLYMERIC FLUID IN A CYLINDRICAL CHANNEL IN CASE OF AN ABSOLUTE CONDUCTIVITY (GENERALIZED VINOGRADOV-POKROVSKI MODEL)

D.L. TKACHEV
AND **E.A. BIBERDORF**

Abstract: We study the linear stability of a resting state for flows of incompressible viscoelastic polymeric fluid under the influence of homogenous magnetic field in an infinite cylindrical channel in axisymmetric perturbation class. The tension vector of the magnetic field is parallel to the cylinder axis. We use structurally-phenomenological Vinogradov-Pokrovski model as our mathematical model.

We formulate the equation that define the spectrum of the problem. Our numerical experiments show that with the growth of perturbations frequency along the channel axis there appear eigenvalues with positive real part for the radial velocity component of the first spectral equation. That guarantees linear Lyapunov instability of the resting state. However for large Reynolds and

---

Spectrum of a linear problem about the MHD flows of a polymeric fluid in a cylindrical channel in case of an absolute conductivity (generalized Vinogradov-Pokrovski model).

The study was carried out within the framework of the state contract of the Sobolev Institute of Mathematics (project no. FWNF-2022-0008).

Поступила 1 января 2023 г., опубликована 31 декабря 2023 г.

Weissenberg numbers the exponential growth rate of the amplitude for high frequencies can be suppressed to quite low values by increasing the magnetic pressure.

Keywords: incompressible viscoelastic polymeric medium, external homogenous magnetic field, rheological correlation, resting state, linearized mixed problem, Lyapunov stability.

1 Introduction

To study the flows of an incompressible viscoelastic polymeric fluid in an infinite cylindrical channel under the influence of a homogenous external field we use structural-phenomenological Vinogradov-Pokrovski model as a base [1, 2]. This model interprets polymeric medium as a suspension of polymer macromolecules moving in an anysotropic fluid consisting of, e.g., solvent and other macromolecules. The environment effects on a chosen macromolecule is approximated by the impact on a chain of brownian particles, each of which is a a sufficiently large part of the macromolecule. It turns out that the formulated physical model is an effective way of describing slow relaxation processes in systems with linear polymers.

Using a mechanical analogy we call the brownian particles "beads" and the analogue of the elastic powers between the particles "springs". In the simplest case when the macromolecule is modelled as a "dumbbell" ("dumbbell" is two beads connected by a spring), we formulate the system of differential correlations (Vinogradov-Pokrovski model):

$$\rho(\frac{\partial}{\partial t}v_i + v_k\frac{\partial}{\partial x_k}v_i) = \frac{\partial}{\partial x_k}\sigma_{ik}, \quad \frac{\partial v_i}{\partial x_i} = 0, \tag{1}$$

$$\sigma_{ik} = -p\delta_{ik} + 3\frac{\eta_0}{\tau_0}a_{ik}, \tag{2}$$

$$\frac{d}{dt}a_{ik} - v_{ij}a_{jk} - v_{kj}a_{ji} + \frac{1+(k-\beta)I}{\tau_0}a_{ik} = \frac{2}{3}\gamma_{ik} - \frac{3\beta}{\tau_0}a_{ij}a_{jk}, \tag{3}$$

$$I = a_{11} + a_{22} + a_{33}, \quad \gamma_{ik} = \frac{v_{ik}+v_{ki}}{2}, \quad i,k = 1,2,3. \tag{4}$$

Here ρ is polymer density, v_i is i-th velocity component, σ_{ik} is stress tensor, p is hydrodynamic pressure; η_0, τ_0 are initial values of shear viscosity and relaxation time for viscoelastic component correspondenly, v_{ij} is velocity gradient tensor, a_{ik} is symmetric anisotropy stress tensor; γ_{ik} is symmetrized velocity gradient tensor, where components of the velocity gradient tensor $\nabla \otimes v$ are calculated as follows: $v_{ik} = \frac{\partial v_i}{\partial x_k}$, $i,k = 1,2,3$; k and β are phenomenological parameters that take into account the size and the form of a macromolecule ball. Equations (1) are motion equation and incompressibility condition, and equations (2)-(3) are rheological correlation, that connects kinematic characteristics of the flow with its thermodynamic parameters; each component a_{ik} is the sum of the first three terms in the left part of equality (3), the so-called upper convective derivative or Oldroyd derivative [3], $\frac{d}{dt} = \frac{\partial}{\partial t} + (\vec{v}, \nabla)$ is material derivative.

Note that the model (1)-(4) allows the formal passage to the limit. If $k, \beta \to 0$ we get the so called contravariant Oldroyd-B model [3]. If in addition $\tau_0 \to 0$ then we get the model for a viscous Newtonian fluid in a form of Navier-Stokes equations [28].

Also note that the accepted physical representation of a polymeric medium allows us to describe its main rheological properties: the decrease of viscosity and the first difference of normal stresses with the growth of shear velocity, the growth of stretching viscosity to a certain limit with the growth of deformation velocity.

Moreover, unlike the known models FENE-R [4], FENE-CR [5] that take into account additional physical mechanisms reflecting the behaviour features of a studied material: boundedness and nonlinearity of a spring elongation, connected to the finite length of a macromolecule and the existence of weaves and engagements in it, which obstruct its uniform and infinite elongation (instead of a Hooke law the nonlinear law of a spring elasticity is used); or RHL-model [6] that take into account potential barriers, that slow down the transition from one equilibrium configuration to the other (additional force of an inner resistance is introduced), the Pokrovski-Vinogradov model allows us to acquire nonzero values of the second difference for normal stresses [7]. Specifically, it tries to take into account the anysotropy effect of the chosen molecule environment that is caused by its elongation and orientation in space during the flow process of its macromolecule chains.

Rheological properties, predicted by the Pokrovski-Vinogradov model with parameters $k = 1, 2\beta$, that guarantee monotone of a flow curve, are qualitatively and quantitatively agree with the experimental data for melts and solutions of polymers [8, 9].

A number of works [10, 11, 12] studied the linear Lyapunov stability of Poiseuille-type flows in an infinite plane channel (the pressure drop on a segment doesn't depend on time) for the model (1)-(4), as well as for its generalization on the case of nonisometric flow of an incompressible weakly conducting polymeric fluid with the existence of a negative space charge [13, 14, 15] and on the case of nonisothermic model with the additional external interaction of a uniform magnetic field [16, 17, 18, 19].

The question of stability of the resting state for nonisothermic model of the polymeric fluid flow in an infinite plane channel under the influence of an external magnetic field was studied in works [20, 21, 22]. The main result being that the resting state in the case of an absolute conductivity, i.e. when $b_m = 0$ (its definition is given in the second section), and vanishing of one of the dissipative coefficients in the righthand part of the equation of the analogue to the heat equation is linearly unstable by Lyapunov.

The result of the work [23] was refined in the works [24, 25]. It states that the spectrum of a linearized with respect to the resting state mixed problem for the system (1)–(4) does not lie in an open right half-plane.

One of the main results of these works is that the mixed problem has periodic solutions which have the more then exponential growth $e^{\mathrm{Re}\lambda t}$, $\mathrm{Re}\lambda > 0$, $t \to +\infty$.

In the current work we study the linear stability of the resting state for the generalized Vinogradov-Pokrovski model (1)-(4). We consider the case when the flow of a polymeric fluid in an infinite cylindrical channel is affected by an external homogenous magnetic field. We also assume that the tension vector of the main magnetic field is parallel to the cylinder axis. From the physical point of view the flow of a polymeric fluid in a cylindrical channel is more real process than the flow in a plane channel.

In a short form the model is formulated as follows:

$$div\mathbf{u} = 0, \tag{5}$$

$$div\mathbf{H} = 0, \tag{6}$$

$$\frac{d\mathbf{u}}{dt} + \nabla P = div\Pi + \sigma_m(\mathbf{H}, \nabla)\mathbf{H}, \tag{7}$$

$$\frac{d\mathbf{H}}{dt} - (\mathbf{H}, \nabla)\mathbf{u} - b_m \triangle \mathbf{H} = 0, \tag{8}$$

$$\overset{\nabla}{\Pi} + \frac{1 + (k - \beta)I}{\tau_0}\Pi = \frac{2}{3}D - \frac{3\beta}{\tau_0}\Pi^2. \tag{9}$$

Here in addition to the notation for the base Vinogradov-Pokrovski model (1)-(4) we also use the new one: Π is the anisotropy tensor, $\mathbf{H}$ is the magnetic field strength vector, σ_m, b_m are some constants that are described further in §2, $\overset{\nabla}{\Pi} = \frac{d\Pi}{dt} - \left[(\nabla\mathbf{u})^T\Pi + \Pi(\nabla\mathbf{u})\right]$ is the Oldroyd derivative, Π^2 is calculated as a square of a matrix.

The study of the flows of fluids of different nature in domains with cylindrical boundaries is fundamentally important not only to study the properties of said flows such as an emergence and development of instabilities (see e.g. classic work [26]) which are described by other already accepted models, but also to test the new models.

Note that the work [29] shows, based on the numerical experiments, that the resting state for the axisymmetric flows of a polymeric fluid in an infinite cylindrical channel with circular section (in the case of the base Vinogradov-Pokrovski model 1)-(4) is linearly unstable by Lyapunov. Moreover we can state with a reasonable degree of confidence that the linear problem allows the construction of a Hadamard-type example [30, 31].

So the current work states the question if it is possible to use a magnetic field to dampen or at least slow down this instability for flows of polymeric fluid in a cylindrical channel.

The result of this work allows to conclude that for some parameters of the medium and flows and for some class of perturbations this is indeed possible.

The work is structured in the following way.

In the second paragraph we transform the system (5)-(9) from a Cartesian coordinate system into a cylindrical one and also formulate the linearized problem. We finish the second paragraph by formulating and proving a theorem that describes the spectrum of a problem for special cases and formulating the main theorem 2.

The next paragraph is dedicated to justification of the results of the theorem 2.

Finally the last paragraph is about the results of numerical experiments and their analysis.

2 Quasilinear and linearized models. Formulation of the main results

Following the monographs [1, 2, 27, 28, 42] and works [43, 44], we formulate the mathematical model for describing magnetohydrodynamic flows of an incompressible polymeric fluid in an infinite cylinder channel with round section directed along the cylinder axis (see Fig. 1).

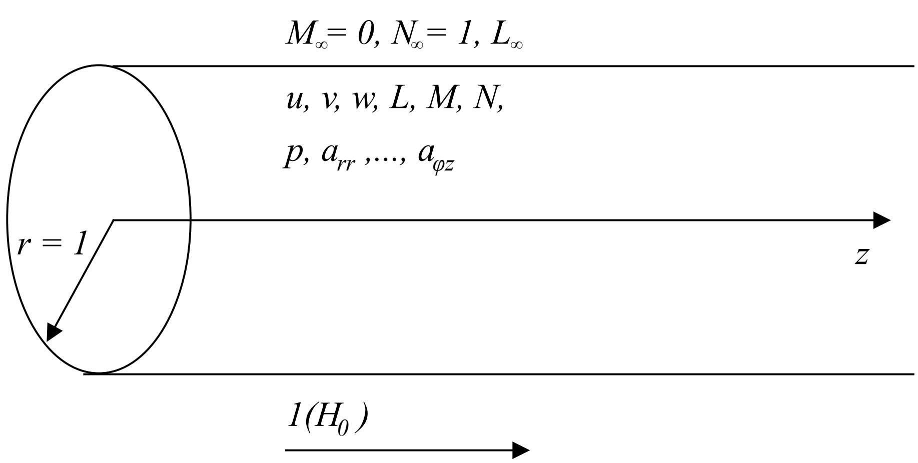

FIGURE 1. Cylinder channel and its base values, defining magnetohydrodynamic flow of a polymeric fluid

We can write the model (5)-(9) in a dimensionless form and in a cylindrical coordinate system as follows:

$$div\mathbf{u} = \frac{1}{r}\frac{\partial(ru)}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \varphi} + \frac{\partial w}{\partial z} = 0, \tag{10}$$

$$div\mathbf{H} = \frac{1}{r}\frac{\partial(rL)}{\partial r} + \frac{1}{r}\frac{\partial M}{\partial \varphi} + \frac{\partial N}{\partial z} = 0, \tag{11}$$

$$\frac{du}{dt} - \frac{v^2}{r} + \frac{\partial P}{\partial r} = \frac{1}{Re}\left(\frac{\partial a_{rr}}{\partial r} + \frac{1}{r}\frac{\partial a_{r\varphi}}{\partial \varphi} + \frac{\partial a_{rz}}{\partial z} + \frac{a_{rr} - a_{\varphi\varphi}}{r}\right) + \\ + \sigma_m\left(L\frac{\partial L}{\partial r} + \frac{M}{r}\frac{\partial L}{\partial \varphi} + N\frac{\partial L}{\partial z} - \frac{M^2}{r}\right), \tag{12}$$

$$\frac{dv}{dt} + \frac{uv}{r} + \frac{1}{r}\frac{\partial P}{\partial \varphi} = \frac{1}{Re}\left(\frac{\partial a_{r\varphi}}{\partial r} + \frac{1}{r}\frac{\partial a_{\varphi\varphi}}{\partial \varphi} + \frac{\partial a_{\varphi z}}{\partial z} + \frac{2a_{r\varphi}}{r}\right) + \\ + \sigma_m\left(L\frac{\partial M}{\partial r} + \frac{M}{r}\frac{\partial M}{\partial \varphi} + N\frac{\partial M}{\partial z} + \frac{LM}{r}\right), \quad (13)$$

$$\frac{dw}{dt} + \frac{\partial P}{\partial z} = \frac{1}{Re}\left(\frac{\partial a_{rz}}{\partial r} + \frac{1}{r}\frac{\partial a_{\varphi z}}{\partial \varphi} + \frac{\partial a_{zz}}{\partial z} + \frac{a_{rz}}{r}\right) + \\ + \sigma_m\left(L\frac{\partial N}{\partial r} + \frac{M}{r}\frac{\partial N}{\partial \varphi} + N\frac{\partial N}{\partial z}\right), \quad (14)$$

$$\frac{dL}{dt} - \left(L\frac{\partial u}{\partial r} + \frac{M}{r}\frac{\partial u}{\partial \varphi} + N\frac{\partial u}{\partial z}\right) - b_m(\triangle_{r,\varphi,z}L + \frac{2}{r^2}\frac{\partial M}{\partial \varphi} - \frac{L}{r^2}) = 0, \quad (15)$$

$$\frac{dM}{dt} + \frac{vL}{r} - \left(L\frac{\partial v}{\partial r} + \frac{M}{r}\frac{\partial v}{\partial \varphi} + N\frac{\partial v}{\partial z}\right) - b_m(\triangle_{r,\varphi,z}M + \frac{2}{r^2}\frac{\partial L}{\partial \varphi} - \frac{M}{r^2}) = 0, \quad (16)$$

$$\frac{dN}{dt} - \left(L\frac{\partial w}{\partial r} + \frac{M}{r}\frac{\partial w}{\partial \varphi} + N\frac{\partial w}{\partial z}\right) - b_m\triangle_{r,\varphi,z}N = 0, \quad (17)$$

$$\frac{da_{rr}}{dt} - 2\left(A_r\frac{\partial u}{\partial r} + \frac{a_{r\varphi}}{r}\frac{\partial u}{\partial \varphi} + a_{rz}\frac{\partial u}{\partial z}\right) + L_{rr} = 0, \quad (18)$$

$$\frac{da_{\varphi\varphi}}{dt} + 2\left(\frac{v}{r} - \frac{\partial v}{\partial r}\right)a_{r\varphi} - 2\left(\frac{1}{r}(u + \frac{\partial v}{\partial \varphi})A_\varphi + a_{\varphi z}\frac{\partial v}{\partial z}\right) + L_{\varphi\varphi} = 0, \quad (19)$$

$$\frac{da_{zz}}{dt} - 2\left(a_{rz}\frac{\partial w}{\partial r} + \frac{a_{\varphi z}}{r}\frac{\partial w}{\partial \varphi} + A_z\frac{\partial u}{\partial z}\right) + L_{zz} = 0, \quad (20)$$

$$\frac{da_{r\varphi}}{dt} + \left(\frac{v}{r} - \frac{\partial v}{\partial r}\right)A_r + \left(a_{r\varphi}\frac{\partial w}{\partial z} - a_{rz}\frac{\partial v}{\partial z} - \frac{A_\varphi}{r}\frac{\partial u}{\partial \varphi} - a_{\varphi z}\frac{\partial u}{\partial z}\right) + L_{r\varphi} = 0, \quad (21)$$

$$\frac{da_{rz}}{dt} - a_{rz}\left(\frac{\partial u}{\partial r} + \frac{\partial w}{\partial z}\right) - \left(A_r\frac{\partial w}{\partial r} + \frac{a_{r\varphi}}{r}\frac{\partial w}{\partial \varphi} + \frac{a_{\varphi z}}{r}\frac{\partial u}{\partial \varphi} + A_z\frac{\partial u}{\partial z}\right) + L_{rz} = 0, \quad (22)$$

$$\frac{da_{\varphi z}}{dt} + \left(\frac{v}{r} - \frac{\partial v}{\partial r}\right)a_{rz} - \left(a_{\varphi z}\frac{\partial u}{\partial r} + A_z\frac{\partial v}{\partial z} + a_{r\varphi}\frac{\partial w}{\partial r} + \frac{A_\varphi}{r}\frac{\partial w}{\partial \varphi}\right) + L_{\varphi z} = 0. \quad (23)$$

In equations (10)–(23) t is time, u, v, w, L, M, N are components of a velocity vector $\mathbf{u}$ and of the magnetic field strength vector $\mathbf{H} = (L, M, N)$ in a cylindrical coordinate system, $P = p + \sigma_m\|\mathbf{H}\|^2/2$, p is hydrodynamic pressure, $\|\mathbf{H}\|^2 = (\mathbf{H}, \mathbf{H})$, $a_{rr}, \ldots, a_{\varphi z}$ are components of a symmetrical anisotropy tensor Π of a second rank [1, 2];

$$L_{rr} = K_I a_{rr} + \beta\|a_r\|^2, \quad L_{\varphi\varphi} = K_I a_{\varphi\varphi} + \beta\|a_\varphi\|^2,$$
$$L_{zz} = K_I a_{zz} + \beta\|a_z\|^2, \quad L_{r\varphi} = K_I a_{r\varphi} + \beta(a_r, a_\varphi),$$
$$L_{rz} = K_I a_{rz} + \beta(a_r, a_z), \quad L_{\varphi z} = K_I a_{\varphi z} + \beta(a_\varphi, a_z),$$
$$a_r = (a_{rr}, a_{r\varphi}, a_{rz}), \quad a_\varphi = (a_{r\varphi}, a_{\varphi\varphi}, a_{\varphi z}), \quad a_z = (a_{rz}, a_{\varphi z}, a_{zz}),$$

$$A_r = a_{rr} + Wi^{-1}, \quad A_\varphi = a_{\varphi\varphi} + Wi^{-1}, \quad A_z = a_{zz} + Wi^{-1},$$
$$K_I = Wi^{-1} + \bar{k}I/\beta, \quad I = a_{rr} + a_{\varphi\varphi} + a_{zz}, \quad \bar{k} = k - \beta,$$

k, β, $0 < \beta < 1$ are phenomenological parameters of a rheological model [1, 2],
$Re = (\rho u_H l)/\eta_0$ is the Reynolds number, $Wi = (\tau_0 u_H)/l$ is the Weisenberg number,
$\rho(= const)$ is the density of the medium, η_0, τ_0 are initial values of shear viscosity and relaxation time [1, 2],
l is the characteristic length, u_H is the characteristic velocity,
$\sigma_m = (\mu\mu_0 H_0^2)/(\rho u_H^2)$ is the magnetic pressure coefficient, $b_m = \frac{1}{Re_m}$, $Re_m = \sigma\mu\mu_0 u_H l$ is the magnetic Reynolds number, μ_0 is the magnetic permeability in a vacuum, μ is the magnetic permeability, σ is the electrical conductivity of the medium,
$\triangle_{r,\varphi,z} = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$ is the Laplace operator,
$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial\varphi} + w\frac{\partial}{\partial z}$.

The system (10)–(23) is written in a dimensionless form: variables $t, r, z, u, v, w, p, L, M, N, a_{rr}, \ldots, a_{\varphi z}$ are divided correspondingly by $l/u_H, l, u_H, \rho u_H^2, H_0, \frac{Wi}{3}$.

Remark 1. *When deriving the magnetohydrodynamic equations we used the Maxwell equations system* [27, 32]*. We assume that the electromagnetic field is small enough, which allows us to put* $\mathbf{B} = \mu\mu_0\mathbf{H} = (1+\chi)\mu_0\mathbf{H}$ *where* $\mathbf{B}$ *is the magnetic induction vector,* χ *is the magnetic susceptibility.* $\chi = \chi_0/Y$*,* χ_0 *is the magnetic susceptibility under the room temperature* $T = T_0(= 300K)$*,* $Y = T/T_0$*,* T *is the temperature* [33, 34].

External with respect to the channel medium (see Fig. 1) is a dielectric and is under an effect of a homogenous magnetic field $\mathbf{H}$, where $M_\infty = 0$, $N_\infty = 1$ and L_∞ satisfies the following equation:

$$\triangle_{r,\varphi,z} L_\infty = 0, \quad r > 1, \tag{24}$$

which guarantees that the normal component of the vector $\mathbf{B}$ and the tangent component of the vector $\mathbf{H}$ are continuous.

Then the boundary no-slip conditions for the system (10)-(24) $(r = 1)$ are as follows:

$$u = 0, \tag{25}$$

$$L_\infty = \frac{1+\chi}{1+\chi_\infty} L, \tag{26}$$

$$N_\infty = 1, \quad M_\infty = 0. \tag{27}$$

As a base solution we choose the resting state

$$u = 0,\ p = p_0 - const,\ \alpha_{rr} = 0,\ \ldots,\ \alpha_{\varphi z} = 0,\ L = M = 0,\ N = 1$$

both inside the channel, i.e. $r < 1$, and outside it, i.e. $r > 1$, which means that the vector of the main magnetic field strength is directed along the z axis.

Linearizing the boundary problem (10)-(24), (25)-(27) with respect to the chosen solution results in a following problem (small perturbations of the components of the solution are written the same as initial variables) leads us to the following problem:

$$Ru + \frac{1}{r}v_\varphi + w_z = 0, \tag{28}$$

$$u_t + \Omega_r = \frac{1}{r}(\alpha_{r\varphi})_\varphi + (\alpha_{rz})_z + \frac{\alpha_{rr} - \alpha_{\varphi\varphi}}{r} + \sigma_m L_z, \tag{29}$$

$$v_t + \frac{1}{r}\Omega_\varphi = (\alpha_{r\varphi})_r + \frac{1}{r}(\alpha_{\varphi\varphi} - \alpha_{rr})_\varphi + (\alpha_{\varphi z})_z + \frac{2\alpha_{r\varphi}}{r} + \sigma_m M_z, \tag{30}$$

$$w_t + \Omega_z = (\alpha_{rz})_r + \frac{1}{r}(\alpha_{\varphi z})_\varphi + (\alpha_{zz} - \alpha_{rr})_z + \frac{\alpha_{rz}}{r} + \sigma_m N_z, \tag{31}$$

$$L_t - u_z - b_m\left\{D_1 L - \frac{M_\varphi}{r^2}\right\} = 0, \tag{32}$$

$$M_t - v_z - b_m\left\{D_1 M - \frac{L_\varphi}{r^2}\right\} = 0, \tag{33}$$

$$N_t - w_z - b_m D_0 N = 0, \tag{34}$$

$$\Lambda\alpha_{rr} = 2\varkappa^2 u_r, \tag{35}$$

$$\Lambda\alpha_{\varphi\varphi} = \frac{2}{r}(u + v_\varphi)\varkappa^2, \tag{36}$$

$$\Lambda\alpha_{zz} = 2\varkappa^2 u_z, \tag{37}$$

$$\Lambda\alpha_{r\varphi} = \frac{\varkappa^2}{r}u_\varphi - \varkappa^2\left(\frac{v}{r} - v_r\right), \tag{38}$$

$$\Lambda\alpha_{rz} = \varkappa^2(w_r + u_z), \tag{39}$$

$$\Lambda\alpha_{\varphi z} = \varkappa^2(v_z + \frac{w_\varphi}{r}). \tag{40}$$

Here $\alpha_{rr} = \frac{a_{rr}}{Re}, \dots, \alpha_{\varphi z} = \frac{a_{\varphi z}}{Re}$, $\varkappa^2 = \frac{1}{WRe}$, $R = \frac{\partial}{\partial r} + \frac{1}{r}$,
$D_0 = \triangle = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2} = R^2 + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2}$,
$D_1 = D_0 - \frac{1}{r^2}$, $\Lambda = \frac{\partial}{\partial t} + \frac{1}{W}$, $\Omega = p - \alpha_{rr}$.

Outside the channel the condition (24) still holds

$$D_0 L_\infty = 0 \text{ for } r > 1, \tag{41}$$

and the boundary conditions (25)-(27) transform into

$$\begin{cases} u = 0, \\ L_\infty = \dfrac{1+\chi}{1+\chi_\infty}L, \quad r = 1, \\ N = M = 0. \end{cases} \tag{42}$$

Remark 2. *For the function Ω, i.e. generalized "pressure", the following holds*

$$D_0\Omega = \left(\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} - \frac{1}{r}\frac{\partial}{\partial r}\right)(\alpha_{\varphi\varphi} - \alpha_{rr}) + \\ + 2R\left(\frac{\partial}{\partial z}\alpha_{rz} + \frac{1}{r}\frac{\partial}{\partial\varphi}\alpha_{r\varphi} + \frac{2}{r}\frac{\partial}{\partial\varphi}\left(\frac{\partial}{\partial z}\alpha_{\varphi z} + \frac{1}{r}\alpha_{r\varphi}\right)\right), \quad (43)$$

Remark 3. *Due to the (28), (31-33), the function*

$$d = L_r + \frac{1}{r}L + \frac{1}{r}M_\varphi + N_z$$

equals zero for all r, φ, z if

$$d\big|_{t=0} = 0, \quad 0 < r < 1, \quad |z| < \infty,$$

so we can use equation (34) and equation

$$d = 0$$

interchangeably.

We will be looking for a solution of the problem (28)–(41) in the special form:

$$u(t, r, \varphi, z) = u(r)\exp\{\lambda t + inz + im\varphi\}, \dots, \\ L_\infty(t, r, \varphi, z) = L_\infty \exp\{\lambda t + inz + im\varphi\}, \dots, \\ \alpha_{\varphi z}(t, r, \varphi, z) = \alpha_{\varphi z}(r)\exp\{\lambda t + inz + im\varphi\}, \quad (44)$$

where $\lambda = \eta + i\xi$, $\xi, \eta \in R^1$, $n, m \in Z$ are some parameters.

Taking into the account **remark 3** and assuming $b_m \neq 0$, $\lambda \neq \frac{1}{Wi}$ we get the equation system for the functions $u(r)$, ..., $L_\infty(r)$ with parameters λ, n, m

$$\begin{cases} \alpha_{rr} = \dfrac{u'}{\bar\lambda}, \quad \alpha_{\varphi\varphi} = \dfrac{u + imv}{r\bar\lambda}, \quad \alpha_{zz} = \dfrac{inu}{\bar\lambda}, \\ \alpha_{r\varphi} = \dfrac{1}{2\bar\lambda}\left\{\dfrac{imu}{r} + v' - \dfrac{v}{r}\right\}, \quad \alpha_{rz} = \dfrac{w' + inu}{2\bar\lambda}, \\ \alpha_{\varphi z} = \dfrac{i(nv + \frac{1}{r}mw)}{2\bar\lambda}, \quad \bar\lambda = \dfrac{\lambda + Wi^{-1}}{2ae^2}. \end{cases} \quad (45)$$

The other for equations of the system (28)–(41) are rewritten as follows:

$$Ru + i(\frac{m}{r}v + nw) = 0, \tag{46}$$

$$\Omega' = -\left(\frac{m^2+2}{2r^2\bar{\lambda}} + \frac{n^2}{2\bar{\lambda}} + \lambda\right)u + \frac{1}{r\bar{\lambda}}u' + \\ + \frac{im}{2\bar{\lambda}r}(v' - \frac{3v}{r}) + \frac{in}{2\bar{\lambda}}w' + in\sigma_m L, \tag{47}$$

$$R^2 v - (n^2 + \frac{1+3m^2}{r^2} + 2\lambda\bar{\lambda})v = \frac{2nm}{r}w - \frac{4im}{r^2}u - \\ - 2in\bar{\lambda}\sigma_m M + \frac{2im\bar{\lambda}}{\Omega}, \tag{48}$$

$$R^2 w - (n^2 + \frac{m^2}{r^2} + 2\lambda\bar{\lambda})w = 2in\bar{\lambda}\Omega - 2in\bar{\lambda}\sigma_m N + \\ + 2mnv + 2n^2 u - \frac{2inu}{r}, \tag{49}$$

$$RL + i\left(\frac{m}{r}M + nN\right) = 0, \tag{50}$$

$$d_1 M + \frac{2imL}{r^2} - \frac{\lambda}{b_m}M + \frac{in}{b_m}v = 0, \tag{51}$$

$$d_0 N - \frac{\lambda}{b_m}N + \frac{in}{b_m}w = 0, \tag{52}$$

$$d_0 L_\infty = 0. \tag{53}$$

Here we use the notation

$$d_1 = d_0 - \frac{1}{r^2}, \quad d_0 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m}{r^2} - n^2 = R^2 - \frac{m^2}{r^2} - n^2$$

From correlations (29), (43) and (45) it follows that

$$d_0\Omega = \left(-\frac{m^2}{r^2} - \frac{1}{r}\frac{d}{dr}\right)\left(\frac{u+imv}{r\bar{\lambda}} - \frac{u'}{\bar{\lambda}}\right) + \\ + R\left(in\frac{w'+inu}{\bar{\lambda}} + \frac{im}{r}\frac{\frac{imu}{r} + v' - \frac{v}{r}}{\bar{\lambda}}\right) + \\ + \frac{im}{r}\left(-n\frac{nv + \frac{1}{r}mw}{\bar{\lambda}} + \frac{1}{r}\frac{\frac{imu}{r} + v' - \frac{v}{r}}{\bar{\lambda}}\right) - n^2\left(\frac{inu}{\bar{\lambda}} - \frac{u'}{\bar{\lambda}}\right), \tag{54}$$

$$\widetilde{\Omega}' = \frac{imv'}{2\bar{\lambda}} - \frac{inw'}{2\bar{\lambda}} + \sigma_m inL \quad \text{for } r = 1, \tag{55}$$

where

$$\widetilde{\Omega} = \Omega - \frac{u}{r\bar{\lambda}} - \frac{in}{\bar{\lambda}}w. \tag{56}$$

In an axisymmetric case, when $m = 0$, which is the main interest to us, the system (46)–(53) is simplified by splitting into two independent subsystems.

That and the boundary conditions (42) leads us to the following two spectral boundary problems:

$$\begin{cases} (d_1 - 2\lambda\bar{\lambda})v + 2in\bar{\lambda}\sigma_m M = 0, \\ (d_1 - \dfrac{\lambda}{b_m})M + \dfrac{in}{b_m}w = 0 \\ v = M = 0 \quad \text{for } r = 1. \end{cases} \tag{57}$$

$$\begin{cases} Ru + inw = 0, \\ (d_0 - 2\lambda\bar{\lambda})w - 2in\bar{\lambda}\hat{\Omega} + 2in\bar{\lambda}\sigma_m N - 2n^2u + n^2w = 0, \\ RL + inN = 0, \\ (d_0 - \dfrac{\lambda}{b_m})N + \dfrac{inw}{b_m} = 0, \\ \hat{\Omega}' + \dfrac{2\lambda\bar{\lambda} + n^2}{2\bar{\lambda}}u - in\sigma_m L = 0, \\ u = w = N = 0 \quad \text{for } r = 1, \\ d_0 L_\infty = 0, \quad \text{for } r > 1, \\ L_\infty = \dfrac{1+\chi}{1+\chi_\infty}L, \quad \text{for } r = 1. \end{cases} \tag{58}$$

Here

$$d_1 = R^2 - n^2 - \frac{1}{r^2}, \quad d_0 = R^2 - n^2, \quad \hat{\Omega} = \widetilde{\Omega} + \frac{in}{2\bar{\lambda}}w,$$

function $\widetilde{\Omega}$ can be represented through Ω due to (56).

The correlation (54) takes the following form:

$$d_0\widetilde{\Omega} = \frac{in^3}{\bar{\lambda}}(w - u). \tag{59}$$

For the case of absolute conductivity $b_m = 0$ in which we are interested the most, the problems (57), (58) take a simpler form

$$\begin{cases} (d_1 - 2\lambda\bar{\lambda})v + 2in\bar{\lambda}\sigma_m M = 0, \\ \lambda M - inv = 0, \\ v = M = 0 \quad \text{for } r = 1. \end{cases} \tag{60}$$

$$\begin{cases} Ru + inw = 0, \\ (d_0 - 2\lambda\bar{\lambda})w - 2in\bar{\lambda}\hat{\Omega} + 2in\bar{\lambda}\sigma_m N - 2n^2 u + n^2 w = 0, \\ RL + inN = 0, \\ \hat{\Omega}' + \dfrac{2\lambda\bar{\lambda} + n^2}{2\bar{\lambda}} u - in\sigma_m L = 0, \\ u = w = N = 0 \quad \text{for } r = 1, \\ d_0 L_\infty = 0, \quad \text{for } r > 1, \\ L_\infty = \dfrac{1+\chi}{1+\chi_\infty} L, \quad \text{for } r = 1. \end{cases} \tag{61}$$

Cross differentiating (61) gives us equivalent problem for the radial velocity component:

$$u^{(V)} + u^{(IV)}\frac{3}{r} - u'''\left(\frac{3}{r^2} + \tau\right) + u''\left(\frac{6}{r^2} - \frac{2\tau}{r} + 2in^3\right) +$$
$$+ u'\left(-\frac{9}{r^4} + \frac{\tau}{r^2} + n^2(\tau + n^2) + i\frac{2n^3}{r}\right) + u\left(\frac{9}{r^5} - \frac{\tau}{r^3} + \frac{n^2\tau}{r}\right) = 0, \tag{62}$$

$$\begin{cases} |u(0)| = 0, \\ \left|(u' + \dfrac{1}{r}u)\big|_{r=0}\right| < \infty, \\ \left|(u''' + \dfrac{2}{r}u'' - \dfrac{1}{r^2}u' - \dfrac{1}{r^3}u)\big|_{r=0}\right| < \infty, \end{cases} \tag{63}$$

$$u(1) = u'(1) = 0, \tag{64}$$

where

$$\tau = 2\lambda\bar{\lambda} + 2n^2\sigma_m\frac{\bar{\lambda}}{\lambda}. \tag{65}$$

Boundary problem (60), naturally, is equivalent to the boundary problem for the angle component v:

$$\begin{cases} (d_1 - 2\lambda\bar{\lambda})v - 2n^2\sigma_m\dfrac{\bar{\lambda}}{\lambda}v = 0, \\ |v(0)| < \infty, \quad v(1) = 0. \end{cases} \tag{66}$$

While formulating problem (62)-(64) and (60), we of course assume that n and $\lambda \neq 0$.

First we study the limit cases: $\lambda = 0$, $\lambda = -\frac{1}{Wi}$ and $n = 0$.
If $\lambda = 0$, $n \neq 0$ ($\lambda \neq -\frac{1}{Wi}$) then due to the system (28)-(40) and correlations (45) we get

$$u = v = w = M = 0, \quad \hat{\Omega} = \widetilde{\Omega} = \Omega = \sigma_m N, L = \frac{1}{in}N'$$

and the component N satisfies equation

$$N'' + \frac{N'}{r} - n^2 N = 0 \tag{67}$$

under the condition

$$|N(0)| < \infty,$$

which gives us

$$N(r) = C_1 J_0(2in\sqrt{r}), \tag{68}$$

where C_1 is some arbitrary constant [36].

The boundary condition

$$N(1) = 0$$

due to (68) leads to the following correlation

$$n^2 = -\frac{\mu_k^2}{4},\ k = 1, 2, \ldots$$

where μ_k are positive zeroes of the Bessel function $J_0(r)$.

So, it turns out that $\lambda = 0$ for $n \neq 0$ is not a eigenvalue of the corresponding spectral problem for the problem (30)-(42) in an axisymmetric case.

Let $\lambda = -\frac{1}{Wi}$ and $n \neq 0$. Still assuming the absolute conductivity we get [45]

$$\mathbf{u} = 0,\ \mathbf{H} = 0, \tag{69}$$

$$\alpha_{r\varphi} = -\frac{in}{r^2}\int_0^r \xi^2 \alpha_{\varphi z} d\xi \tag{70}$$

$$\begin{aligned} \alpha_{rz}(\sqrt{n}r) = C_2 J_1(\sqrt{n}r) + \frac{i}{\sqrt{n}} J_1(\sqrt{n}r)\times \\ \times \int_0^r \frac{Y_1(\sqrt{n}r)}{W(\sqrt{n}r)}\left[(\alpha_{rr} - \alpha_{zz})\eta + \frac{\alpha_{rr} - \alpha_{\varphi\varphi}}{\eta}\right] d\eta - \\ - \frac{i}{\sqrt{n}} Y_1(\sqrt{n}r)\int_0^r \frac{J_1(\sqrt{n}r)}{W(\sqrt{n}r)}\left[(\alpha_{rr} - \alpha_{zz})\eta + \frac{\alpha_{rr} - \alpha_{\varphi\varphi}}{\eta}\right] d\eta, \end{aligned} \tag{71}$$

$$W(\eta) = \left[Y_1(\eta)J_1'(\eta) = Y_1'(\eta)J_1(\eta)\right]\big|_{\eta=1}\frac{1}{\eta} \text{ is Wronski's determinate,} \tag{72}$$

$$p = \frac{1}{in}[(\alpha_{rz})' + \frac{\alpha_{rz}}{r}] + \alpha_{zz}, \tag{73}$$

where $\alpha_{\varphi z}, \alpha_{rr}, \alpha_{zz}, \alpha_{\varphi\varphi}$ are arbitrary functions, $J_1(\xi), Y_1(\xi)$ are Bessel functions of the first order of the first and second kind, C_2 is an arbitrary constant.

If additionally $n = 0$, then instead of solution (69)-(73) we get

$$u = 0,\ H = 0, \tag{74}$$

$$\alpha_{rz} = \alpha_{r\varphi} = 0, \tag{75}$$

$$p = \alpha_{rr} + \int_0^r \frac{\alpha_{rr} - \alpha_{\varphi\varphi}}{\xi} d\xi. \tag{76}$$

where $\alpha_{\varphi z}, \alpha_{rr}, \alpha_{zz}, \alpha_{\varphi\varphi}$ are arbitrary functions.

Of course, the anisotropy tensor components in the right-hand side of equations (70), (71), (75) need to be chosen in a such a way that they are bounded for $r = 0$ and the left-hand side is also bounded.

Remark 4. *For now we will not be discussing the conditions that guarantee boundedness of all the anisotropy tensor components for $r = 0$ in detail because, firstly, we don't know if the point $\lambda = -\frac{1}{Wi}$ is isolated from the other points of the spectrum (but below we will show that for $n = 0$ it is) which is important for the reverse Laplace transform, and secondly, $\lambda = -\frac{1}{Wi}$ lies strictly in the left half-plane which guarantees vanishing of the corresponding mode for $t \to \infty$ and we are more interested in cases when the instability may arise.*

Finally let $n = 0$, but $\lambda \neq 0$.
In this case from the system (28)-(40) and the notation (45) we get

$$u = L = N = 0, \quad \Omega = p = C_3, \quad C_3 \text{ is an arbitrary constant,} \tag{77}$$

and for the velocity component w the following boundary problem

$$\begin{cases} \left(\dfrac{d^2}{dr^2} + \dfrac{1}{r}\dfrac{d}{dr} - 2\lambda\bar{\lambda} \right) w = 0, \\ w(1) = 0, \quad |w(0)| < \infty. \end{cases} \tag{78}$$

Introducing a new variable

$$\xi = i\sqrt{2\lambda\bar{\lambda}}r \tag{79}$$

and using the boundedness of the solution w for $r = 0$ we get [36]

$$w = C_4 J_0(\xi) = C_4 J_0(i\sqrt{2\lambda\bar{\lambda}}r), \, C_4 \text{ is an arbitrary constant.} \tag{80}$$

The second boundary condition leads us to equality

$$2\lambda\bar{\lambda} = -\mu_k^2, \, k = 1, 2, \ldots \tag{81}$$

where μ_k are roots of the equation

$$J_0(\mu) = 0.$$

They are symmetric so we assume $\mu_k > 0$.

Equation (81) gives us

$$\lambda_{1,2}^{k,0} = \frac{-\frac{1}{Wi} \pm \sqrt{\frac{1}{Wi^2} - 4\varkappa^2\mu_k^2}}{2}, \, k = 1, 2, \ldots \tag{82}$$

It is obvious that $Re\lambda_{1,2}^{k,0} \leq -\sigma < 0$ for some constant $\sigma > 0$.

In its turn the unknowns M and v satisfy the following conditions:

$$M = 0. \tag{83}$$

$$\begin{cases} \left(\dfrac{d^2}{dr^2} + \dfrac{1}{r}\dfrac{d}{dr} - \dfrac{1}{r^2} - 2\lambda\bar{\lambda} \right) v = 0, \\ v(1) = 0, \quad |v(0)| < \infty. \end{cases} \tag{84}$$

The change of variable (79) allows us to find the spectral function

$$v = C_5 J_1(\xi) = C_5 J_1(i\sqrt{2\lambda\bar{\lambda}}r),\ C_5 \text{ is an arbitrary constant.} \tag{85}$$

and the spectrum

$$\lambda_{1,2}^{k,1} = \frac{-\frac{1}{Wi} \pm \sqrt{\frac{1}{Wi^2} - 4\varkappa^2\nu k^2}}{2},\ k = 2, 3\dots \tag{86}$$

where $J_1(\nu k) = 0$.

In equality (86) we excluded the case $\nu_1 = 0$ due to the fact that for $\lambda = 0$ the change of variable (79) becomes degenerate and remembering formulas (45) gives us a way to find other components of the unknown vector-function for $n = 0$.

Remark 5. *If in addition to the condition $n = 0$ we also set $\lambda = 0$ then the components N and M in correlations (77) and (83) are no longer necessarily equal zero but can be arbitrary functions.*

Thus for the case of absolute conductivity $b_m = 0$ the modes acquired for $\lambda = -\frac{1}{Wi}$ and for $n = 0$ (e.g. formulas (77), (80), (82), (83), (85), (86)) decrease exponentially for $t \to \infty$ with the exception of

$$(u, v, w, L, M, M, p, a_{rr}, a_{r\varphi}, a_{rz}, a_{\varphi\varphi}, a_{\varphi z}, a_{zz})^T = \\ = C_6 e^{\lambda t}(0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0)^T \tag{87}$$

where $\lambda \neq 0$ is an arbitrary parameter, C_6 is an arbitrary constant.

Remark 6. *That means that if $C_6 \neq 0$ then there can occur perturbations which grow unlimitedly for $t \to +\infty$. To get the decreasing pressure for $t \to \infty$ we need to set this mode for $Re\lambda < 0$. This situation is similar to the flow of polymeric fluid in an infinite plane channel for the base variant of the Vinogradov-Pokrovski model* [24, 25].

We can unify the results above into the following theorem:

Theorem 1. *In case of absolute conductivity $b_m = 0$ the spectral function of the system (28)-(40) (taking into account (45)) for $\lambda = -\frac{1}{Wi}$, $n \neq 0$ is defined by the formulas (69)-(73). For $\lambda = -\frac{1}{Wi}$, $n = 0$ it is defined by the formulas* (74)-(76). *For $n = 0$, $\lambda \neq 0$ it is defined by the formulas (77), (80), (82), (83), (85), (86) (if $n = 0$ and $\lambda = 0$ it is necessary to tale into account remark 5).*

Not counting the already studied special cases to get the full picture of the spectrum of the problems (57), (58) it is necessary to study the spectral problem (62)-(65) and the spectral problem (66).

The following two theorem hold true.

Theorem 2. *Let $\lambda \neq \frac{1}{Wi}$ and $n \neq 0$. Then the spectral equation for the boundary problem (62)-*(65) *takes the form*

$$u_2'(1)u_1(1) - u_1'(1)u_2(1) = 0.$$

The functions $u_1(r)$ and $u_2(r)$ are defined by the formulas (89), (90) and (91), (92) correspondingly.

The spectrum of the boundary problem (66) is defined by the formulas (104), (101)-(102) and its eigenfunctions by correlation (104) where the limit correlations (105), (106) also hold.

3 Proof of theorem 2

Let us start the proof of theorem 2 by studying the fundamental solutions system of the equation (62). Here the main point is the information about the basis of the defining Euler equation [37, 38].

$$\tilde{u}^{(V)} + \frac{3}{r}\tilde{u}^{(IV)} - \frac{3}{r^2}\tilde{u}''' + \frac{6}{r^3}\tilde{u}'' - \frac{9}{r^4}\tilde{u}' + \frac{9}{r^5}\tilde{u} = 0. \tag{88}$$

The basis of the equation (88) consists of the following functions

$$\tilde{u}_1 = r^3,\ \tilde{u}_2 = r,\ \tilde{u}_3 = r^3 \ln r,\ \tilde{u}_4 = r \ln r,\ \tilde{u}_5 = \frac{1}{r}.$$

Due to the boundary conditions for $r = 0$ the solutions of the equation (62) we are looking for are the first two functions $\tilde{u}_1$, $\tilde{u}_2$. So the first element of the equation (62) basis has the form [38]

$$u_1 = \sum_{k=0}^{\infty} a_k r^{k+3} = a_0 r^3 + a_1 r^4 + \ldots, \tag{89}$$

where u_1 is a whole function, $r \in R$. Coefficients a_i can be found in the following way:

$$a_0 = 1,\ a_1 = 0, a_2 = \frac{\tau}{24},\ a_3 = -\frac{2}{175} i n^3,\ a_4 = \frac{\alpha^2 - n^2(n^2+\tau)}{1152},$$
$$a_k = \frac{\tau}{(k+2)(k+4)} a_{k-2} - \frac{2in^3}{(k+2)^2(k+4)} a_{k-3} - $$
$$- \frac{n^2(n^2+\tau)}{(k+2)^3 k(k+4)} a_{k-4},\ k = 5, 6, 7, \ldots \tag{90}$$

where $\tau = 2\lambda\bar{\lambda} + 2n^2\sigma_m \frac{\bar{\lambda}}{\lambda}$ due to (65).

In its turn

$$u_2 = \sum_{k=0}^{\infty} b_k r^{k+1} = b_0 r + b_1 r^2 + \ldots, \tag{91}$$

where

$$b_0 = 1,\ b_1 = b_2 = 0,\ b_3 = -\frac{2in^3}{45},$$
$$b_k = \frac{\tau}{(k-2)(k+2)} b_{k-2} - \frac{2in^3}{(k+2)k^2} b_{k-3} - \\ - \frac{n^2(n^2+\tau)}{(k-2)k^2(k+2)} b_{k-4},\ k = 4, 5, 6, \dots \quad (92)$$

Thus the set of solutions of the equation (62) which satisfy the boundary conditions for $r = 0$ has the following form:

$$u(r) = C_1 u_1(r) + C_2 u_2(r), \quad (93)$$

where C_1, C_2 are arbitrary constants and $u_1(r)$, $u_2(r)$ are whole functions of the form (89) (with the coefficients from (90)) or (91) (with the coefficients from (92)).

Boundary conditions for $r = 1$ lead us the needed spectral equation from which we find eigenvalues λ

$$u_2'(1)u_1(1) - u_1'(1)u_2(1) = 0. \quad (94)$$

This proves the first statement of theorem 2.

Now moving to the problem (66) and using the variable change analogous to (79) we get the spectral equation

$$\lambda^3 + \frac{1}{Wi}\lambda^2 + \lambda(n^2(\varkappa^2 + \sigma_m) + \nu_k^2\varkappa^2) + n^2\sigma_m\frac{1}{Wi} = 0, \quad (95)$$

where ν_k, $k = 1, 2, 3, \dots$ are positive zeros of the Bessel function J_1. It is obvious that due to the Routh-Hurwitz stability criterion [39]

$$Re\lambda_i < 0,\ i = 1, 2, 3, \dots \quad (96)$$

for fixed μ_k and M.

Now to find roots of the equation (95). As is common for the solutions of the third order equation we will use the following notation:

$$p = -\frac{1}{3Wi^2} + n^2(\varkappa^2 + \sigma_m) + \nu_k^2\varkappa^2,$$
$$q = \frac{2}{27Wi^3} - \frac{1}{3Wi}\left(n^2(\varkappa^2 - 2\sigma_m) + \nu_k^2\varkappa^2\right). \quad (97)$$

Then after the variable change

$$\lambda = \xi - \frac{1}{3Wi}, \quad (98)$$

the equation (95) transforms into

$$\xi^3 + p\xi + q = 0. \quad (99)$$

The form of the roots of the equation (99) depends on the sign of the parameter Q ($-108Q$ is the determinant of equations (95) and (99))[40]

$$Q = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2 = n^6 \frac{1}{27}(\varkappa^2 + \sigma_m)^3 +$$
$$+ n^4 \left(3(\varkappa^2 + \sigma_m)^2 \nu_k^2 \varkappa^2 + \frac{1}{36Wi^2}(\varkappa^2 - 2\sigma_m)^2 - \frac{1}{27Wi^3}(\varkappa^2 + \sigma_m)^2 \right) +$$
$$+ n^2 \left(\frac{1}{81Wi^4}(\varkappa^2 + \sigma_m) - \frac{2}{27Wi^2}(\varkappa^2 + \sigma_m)\nu_k^2 \varkappa^2 + \right.$$
$$\left. + \frac{1}{9}\nu_k^4 \varkappa^4 (\varkappa^2 + \sigma_m) - \frac{1}{81Wi^4}(\varkappa^2 - 2\sigma_m) + \frac{1}{18Wi^2}(\varkappa^2 - 2\sigma_m)\nu_k^2 \varkappa^2 \right) +$$
$$+ \frac{1}{9Wi^2}\nu_k^4 \varkappa^4 + \frac{1}{27}\nu_k^6 \varkappa^6. \quad (100)$$

If $Q < 0$ then (trigonometry solution)

$$\xi_1 = 2\sqrt{-\frac{p}{3}} \cos\frac{\alpha}{3},$$
$$\xi_{2,3} = -2\sqrt{-\frac{p}{3}} \cos(\frac{\alpha}{3} \pm \frac{2\pi}{3}), \quad (101)$$

where

$$\cos\alpha = -\frac{9}{\sqrt{-(\frac{p}{3})^3}}.$$

If $Q > 0$, $p > 0$ then

$$\xi_1 = -2\sqrt{\frac{p}{3}} \cot(2\alpha),$$
$$\xi_{2,3} = \sqrt{\frac{p}{3}} \cot(2\alpha) \pm i\sqrt{3} \csc(2\alpha), \quad (102)$$

where

$$\tan\alpha = \sqrt[3]{\tan\frac{\beta}{2}},\ (|\alpha| \le \frac{\pi}{4}),\ \tan\beta = \frac{2}{q}\sqrt{(\frac{p}{3})^3},\ (|\beta| \le \frac{\pi}{2}).$$

If $Q \ge 0$, $p < 0$ then

$$\xi_1 = -2\sqrt{-\frac{p}{3}} \csc(2\alpha),$$
$$\xi_{2,3} = \sqrt{-\frac{p}{3}} \csc(2\alpha) \pm i\sqrt{3} \cot(2\alpha), \quad (103)$$

where

$$\tan\alpha = \sqrt[3]{\tan\frac{\beta}{2}},\ (|\alpha| \le \frac{\pi}{4}),\ \sin\beta = -\frac{2}{q}\sqrt{(-\frac{p}{3})^3},\ (|\beta| \le \frac{\pi}{2}).$$

This means that due to (98) the roots of the equation are as follows:

$$\lambda_{1,2,3} = \xi_{1,2,3} - \frac{1}{3Wi}, \tag{104}$$

where $\xi_{1.2.3}$ are defined by the formulas (101)-(103). Note that if $n \to \infty$ then (for the fixed ν_k)

$$\lambda_1 \to -\frac{1}{Wi}\frac{\sigma_m}{\varkappa^2 + \sigma_m},$$
$$\lambda_{2,3} \to -\frac{1}{2Wi}(\varkappa^2 + \sigma_m) \pm \frac{1}{\sqrt{3}Wi}\left(\frac{\varkappa^2 - 2\sigma_m}{\varkappa^2 + \sigma_m}\right)i. \tag{105}$$

Remark 7. *If n is fixed and $\nu_k \to +\infty$ then*

$$\lambda_1 \to 0,\ \lambda_{2,3} \to -\frac{1}{2Wi} \pm \frac{1}{6Wi}i. \tag{106}$$

For the finite values of n and ν_k eigenfunctions of the problem (66) are defined the following way:

$$V_{k,n}(r) = CJ_1(i\sqrt{2\lambda\bar{\lambda} + 2n^2\sigma_m\frac{\bar{\lambda}}{\lambda}}r), \tag{107}$$

where C is an arbitrary constant.

4 Numerical study of the spectrum

The study of the spectrum with numerical methods is about solving two main problems. First is that to construct a discrete analogue of the operator with the spectrum that approximates the spectrum of the original accurately enough. Lately the most frequent approach for this is the collocation discretization method in Gauss-Lobato points. We should keep in mind that the end result of this method can be sensitive to the way we take into account the boundary conditions. In our case the most complications and interest arise from the boundary condition in zero. Below we show two ways to take them into account in the matrix operators.

Another problem is the evaluation of the credibility of our results. The possible errors have two sources. The first is the accumulation of the rounding errors in the process of calculations. Note that since we are talking about non-symmetric spectral problem the size of the computational error can be unpredictably large. The second source is the discretization itself. And we are talking not only about the difference between the original and discrete operators, which decreases with the increase of the number of collocation points. We can also expect the appearance of the so called "parasite" eigenvalues. Similar facts can be found for example in work [41]. These eigenvalues appear for the discrete operators and don't converge to the spectrum of the original differential operator. Usually they differ from "true" eigenvalues by saw-like structure of the eigenfunction. Our work, as was already said, uses two methods for constructing discrete operators. We use them for calculations for different discretization parameters and study the smoothness of eigenfunctions for questionable eignevalues.

Now we will construct the finitely dimensional approximation of differential operators by pseudo-spectral method. The collocation nodes are Gauss - Lobato points ξ_k, $k = 0, 1, \dots, N$, ordered by ascension. We note collocation matrix as $\widetilde{D}_1$. This matrix approximates the functions derivative in points ξ_k on the interval [-1,1]. We will map Gauss-Lobato linearly to the interval [0,1]: $\xi_k = 2r_k - 1$. Then according to the rule for differentiating complex function the approximation of the derivative has the from $D_1 = 2\widetilde{D}_1$. The discrete representation of the derivatives of higher order is represented as orders of the matrix of the first derivative. For example $D_2 = D_1 \times D_1$ is the approximation of the operator $\frac{d^2}{dy^2}$.

Now we will describe how do we take into account the boundary conditions in the discrete approximation of the differential operator.

First we will consider the Dirichlet condition $u(0) = u(1) = 0$. To take into account these conditions we will delete the first and the last row and column from matrices D_k. We will denote these matrices as $\widetilde{D}_k$. Also in further calculations the boundary points $r_0 = 0$, $r_N = 1$ are not considered and instead we use "shortened" vectors $u = (u_1, \dots, u_{N-1})^T$.

Note that the boundary conditions of the form (63) can be often found in descriptions of viscous fluids. To represent their influence on collocation derivative matrices we use the common approach described in [46]. For that the functions u is represented as a product $u(r) = w(r)v(r)$, where both multipliers equal zero on the boundary and have the necessary number of bounded derivatives. Then $u' = w'v + wv' = w'(u/w) + w(u/w)'$ and the matrix of the first derivative which acts of the discrete analogue of the function u can be transformed to the following form

$$\widetilde{D}_1 = diag\left(\frac{w'(r_j)}{w(r_j)}\right) + diag(w(r_j))\widetilde{D}_1 diag\left(\frac{1}{w(r_j)}\right). \tag{108}$$

To construct the second derivative we use equality $u'' = w''v + 2w'v' + wv'' = w''(u/w) + 2w'(u/w)' + w(u/w)''$. From it we get the following:

$$\widetilde{D}_2 = diag\left(\frac{w''(r_j)}{w(r_j)}\right) + 2diag(w'(r_j))\widetilde{D}_1 diag\left(\frac{1}{w(r_j)}\right) + \\ + diag(w(r_j))\widetilde{D}_2 diag\left(\frac{1}{w(r_j)}\right). \tag{109}$$

By continuing to follow this principle we can construct derivatives of any order. As a function w we choose linear function $w(r) = r - 1$.

Of special interest are conditions on the left boundary (63). The mentioned above work [46] proposed a certain way of constructing matrices that approximate derivatives that are bounded in zero. For that the author offers to continue the function into the domain of negative values of the independent parameter r in an even way and then use the usual differentiating operator $\widetilde{D}_1$ on the symmetrical with respect to zero interval. Note that this approach demonstrates high accuracy for functions with sufficient smoothness after the even continuation. This approach with little change can be used for functions with sufficient smoothness after the odd continuation. In our case the asymptotic representation of eigenfunctions in zero (89)-(92) show that they cannot be continued in either way. Which means that the described method is not suitable for our purposes. However the same asymptotic representation allow us to get two variants of derivatives that take into account boundary conditions (63).

First variant (was first proposed in [29]). According to the acquired representations (89)-(92) there are following equivalences

$$1 = \lim_{r\to 0} \frac{u_1(r)}{r^l} = \lim_{r\to 0} \frac{u_2(r)}{r^m}$$

or

$$u_1(r) = r^l(1 + r\varphi_1(r)), \quad u_2(r) = r^m(1 + r\varphi_2(r)),$$

where $\varphi_i \neq 0$, $l = 1$, $m = 3$.

Note that functions r^l and r^m satisfy the Euler equation of the second order

$$Lq = 0, \text{ where } L = r^2 \frac{d^2}{dr^2} - (l + m - 1)r\frac{d}{dr} + lm = 0,$$

and substituting q with u_1 and u_2 gives us relative error in the form

$$\frac{Lu_i}{u_i} = r\Phi_i(r) \to 0, \text{ for } r \to 0.$$

Based on these facts we conclude that satisfying boundary conditions on the left boundary for the solution u of the problem (62)-(65) is equivalent to satisfying the following equality:

$$r^2 \frac{d^2u}{dr^2} - (l + m - 1)r\frac{du}{dr} + lmu = 0 \text{ for } r \to 0. \tag{110}$$

This equality is what we will be using as a boundary condition.

Second variant. To construct derivatives by the second variant we will note that due to the asymptotic representation solutions that satisfy boundary condition in zero differ from others in the way that them and all their derivatives are bounded in zero. To use this fact we modify the approach above to also take into account the Neumann right boundary condition.

If the function is bounded $|u(0)| < \infty$ then it can be represented as a product $u(r) = \frac{1}{r}w(r)$ where $w(r) = ru(r)$ and, obviously, it satisfies the homogeneous Dirichlet condition $w(0) = 0$. To approximate derivative of the function w we can use collocation matrix $\widetilde{D}_1$ that takes into account this simple condition. So for numerical differentiating of the function u we use the matrix

$$\widetilde{D}_1 = diag\left(\frac{-1}{r_k}\right) + diag\left(\frac{1}{r_k}\right)\widetilde{D}_1 diag(r_k),$$

since $u' = \left(\frac{1}{r}\right)'(ru) + \frac{1}{r}(ru)'$.

If the boundedness condition holds for derivatives we use equalities $u^{(j)} = \left(\frac{1}{r}\right)'(ru^{(j-1)}) + \frac{1}{r}(ru^{(j-1)})'$ to obtain a sequence of approximations of higher-order derivatives:

$$\widetilde{D}_j = \left[diag\left(\frac{-1}{r_k}\right) + diag\left(\frac{1}{r_k}\right)\widetilde{D}_1 diag(r_k)\right]\widetilde{D}_{j-1}. \tag{111}$$

Now we move to **constructing matrix beams**.

Matrix beams that approximate initial spectral problems are formed in different ways depending on which variant we have chosen for taking into account boundary conditions.

We consider the first variant first. We assume equation (110) to hold near zero and equation (62) to hold on the rest of the interval. Discretazing both differential operators and using matrices D_j (see, e.g. (108), (109)) which already satisfy homogenous Neumann and Dirichlet conditions gives us

$$\left[\widetilde{D}_2 - diag\frac{3}{r_k}\widetilde{D}_1 + diag\frac{3}{r_k^2}\right]u = 0,$$

$$\left[\widetilde{D}_5 + diag\frac{3}{r_k}\widetilde{D}_4 - diag\left(\frac{3}{r_k^2} + \tau\right)\widetilde{D}_3 + diag\left(\frac{6}{r_k^3} - \tau\frac{2}{r_k} + 2in^3\right)\widetilde{D}_2 + \right.$$
$$+ diag\left(\frac{-9}{r_k^4} + \tau\frac{1}{r_k^2} + n^2(\tau + n^2) + \frac{2in^3}{r_k}\right)\widetilde{D}_1 +$$
$$\left. + diag\left(\frac{9}{r_k^5} - \tau\frac{1}{r_k^3} + \tau\frac{n^2}{r_k}\right)\right]u = 0.$$

Note that in the first equation, that corresponds to the equation (110), there is no spectral parameter. Also it can be written as

$$(A_0 - \tau B_0)u = 0,$$

where

$$A_0 = \widetilde{D}_2 - diag\frac{3}{r_k}\widetilde{D}_1 + diag\frac{3}{r_k^2}, \quad B_0 = 0,$$

B_0 is zero matrix of size $(N-1)\times(N-1)$.

The second matrix equation, that corresponds to equation (62) can be written as

$$(A_1 - \tau B_1)u = 0,$$

where

$$A_1 = \widetilde{D}_5 + diag\frac{3}{r_k}\widetilde{D}_4 - diag\left(\frac{3}{r_k^2}\right)\widetilde{D}_3 + diag\left(\frac{6}{r_k^3} + 2in^3\right)\widetilde{D}_2 +$$
$$+ diag\left(\frac{-9}{r_k^4} + n^4 + \frac{2in^3}{r_k}\right)\widetilde{D}_1 + diag\left(\frac{9}{r_k^5}\right),$$

$$B_1 = \widetilde{D}_3 + diag\left(\frac{2}{r_k}\right)\widetilde{D}_2 - diag\left(\frac{1}{r_k^2} + n^2\right)\widetilde{D}_1 + diag\left(\frac{1}{r_k^3} - \frac{n^2}{r_k}\right).$$

Since the equality (110) has asymptotic character we can combine both matrix equalities:

$$A = \begin{pmatrix} \bar{A}_0 \\ \underline{A}_1 \end{pmatrix}, \quad B = \begin{pmatrix} \bar{B}_0 \\ \underline{B}_1 \end{pmatrix}, \tag{112}$$

where $\bar{A}_0$, $\bar{B}_0$ are first rows of matrices A_0, B_0 and $\underline{A}_1$, $\underline{B}_1$ are rows of matrices A_1, B_1 starting form second. Thus we indicate that in the closest to zero point r_1 the correlation (110) holds and the correlation (62) holds in all others. This gives us the matrix beam $A - \tau B$.

If we use the second variant of construrcting derivatives (111) then with the use of these matrices we form algebraic spectral problem for the matrix beam $\tilde{A} - \tau \tilde{B}$ where

$$\tilde{A} = \widetilde{D}_5 + diag\frac{3}{r_k}\widetilde{D}_4 - diag\left(\frac{3}{r_k^2}\right)\widetilde{D}_3 + diag\left(\frac{6}{r_k^3} + 2in^3\right)\widetilde{D}_2 + \\ + diag\left(\frac{-9}{r_k^4} + n^4 + \frac{2in^3}{r_k}\right)\widetilde{D}_1 + diag\left(\frac{9}{r_k^5}\right), \tag{113}$$

$$\tilde{B} = \widetilde{D}_3 + diag\left(\frac{2}{r_k}\right)\widetilde{D}_2 - diag\left(\frac{1}{r_k^2} + n^2\right)\widetilde{D}_1 + diag\left(\frac{1}{r_k^3} - \frac{n^2}{r_k}\right).$$

Now it's time to describe the technology for obtaining numerical results.

We use the function *eig* of the freely available packet Octave to the matrix beam defined in (112), (113) to calculate their eigenvalues τ_j and then solve a cubic equation (65) for each τ_j we found. The set of all such roots is the spectrum of the operator (62).

First things first, we note that the two proposed discretezations give similar results both for large scales and local structures (see fig. 2,3). This is also confirmed by calculations for different number of points N. This coincidence is a further confirmation of the idea that these discrete operators represent the structure of the spectrum of the initial differential operator adequately.

Next is the fact that for large values of Re, Wi and σ part of the spectrum is situated in the left half-plane and another part snuggles the imaginary axis. On fig. 4 we see the images of functions N_{re} which is the number of eigenvalues in the right half-plane and $M_{re} = \log_{10} \max(Re(\lambda_j))$ which is the decimal order of the maximum of the real part among all eigenvalues. The graphs show that with the growth of σ the number of eigenvalues in the right half-plane is conserved but the values of their real parts lower significantly. In this case both discretezations also demonstrate a high level of consistency.

Figures 5,6 show some eigenfunctions corresponding to eigenvalues with positive real parts. Among them there are smooth ones and both unimodal and oscillating ones (see graphs in the left part). Smooth functions correspond to eigenvalues with rather small real part 10^{-9}. Generally speaking the question about whether or not eigenvalues are in the right half-plane

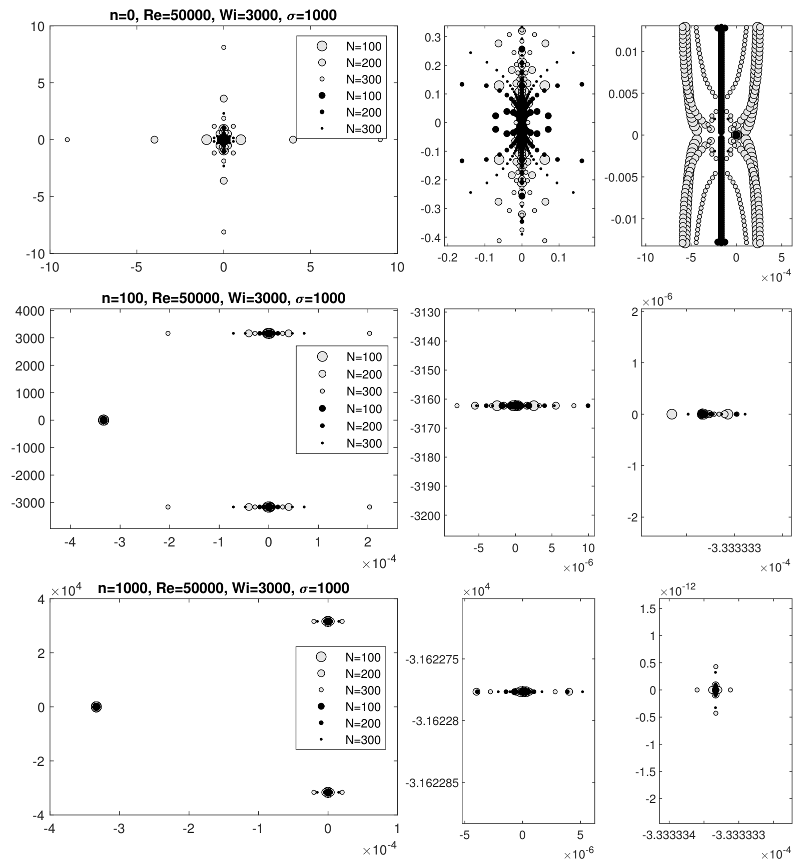

FIGURE 2. Numerical spectrums of beams (112) (black circles), (113) (light circles)

is still open because of the computational errors. The strict answer to it requires additional study.

We emphasize that the possible doubts in the accuracy of stated algebraic spectral problems in this case is valid. Since we deal with matrices of a rather high sizes and norms we can expect a significant accumulation of numerical errors. However the similarities between the computations from two different methods indirectly confirm the sufficient level of accuracy during the solution of the matrix spectral problem.

Some eigenfunctions are entirely saw-like (see bottom right graphs on fig. 5,6). As was already mentioned, the functions with this property usually correspond to "parasitic" eigenvalues that appear due to discretezation. Also among eigenfunctions there are some of the form that can be seen on the top right. They can be called quasisaw-like. This form of eigenfunctions means that the number of points we use for the discrete approximation of the differential operator is insufficient to describe a smooth but quickly oscillating function.

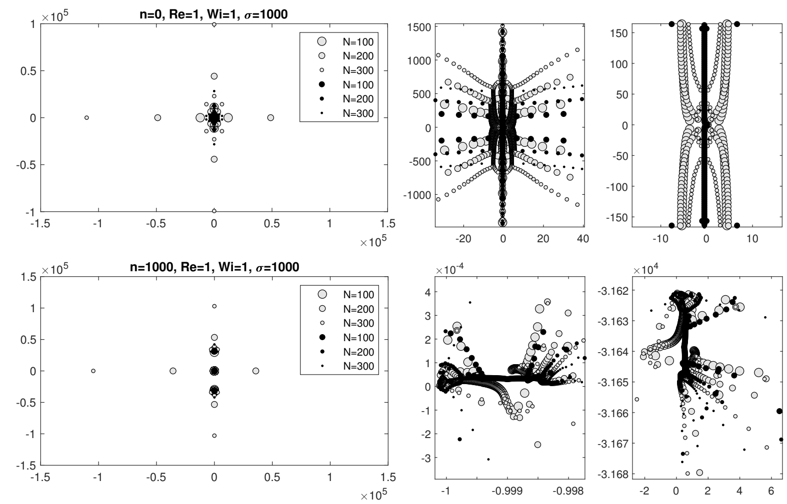

FIGURE 3. Numerical spectrums of beams (112) (black circles), (113) (light circles)

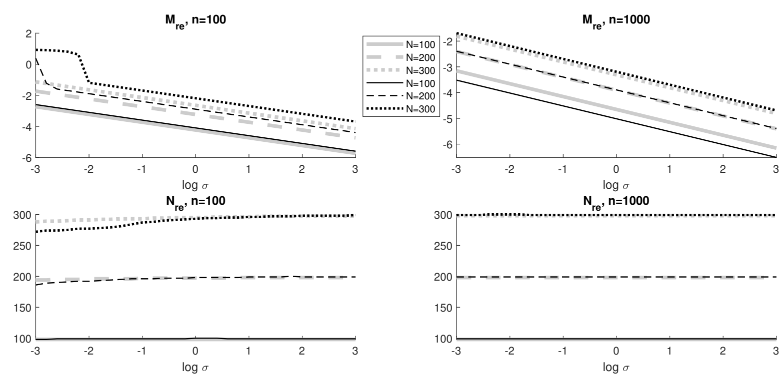

FIGURE 4. Orders of the maximal real part of eigenvalues $M_{re}(\sigma)$ and number of eigenvalues in the right half-plane $N_{re}(\sigma)$ for beams (112) (black lines), (113) (grey lines). $Re = 50000$, $Wi = 3000$, $n = 1000$

So as a result of numerical computations we have established that, generally speaking, the considered solutions are stable by Lyapunov. However for large values of Reynolds and Weisenberg numbers the exponent of the amplitude growth for the perturbations with high enough frequencies along the channel length can be suppressed to rather small values by increasing the magnetic pressure. Note that in the work [29] for the resting state without the magnetic field we showed that the velocity of the growth of amplitudes for similar perturbations grows to infinity with the growth of their frequency.

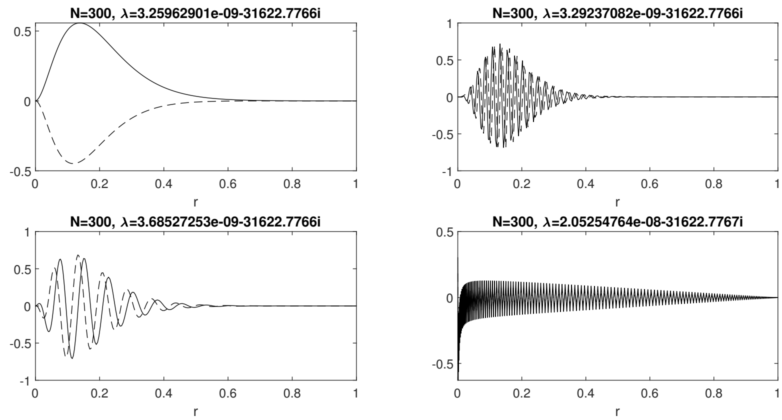

FIGURE 5. Some eigenfunctions of the beam (112) corresponding to eigenvalues in the right half-plane. Real part is smooth line, imaginary part is dashed line. $Re = 50000$, $Wi = 3000$, $n = 1000$

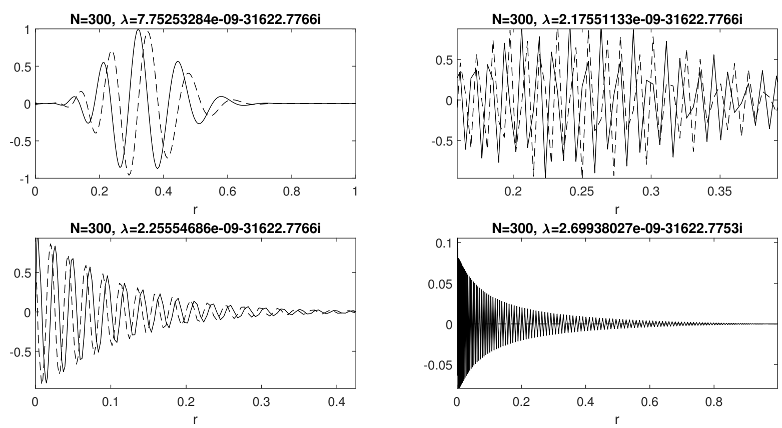

FIGURE 6. Some eigenfunctions of the beam (113) corresponding to eigenvalues in the right half-plane. Real part is smooth line, imaginary part is dashed line. $Re = 50000$, $Wi = 3000$, $n = 1000$

These results confirms the hypothesis that the magnetic field can be used as a stabilizing factor on the flows of polymeric fluid.

References

[1] V.N. Pokrovskii, *The mesoscopic theory of polymer dynamics*, Springer Ser. Chem. Phys., 95, Springer, Dordrecht, 2010.

[2] Yu.A. Altukhov, A.S. Gusev, G.V. Pishnograi, *Introduction into mesoscopic theory of flowing polymeric systems*, Alt. GPA, Barnaul, 2012 (in Russian).

[3] J.G. Oldroyd, *On the formulation of rheological equations of state*, Proc. R. Soc., **200** (1950), 523–541.

[4] R.B. Bird, P.J. Dotson, N.L. Johnson, *Polymer solution rheology based on a finitely extensible bead-spring chain model*, Journal of Non-Newtonian Fluid Mechanics, **7**: 2/3 (1980), 213–235.

[5] M.D. Chilcott, J.M. Ralliston, *Creeping flow of dilute polymer solutions past cylinders and spheres*, Journal of Non-Newtonian Fluid Mechanics, **29**: 3 (1988), 381–432.

[6] J. Remmelgas, G. Harrison, L.G. Leal, *A differential constitutive equation for entangled polymer solutions*, Journal of Non-Newtonian Fluid Mechanics, **80**:2 (1999), 115–134.

[7] V.N. Pokrovski, *Statistical mechanics of dilute suspension*, Nauka, Moscow, 1978.

[8] S.A. Zinovich, I. E. Golovicheva, G.V. Pyshnograi, *Influence of the molecular mass on shear and lateral viscosity of linear polymers*, Applied mechanics and technical physics, **200**:41 (2000), 154–160. (in Russian)

[9] G. Pyshnograi, D. Merzlikina, P. Filip, R. Pivokonsky, *Mesoscopic single and multi-mode rheological models for polymeric melts viscometric flow description*, WSEAS transactions on heat and mass transfer, **13** (2018), 49–65.

[10] A.M. Blokhin, D.L. Tkachev, *Linear asymptotic instability of a stationary flow of a polymeric medium in a plane channel in the case of periodic perturbations*, J. Appl. Ind. Math. **8**:4 (2014), 467–478.

[11] A.M. Blokhin, A.V. Yegitov, D.L. Tkachev, *Linear instability of solutions in a mathematical model describing polymer flows in an infinite channel*, Comput. Math. Math. Phys., **55**:5 (2015), 848–873.

[12] Alexander Blokhin, Dmitry Tkachev, *Spectral asymptotics of a linearized problem about flow of an incompressible polymeric fluid. Base flow is analogue of a Poiseuille flow*, AIP Conf. Proc., **2027** (2018), 030028.

[13] A.M. Blokhin, D.L. Tkachev, *Analogue of the Poiseuille flow for incompressible polymeric fluid with volume charge. Asymptotics of the linearized problem spectrum*, J. Phys. Conf. Ser., **894**:012096 (2017), 1–6.

[14] A.M. Blokhin, A.V. Yegitov, D.L. Tkachev, *Asymptotics of the spectrum of a linearized problem of the stability of a stationary flow of an incompressible polymeric fluid with a space charge*, Comput. Math. Math. Phys., **56**:1 (2018), 102–117.

[15] Alexander Blokhin, Dmitry Tkachev, Aleksey Yegitov, *Spectral asymptotics of a linearized problem for an incompressible weakly conducting polymeric fluid*, Z. Angrew. Math. Mech., **98**:4 (2018), 589–601.

[16] A.M. Blokhin, D.L. Tkachev, *Stability of Poiseuille-type flows for an MHD model of an incompressible polymeric fluid*, J. Hyperbolic Differ. Equ., **4**:4 (2019), 1–25.

[17] A.M. Blokhin, D.L. Tkachev, *Stability of the Poiseuille-type flow for a MHD model of an incompressible polymeric fluid*, Eur. J. Mech. B. Fluids **80** (2020), 112–121.

[18] A.M. Blokhin D.L. Tkachev, *Stability of Poiseuille-type flows for an MHD model of an incompressible polymeric fluid*, Fluid Dyn., **54**:8 (2019), 1051–1058.

[19] A.M. Blokhin, D.L. Tkachev, *Stability of Poiseuille-type flows in an MHD model of an incompressible polymeric fluid*, Sb. Math., **211**:7 (2020), 901–921.

[20] A.M. Blokhin, D.L. Tkachev, *MHD model of incompressible polymeric fluid. Linear instability of the resting state*, Complex Variables and Elliptic Equations, **66**:6-7 (2021), 929–944.

[21] A.M. Blokhin, D. Tkachev, *Linear instability of the resting state for the MHD model of an incompressible polymeric fluid*, AIP Conference Proceedings. 20th International Conference on the Methods of Aerophysical Research, ICMAR 2020, **2351** (2021), 040057.

[22] A.M. Blokhin, D.L. Tkachev, *On Linearly Unstable Steady States of an MHD Model of an Incompressible Polymeric Fluid in the Case of Absolute Conductivity*, Siberian Advances in Mathematics, **32** (2022), 1–12.

[23] A.M. Blokhin, A.Yu. Goldin, *On linear stability of an incompressible polymer liquid at rest*, J. Math. Sci., **230**:1 (2018), 14–24.

[24] D.L. Tkachev, *Spectrum and linear Lyapunov instability of a resting state for flows of an incompressible polymeric fluid*, J. Math. Anal. Appl., **522** (2023), 126914.

[25] D.L. Tkachev, *The spectrum and Lyapunov linear instability of the stationary state for polymer fluid flows: the Vinogradov-Pokrovski model*, Siberian Math. Jorunal, **64**:2 (2023), 407–423.

[26] G.I. Taylor, *Stability of a Viscous Liquid contained between Two Rotating Cylinders. Philosophical transactions of the Royal Society A*, **223**:605-615 (1923), 289–343.

[27] L.I. Sedov, *Mechanics of Continuous Medium. Vol. 1*, Moscow, Science, 1970 (in Russian).

[28] L.G. Loytsyanskiy, *Mechanics of Fluids and Gas*, Moscow, Science, 1978 (in Russian).

[29] D.L. Tkachev and E.A. Biberdorf, *Spectrum of a problem about the flow of a polymeric viscoelastic fluid in a cylindrical channel (Vinogradov-Pokrovski model)*, Siberian Electronic Mathematical Reports, **20**:2 (2023), 1269-1283.

[30] J. Hadamard, *Lectures on Cauchy's problem in linear partial differential equations*, New Have Yale University Press, 1923.

[31] S.K. Godunov, *Equations of mathematical physics (2nd ed.)*, Moscow, Science, 1979.

[32] A.B. Vatzhin, G.A. Lyubimov, S.A. Regirer, *Magnetohydrodynamic flows in channels*, Moscow, Science, 1970 (in Russian).

[33] A.I. Achiezer, I.A. Achiezer, *Electromagnetism and electromagnetic waves*, Moscow, Higher School, 1985 (in Russian).

[34] C. Nordling, J. Osterman, *Physics Handbook for Science and Engineering*, Professional Publishing House, 2006.

[35] L.D. Landau, E.M. Lifshitz, *Electrodynamics of continuous mediums*, Moscow, Physmathlit, 1959 (in Russian).

[36] M. Abramowitz, I. Stegun, *Handbook of Mathematical Functions*, NBS, 1964.

[37] P. Hartman, *Ordinary Differential Equations: Second Edition*, SIAM, Philadelphia, 1982.

[38] E.A. Coddington, N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, 1955.

[39] F.R. Gantmaher, *Matrix theory (5th ed.)*, Moscow, Physmathlit, 2010.

[40] G.A. Korn, T.M. Korn, *Handbook of mathematics for scientist and engineers. Definitions, theorems, formulas (5th ed.)*, Moscow, Science, 1984.

[41] S.K. Godunov, V.T. Zhukov, O.B. Feodoritova, *A method for calculating invariant subspaces of symmetric hyperbolic equations*, Comput. Math. and Math. Phys., **46**, (2006), 971–982.

[42] Shih-I Pai, *Introduction to the Theory of Compressible Flow*, Princeton: D. Van Nostrand Co., 1959.

[43] A.M. Blokhin, A.S. Rudometova, *Stationary flows of a weakly conducting incompressible polymeric liquid between coaxial cylinders*, Journal of Applied and Industrial Mathematics, **11**:4 (2017), 486-493.

[44] A.M. Blokhin, R.E. Semenko, A.S. Rudometova, *Magnetohydrodynamic Vortex Motion of an Incompressible Polymeric Fluid*, Journal of Applied and Industrial Mathematics, **15**:(1), 7–16.

[45] E. Kamke, *Handbook on Ordinary Differential Equations*, Moscow, Nauka, 1976 (in Russian).
[46] L.N. Trefethen, *Spectral Methods in MATLAB*, SIAM, Philadelphia, 2000.

Dmitry Leonidovich Tkachev
Sobolev Institute of Mathematics,
pr. Koptyuga, 4,
630090, Novosibirsk, Russia
Email address: tkachev@math.nsc.ru

Elina Arnoldovna Biberdorf
Sobolev Institute of Mathematics,
pr. Koptyuga, 4,
630090, Novosibirsk, Russia
Email address: ermolova@math.nsc.ru