WEAKLY PERIODIC MATRICES OVER ALGEBRAICALLY CLOSED FIELDS

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Abstract. We explore the situation where all square $n \times n$ matrices over an algebraically closed field $\mathbb{F}$ are weakly periodic of index of nilpotence 2 and prove that this can be happen uniquely when $\mathbb{F}$ is a countably infinite field, which is a countable union of its finite subfields of order greater than n. In particular, in the commuting case of weak periodicity, we show that such a decomposition is absolutely impossible.

Our obtained by us results somewhat generalize those obtained by Breaz-Modoi in Lin. Algebra & Appl. (2016).

1. Introduction and Principalities

Let $\mathbb{F}$ be an arbitrary field and n an arbitrary non-negative integer, say $n \geq 1$. We denote by $\mathrm{M}_n(\mathbb{F})$ the matrix ring consisting of all square $n \times n$ matrices over $\mathbb{F}$.

The matrix $M \in \mathrm{M}_n(\mathbb{F})$ is called *potent* if there exists an integer $k \geq 1$ such that $M^{k+1} = M$. So, let ST_n be the set of traces of all potent companion matrices $C \in \mathrm{M}_n(\mathbb{F})$.

Let R be a ring. An element $x \in R$ is said to be *potent* if there exists a non-negative integer $q \geq 1$ with $x^{q+1} = a$, and $x \in R$ is said to be *weakly periodic with nilpotence index* 2, provided x is the sum of a potent and a square-zero nilpotent of R. Furthermore, we shall say that the ring R is *weakly periodic with nilpotence index* 2, provided every element of R is weakly periodic with nilpotence index 2.

Historically, the concept of weak periodicity arisen quite normally in the existing on the subject literature. In fact, it was showed in [3] that an element x of a ring R is *periodic*, i.e., $x^n = x^m$ for some two different positive integers m, n, if and only if x can be written as a sum of a potent element and a nilpotent element which commute each other. Thus, by removing the "commuting property", it is rather natural to consider the sum of such two elements. In addition, a ring R is called *weakly periodic* if all its elements are weakly periodic.

On the other hand, Diesl defined in [5] the notion of a *nil-clean* ring R as the ring for which, for every $a \in R$, there are an idempotent e and a nilpotent b such that $a = e+b$. Moreover, Ye introduced in [8] the notion of *semi-clean* ring R as the ring for which, for each $a \in R$, there are a potent c and a unit u such that $a = c + u$. Henceforth, it is pretty obvious that the weakly periodic rings are properly situated between nil-clean and semi-clean rings. Indeed, there is a nil-clean ring (and thus a weakly periodic ring) that is *not* periodic.

So, they really need a detailed exploration in the matrix case, being our basic motivation, to which is devoted the present article. Our major purpose here is to decide what is the power structure of the base field, provided that it is algebraically closed, when all square matrices over it are either arbitrary weakly periodic with

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index of nilpotence 2 or just commuting weakly periodic with index of nilpotence 2, respectively. In what follows, we shall show that such a field is either countably infinite or that such a matrix decomposition is quite impossible, respectively (see, respectively, Theorems 3.7 and 3.13).

2. Sums of Roots of Unity

Throughout this section, let $\mathbb{F}$ be an algebraically closed field of arbitrary characteristic. By a root of unity, we denote a root of the polynomial over $\mathbb{F}$ which is of the form $g = X^i - 1$, where i is an integer with $i \geq 2$. Let SR_n be the set of sums of at most n roots of unity, which are not necessarily distinct.

For a non-negative integer $m \geq 2$, we denote by τ the cycle $\tau_m = (1\ 2\ \dots\ m) \in S_m$, and by P_{τ_m} the $m \times m$ permutation matrix with only 0s and 1s over $\mathbb{F}$, that is, $P_{\tau_m} = (a_{ij})_{1 \leq i,j \leq m}$, where $(a_{ij}) = 1$ if $j = \tau_m(i)$, and $(a_{ij}) = 0$ if $j \neq \tau_m(i)$. Likewise, we denote by $L_{m,n}$ the set of polynomials having degree at most m and with non-negative integer multiples of unity coefficients such that their sum is no more than n.

We now need the following two technicalities.

Lemma 2.1. *Let $n \geq 1$ be a non-negative integer and $t \in SR_n$. Then, there exists an integer $m > 1$ and $f \in L_{m,n}$ such that*

$$t \in Spec(f(P_{\tau_m})).$$

Proof. Letting $t \in SR_n$, then there exist non-zero integers $1 \leq k \leq n$ and non-negative integers $\alpha_1, \alpha_2, \dots \alpha_k$ with sum no more than n, non-negative integers $m_1, m_2, \dots, m_k$, as well as there are pairwise distinct roots of unity $\lambda_1, \lambda_2, \dots \lambda_k$, such that $\lambda_1^{m_1} = \lambda_2^{m_2} = \dots = \lambda_k^{m_k} = 1$ and such that

$$t = \alpha_1\lambda_1 + \alpha_2\lambda_1 + \dots \alpha_k\lambda_k.$$

If we take m to be the common multiple of $m_1, m_2, \dots, m_k$, then $\lambda_i^m = 1$ for every $i \in \{1, 2, \dots, k\}$. Consequently, $k \leq m$ and

$$\{\lambda_1, \lambda_2, \dots, \lambda_k\} \subseteq \{\epsilon^{a_1}, \epsilon^{a_2}, \dots \epsilon^{a_k}\},$$

where $\{a_1, a_2, \dots, a_k\} \subseteq \{0, 1, \dots, m-1\}$ and ϵ is so that it generates all m-th roots of unity. Without loss of generality, we can assume that $a_1 < a_2 < \dots < a_k$. Therefore,

$$t = \alpha_1\epsilon^{a_1} + \alpha_2\epsilon^{a_2} + \dots + \alpha_k\epsilon^{a_k}.$$

Suppose now that

$$f = \alpha_1X^{a_1} + \alpha_2X^{a_2} + \dots + \alpha_kX^{a_k}.$$

Since $\alpha_1, \alpha_2, \dots, \alpha_k$ are non-negative integers with sum no more than n, it will follow that $f \in L_{m,n}$. We also will have the equality

$$t = f(\epsilon).$$

Since ϵ is a primitive m-th root of unity, one extracts that ϵ is an eigenvalue for P_{τ_m}, and hence $f(\epsilon)$ is an eigenvalue of $f(P_{\tau_m})$. Therefore, $t \in Spec(f(P_{\tau_m}))$, as promised. □

Lemma 2.2. *Let $m \geq 2$ and $n \geq 1$ be integers. If $f \in L_{m,n}$, then, for $t \in Spec(f(P_{\tau_m}))$, there exist two integer multiples of unity $u = u(t)$ and $\alpha = \alpha(t)$ such that $(t - \alpha)^m = u$.*

Proof. Let$f \in L_{m,n}$, and $t \in Spec(f(P_{\tau_m}))$. It follows now that t is a root of the polynomial $det(XI_m - f(P_{\tau_m}))$. But as $f \in L_{m,n}$, there exist $1 \leq k \leq m$ and non-zero integers $\alpha_1, \alpha_2, \ldots, \alpha_k$ and non-negative pairwise distinct integers $a_1, a_2, \ldots, a_k$ with sum at most n and $\max a_1, a_2, \ldots, a_k \leq m$, such that

$$f = \alpha_1 X^{a_1} + \alpha_2 X^{a_2} + \ldots + \alpha_k X^{a_k}.$$

Therefore,

$$f(P_{\tau_m}) = \alpha_1 (P_{\tau_m})^{a_1} + \alpha_2 (P_{\tau_m})^{a_2} + \ldots + \alpha_k (P_{\tau_m})^{a_k}.$$

Since τ_m is a cycle in S_m, and $\tau_m = (1\ 2 \ldots\ m)$, it follows that ${\tau_m}^{a_{i_1}}(j) \neq {\tau_m}^{a_{i_2}}(j)$ for every $i_1, i_2, j \in \{1, 2 \ldots, m\}$, with $i_1 \neq i_2$. It can also be derived that τ_m is a product of $m-1$ transpositions, and hence τ_m and m obviously have different parities.

Furthermore, assuming m is even, it follows then that τ_m is odd, and thus $\tau_m^{a_i}$ and a_i have the same parity for every $i \in \{1, 2, \ldots, k\}$. Now, let

$$(b_{ij})_{1 \leq i,j \leq m} := XI_m - f(P_{\tau_m})$$

and $\alpha_i = f(0)$. Therefore,

$$det(f(P_{\tau_m}) - XI_m) = (\alpha_i - X)^m + \sum_{\sigma \neq e, \sigma \ \ even, \sigma = \tau_m^{a_i}, i \in \{1,2,\ldots,k\}} b_{1\sigma(1)} \cdots b_{m\sigma(m)} -$$

$$\sum_{\sigma odd, \sigma = \tau_m^{a_i}, i \in \{1,2,\ldots,k\}} b_{1\sigma(1)} \cdots b_{m\sigma(m)} =$$

$$= (\alpha_i - X)^m + (-1)^{a_1} \alpha_1^m + (-1)^{a_2} \alpha_2^m + \ldots + (-1)^{a_k} \alpha_k^m - (-1)^{a_i} \alpha_i^m.$$

Consequently, for even m, we obtain:

$$(\alpha_i - t)^m = (-1)^{a_1+1} \alpha_1^m + (-1)^{a_2+1} \alpha_2^m + \ldots + (-1)^{a_k+1} \alpha_k^m - (-1)^{a_i+1} \alpha_i^m.$$

Assuming that m is now odd, we then infer that τ_m is even, and so $\tau_m^{a_i}$ is even for each $i \in \{1, 2, \ldots, k\}$. Now, let

$$(b_{ij})_{1 \leq i,j \leq m} := f(P_{\tau_m}) - XI_m$$

and $\alpha_i = f(0)$. Therefore,

$$det(f(P_{\tau_m}) - XI_m) = (\alpha_i - X)^m + \sum_{\sigma \neq e, \sigma = \tau_m^{a_i}, i \in \{1,2,\ldots,k\}} b_{1\sigma(1)} \cdots b_{m\sigma(m)} =$$

$$= (\alpha_i - X)^m + \alpha_1^m + \alpha_2^m + \ldots + \alpha_k^m - \alpha_i^m.$$

Consequently, for odd m, we conclude:

$$(t - \alpha_i)^m = \alpha_1^m + \alpha_2^m + \ldots + \alpha_k^m - \alpha_i^m.$$

Finally, there exist multiple integers of unity, say $\alpha = \alpha_i$ and u, such that $(t - \alpha)^m = u$, as required. □

The next example is worthy of including in order to explain the situation more detailed.

Example 2.3. Let $\mathbb{F}$ be an algebraically closed field which has characteristic $p > 0$. Let $l > 0$ be an integer, and n, m integers with $n \geq 1, m > 1$ such that $p^l \leq n - 1$ and $GF(p^l) \subseteq \mathbb{F}$.

Also, let $x \in \mathbb{F}$ such that $x^m - 1 = 0$. Observe that there exist exactly $d = \gcd(p^l - 1, m)$ pairwise distinct values of x in $GF(p^l)$, namely $1, h, h^2, \ldots, h^{d-1}$, for

$h = g^{\frac{p^l-1}{d}}$, where g is a generator of the multiplicative cyclic group of the field $GF(p^l)$. Since $h^d = 1$, we have

$$1 + h + h^2 + \ldots + h^{d-1} = 0.$$

Therefore,

$$-1 = h + h^2 + \ldots + h^{d-1}.$$

So, -1 is the sum of $d-1$ m-th roots of unity with each of them not equal to 1. Since $d - 1 < d \leq p^l - 1 \leq n - 2$, we detect that

$$-1 = \alpha_1 h + \alpha_2 h^2 + \ldots + \alpha_{d-1} h^{d-1}$$

with

$$\alpha_1 = \alpha_2 = \ldots = \alpha_{d-1} = 1,$$

and hence it follows from the proof of Lemma 2.2 that

$$(-1-0)^m = \alpha_1^m + \alpha_2^m + \ldots + \alpha_{d-1}^m - 0.$$

Thus, $(-1)^m = d - 1$, as expected. This substantiates our statement.

We continue our work with the following simple but useful claim.

Proposition 2.4. *Let $n > 0$ be an integer, and let $\mathbb{F}$ be an algebraically closed field of prime characteristic p with $n < p$. Then, every non-zero sum of at most n roots of unity of $\mathbb{F}$ is again a root of unity.*

Proof. Let t be a non-zero sum of at most n roots of unity. According to Lemma 2.1, we get that there exists $f \in L_{m,n}$ such that $t \in Spec(f(P_{\tau_m}))$, whereas Lemma 2.2 enables us that if $t \in Spec(f(P_{\tau_m}))$, then there exist $u \in \mathbb{F}_p$ and $\alpha \in \mathbb{F}_p$ such that $(t-\alpha)^m = u$. Likewise, the proofs of the mentioned two lemmas alluded to above tell us that if t is the sum of at most n m-th roots of unity within there are exactly α values of 1, then $(t-\alpha)^m \in \mathbb{F}_p$. Now, in this in mind, we may assume that $t \neq \alpha$. Since $t \neq 0$ is the sum of at most n roots of unity within there are no values of 1, then $(t-\alpha)^m \in \mathbb{F}_p - \{0\}$. If, however, $\alpha \neq 1$, then t is the sum of two roots of unity not equal to 1, i.e., $t = \alpha + y$ with $y^{m(p-1)} = 1$. So,

$$(t-0)^{m(p-1)} \in \mathbb{F}_p - \{0\}$$

and, therefore, $t^{m(p-1)^2} = 1$. Thus, t is a root of unity.

Now, take $\alpha = 1$. But, as

$$t = 1 \cdot 1 + y =$$
$$= (p-1)\cdot(-1) + y = ((n)\cdot(-1) + y) + (p-n-1)\cdot(-1),$$

we have that $(n)(-1) + y$ is the sum of n roots of unity equal to (-1) and one $m(p-1) - th$ root of unity not equal to 1. It follows now that

$$((n)\cdot(-1) + y - 1\cdot 0)^{2m(p-1)} = 1.$$

Moreover, if

$$(p-n-1)\cdot(-1) = 1,$$

then p obviously divides n. However, we have the restriction $n \leq p-1$, so that we deduce $p \leq p-1$ which is absolutely false. With this contradiction at hand, we derive that t is the sum of two $2m(p-1)$-th roots of unity not equal to 1. Consequently,

$$(t-0)^{2m(p-1)} \in \mathbb{F}_p - \{0\},$$

whence $t^{2m(p-1)^2} = 1$. Finally, t is a root of unity, as desired. □

3. Main Results

We divide our basic results into two subsections as follows:

3.1. **The general case.**

We begin our work with the following plain, but helpful technicality.

Lemma 3.1. *Let $n \geq 1$ be an integer and let $\mathbb{F}$ be an algebraically closed field. Then, any trace of a potent $n \times n$ companion matrix over $\mathbb{F}$ is a sum of at most n roots of unity.*

Proof. Let t be the trace of a potent n by n companion matrix C over $\mathbb{F}$ such that $C^{m+1} = C$ for some integer $m \geq 1$. Therefore, C is similar to the Jordan canonical form $J_{\lambda_1} \oplus J_{\lambda_2} \oplus \dots J_{\lambda_k}$, where $Spec(C) = \{\lambda_1, \lambda_2, \dots, \lambda_k\}$ and $J_{\lambda_i} = J_{\lambda_i}^{m+1}$ is a Jordan cell corresponding to the eigenvalue λ_i, $i \in \{1, 2, \dots, k\}$. Hence, one derives that $\lambda_i^{m+1} = \lambda_i$ and, therefore, λ_i is either zero, or λ_i is an m-th root of unity of $\mathbb{F}$.

But, since similar matrices have the same trace, it follows that there exist positive integers $\alpha_1, \alpha_2, \dots, \alpha_k$ with sum n, where J_{λ_i} is an $\alpha_i \times \alpha_i$ matrix, such that

$$t = \alpha_1\lambda_1 + \alpha_2\lambda_2 + \dots + \alpha_k\lambda_k.$$

Consequently, t is a sum of at most n m-th roots of unity, as pursued. □

We now proceed with a few pivotal for our work statements.

Proposition 3.2. *Let $n \geq 1$ be a non-negative integer and $C \in \mathrm{M}_n(\mathbb{F})$ a companion matrix over an algebraically closed field $\mathbb{F}$. Then, the following two claims are true:*

(1) *If C is weakly periodic with nilpotence index 2, then* trace $C \in SR_n$.
(2) *If the* trace C *is a trace of a potent companion matrix, then C is weakly periodic with nilpotence index 2.*

Proof. (1) Assume $C = M + N$ with M potent and N a square-zero nilpotent such that trace $C \notin SR_n$. So, it is obvious that trace $C \notin ST_n$. Since trace $N = 0$, it follows that trace $C =$ trace M. Thus, one deduces that M is not similar to a companion matrix. We shall now consider the next two cases:

Case 1: M is similar to a direct sum of two companion matrices C_1 of size l and C_2 of size $n - l$, so that there exists an invertible matrix of size n over $\mathbb{F}$ with $M = U(C_1 \oplus C_2)U^{-1}$. Hence,

$$\text{trace } M = \text{trace } C = \text{trace } C_1 + \text{trace } C_2.$$

Since the direct sum $C_1 \oplus C_2$ is obviously potent, one sees that trace $C_1 \in ST_l$ and trace $C_2 \in ST_{n-l}$. By virtue of Lemma 3.1, one gets that trace $C_1 \in SR_l$ and trace $C_2 \in SR_{n-l}$. Therefore,

$$\text{trace } C_1 + \text{trace } C_2 \in SR_n,$$

and hence trace $M \in SR_n$. But trace $C =$ trace M, and so we obtain that trace $C \in SR_n$.

Case 2: M is similar to a direct sum of u companion matrices, where $3 \leq u \leq n$. It can easily be seen that with induction on u we will have trace $C \in SR_n$.

(2) Since trace C is a trace of a potent companion matrix, we get that there is a potent companion matrix C' such that trace $C' =$ trace C, whence $(C - C')^2 = 0$. In conclusion, C is weakly periodic with nilpotence index 2, as claimed. □

Proposition 3.3. *Let $n \geq 1$ be an integer. If all $n \times n$ companion matrices over an algebraically closed $\mathbb{F}$ are weakly periodic with nilpotence index 2, then $\mathbb{F} \subseteq SR_n$.*

Proof. This follows immediately with the aid of Proposition 3.2 by simple induction on n. □

The next assertion is closely related to assertions from [4].

Lemma 3.4. *Every diagonalizable matrix over the finite Galois field $GF(p^l)$, for some $l \geq 1$, with no multiple eigenvalues is a non-derogatory potent matrix.*

Proof. Since every element $x \in GF(p^l)$ satisfies the equation $x^{p^l} = x$, it follows that each diagonal over $GF(p^l)$ is potent. Hence, any diagonalizable matrix over $GF(p^l)$ is necessarily potent as well.

Moreover, since the matrix has no multiple eigenvalues, the algebraic multiplicity of every eigenvalue is exactly 1. And since the matrix is diagonalizable, the geometric multiplicity of any eigenvalue equals to the algebraic multiplicity, so that it will be precisely 1 too. Therefore, the matrix is non-derogatory as well. In conclusion, the matrix is a non-derogatory potent matrix, as stated. □

The following technical claim from field theory is our key to establish the chief result formulated below. Although it is somewhat a folklore, we shall provide a full proof for the readers convenience and completeness of the exposition (for a more detailed information see, for instance, [7]).

Proposition 3.5. *Each infinite field, which is an algebraic extension of its minimal simple (finite) subfield, is an infinite countable union of its finite subfields, and thus it is countable. In addition, these finite subfields can be taken of the sort $GF(p^{l_i})$ such that $GF(p^{l_i})$ is contained in $GF(p^{l_{i+1}})$ and l_i divides l_{i+1} for each $i \geq 1$.*

Proof. Given $\mathbb{F}$ is an infinite field of characteristic $p > 0$, which is an algebraic extension of its simple subfield $\mathbb{F}_p$ of p elements, and u is an arbitrary element of $\mathbb{F}$. Hence, each $\mathbb{F}_p(u)$ which means the extension of $\mathbb{F}_p$ generated by u, is a finite extension of $\mathbb{F}_p$ and so it is a finite subfield of $\mathbb{F}$. Precisely, all finite subfields of $\mathbb{F}$ are of this type, because for the finite extensions of $\mathbb{F}_p$ is valid the classical theorem for the "primitive element". We also know that

$$\mathbb{F}_p(u) \cong F_p[X]/(g_u(X)),$$

where X is a transcendental element over $\mathbb{F}_p$, and $g_u(X)$ is the minimal polynomial of u over $\mathbb{F}_p$. To ensure an uniqueness of $g_u(X)$, as usual we will assume that it is normed, that is, its chief coefficient is exactly 1. Also, the degree $[\mathbb{F}_p(u) : \mathbb{F}_p]$ of the extension $\mathbb{F}_p(u)/\mathbb{F}_p$ coincides with the degree of the polynomial $g_u(X)$.

Furthermore, the extensions $\mathbb{F}_p(u)/\mathbb{F}_p$ are normal with cyclic Galois group. Moreover, for every positive integer n, at most (exactly) one field of the form $\mathbb{F}_p(u)$ is an extension of the field $\mathbb{F}_p$ of degree n. In other words, there is a bijection between the set of all subfields of $\mathbb{F}$ of the kind $\mathbb{F}_p(u)$ onto some subset of the set $\mathbb{N}$ consisting of all natural numbers. Concretely, this subset consists of those elements of $\mathbb{N}$ which are equal to the degree $[\mathbb{F}_p(u) : \mathbb{F}_p]$ for some element $u \in \mathbb{F}$. Since any subset of $\mathbb{N}$ is either finite or infinitely countable, our arguments so far allow us to conclude that $\mathbb{F}$ can be presented as a finite or countably infinite union of finite (sub)fields. This union is properly countable uniquely when $\mathbb{F}$ is an infinitely countable field, as pursued.

Next, by what we have already shown, the base field $\mathbb{F}$ is countable, and hence the elements of $\mathbb{F}$ can be linearly ordered as $f_1, \dots, f_n, \dots$. Set $\mathbb{F}_n = \mathbb{F}_p(f_1, \dots, f_n)$ for every $n \in \mathbb{N}$. It is now easily inspected that $\mathbb{F}$ is equal to the countable union of all its subfields $\mathbb{F}_n$, where $n \geq 1$. Besides, it easily follows that $\mathbb{F}_n/\mathbb{F}_p$ is a finite extension, because it is simultaneously an algebraic extension and finitely generated. Likewise, $\mathbb{F}_n$ is obviously a subfield of $\mathbb{F}_{n+1}$ for each natural $n \geq 1$. Thus, we arrive at the conclusion that $\mathbb{F}_n$ is a finite field of order p^{l_n}, where $l_n = [\mathbb{F}_n : \mathbb{F}_p]$. Since $\mathbb{F}_n \leq \mathbb{F}_{n+1}$, one deduces that l_{n+1} is divided by l_n and that the equality

$$l_{n+1}/l_n = [\mathbb{F}_{n+1} : \mathbb{F}_n]$$

holds for all $n \in \mathbb{N}$, as desired.

As for the second part that such a field is necessarily is now an immediate consequence of the first one. □

The next comments are worthwhile to explain that the conditions in the previous statement cannot be weakened.

Remark 3.6. Knowing that $\mathbb{F}_p$ is the simple (finite) field of prime characteristic p, routine arguments show that the field $\mathbb{F}_p(X)$ of rational functions of the variable X with coefficients from $\mathbb{F}_p$, which is actually a transcendental extension of $\mathbb{F}_p$, is an example of a countable field of characteristic p which *cannot* be constructed as a countable union of finite fields. Moreover, arguing in the same manner, one can see that the field of rational numbers, $\mathbb{Q}$, also cannot be presented as a countable union of finite fields.

We now arrive at our first main result in the present paper. Specifically, the following is true:

Theorem 3.7. *Let $n \geq 1$ be an integer and let $\mathbb{F}$ be an algebraically closed field. Then, the following two conditions are equivalent:*

(1) *Every $n \times n$ matrix over $\mathbb{F}$ is the sum of a potent and a square-zero nilpotent over $\mathbb{F}$.*
(2) *$\mathbb{F}$ is a countably infinite field, which is a countable union of its finite subfields of order greater than n.*

Proof. (1) $\implies$ (2). Assume that each $n \times n$ matrix C over $\mathbb{F}$ is weakly periodic with nilpotence index 2. Therefore each $n \times n$ companion matrix $\mathbb{F}$ is weakly periodic with nilpotence index 2. Referring to Proposition 3.2, we have that $t \in \mathbb{F}$, the trace of C is also in SR_n. Likewise, by Proposition 2.1, we have that there exists an integer $m > 1$ and $f \in L_{m,n}$ such that $t \in Spec(f(P(\tau_m))$. Therefore, there exist two integer multiples of unity, say α and u, such that $(t - \alpha)^m = u$ holds owing to Lemma 2.2.

Assume now, in a way of contradiction, that $\mathbb{F}$ has zero characteristic. It then follows that each non-zero integer multiple of unity s is invertible. Set $t = r \cdot s^{-1}$. Thus, there exist two integer multiples of unity α and u such that $(r \cdot s^{-1} - \alpha)^m = u$. However, applying the classical Newton's binomial formula, we find that

$$(r \cdot s^{-1})^m + (-1)^1 \binom{m}{1} (r \cdot s^{-1})^{m-1} \alpha + (-1)^2 \binom{m}{2} (r \cdot s^{-1})^{m-2} \alpha^2 + \dots$$

$$+(-1)^{m-1} \binom{m}{m-1} (r \cdot s^{-1})^1 \alpha^{m-1} + \alpha^m = u$$

and so

$$r^m + (-1)^1\binom{m}{1}r^{m-1}s\alpha + (-1)^2\binom{m}{2}r^{m-2}s^2\alpha^2 + \\ \ldots + (-1)^{m-1}r^1s^{m-1}\alpha^{m-1} + \alpha^m \cdot s^m = u \cdot s^m,$$

which is equivalent to

$$r^m = ((-1)^0\binom{m}{1}r^{m-1} \cdot \alpha + (-1)^1\binom{m}{2}r^{m-2}\alpha^2 s + \\ \ldots + (-1)^{m-2}r\alpha^{m-1}s^{m-2} + \alpha^m \cdot s^{m-1} + us^{m-1}) \cdot s.$$

As by assumption the field is of characteristic zero, we may consider with no harm of generality that the above equation is satisfied in $\mathbb{Z}$, whence it follows that s divides r^m for any $s \in \mathbb{Z}^*$, $r \in \mathbb{Z}$, which is manifestly false; for example, by considering $s = 2$ and $r = 3$. Hence, the characteristic of the field has to be some non-zero prime, say $p > 0$. Besides, Proposition 3.3 ensures that the field is of necessity countable, because SR_n is countable and $\mathbb{F} \subseteq SR_n$.

We now consider $GF(p^l)$ with $GF(p^l) \subseteq \mathbb{F}$. Assume $p^l \leq n$, $2^t \neq p^l - 1$, for any non-negative integer t, and p an odd prime. Thus, there exists an odd proper divisor d of $p^l - 1$. Further, take q a non-negative odd integer such that q and $p^l - 1$ are co-prime. Then, $m = d \cdot q$ is odd and $d = \gcd(p^l - 1, m)$. We are, in fact, in the framework of Example 2.3. So, $(-1)^m = d - 1$. Since m is odd, we derive that p divides d, but d divides $p^l - 1$. The obtained contradiction that d divides 1 leads us to the existence of the non-negative integer t such that $2^t = p^l - 1$, or $p = 2$, or $p^l > n$.

Assume $2^t = p^l - 1$ and $p \neq 2$ for a non-negative integer t. It routinely follows now that $p - 1$ and

$$r = p^{l-1} + p^{l-2} + \ldots + p + 1$$

are powers of 2 and $p-1$ divides r. But since $p-1$ divides $r-l$, we receive that $p-1$ divides l. Since $p \neq 2$, we deduce that $l = 2u$ with u a positive integer. That is why, $(p^u - 1)(p^u + 1) = 2^t$ and hence $p^u - 1$ divides $p^u + 1$. Finally, $p^u - 1$ divides 2 and since $p \neq 2$, we obtain $p = 3$ and $u = 1$. Thus, $l = 2u = 2$ and $GF(p^l) = GF(3^2)$, as required.

Therefore, either $GF(p^l) = GF(3^2)$, or $p = 2$, or $p^l > n$.

Now, in order to demonstrate that $\mathbb{F}$ is an algebraic extension of its minimal simple (finite) subfield, we simply use Proposition 3.3 (from which we know $\mathbb{F} \subseteq SR_n$) combined with Lemma 2.4 (which states that every non-zero sum of at most n roots of unity is a root of unity). However, with the aid of Proposition 3.5, all of this means that $\mathbb{F}$ is a countable union of its finite subfields, as needed.

Assume further that $GF(p^l) = GF(3^2)$. Since 3^2 is the only value of p^l such that $GF(p^l) \subseteq \mathbb{F}$, and since $\mathbb{F}$ is a countable union of its finite subfields, it follows that $\mathbb{F} = GF(3^2)$, which is impossible as $\mathbb{F}$ is an algebraically closed field.

Assume now $p = 2$. If we take, in the framework of Lemma 2.3, $p = 2$ and $m = 2(2^l - 1)$, then one sees that $d = 2^l - 1 = \gcd(2^l - 1, 2(2^l - 1))$ as there we have obtained $(-1)^m = d - 1$. But, in $GF(2^l)$, we have $-1 = 1$. Consequently, $1 = d + 1$ and so d is even. Hence, 2^l is odd and $l = 0$. However, since $l = 0$ is the only integer such that $GF(2^l) \subseteq \mathbb{F}$, and since $\mathbb{F}$ is a countable union of its finite subfields, it follows that $\mathbb{F} = \mathbb{F}_2$, which is a contradiction, because $\mathbb{F}$ is an algebraically closed field.

Therefore, one infers that $p^l > n$ and $\mathbb{F}$ is a countable union of its finite subfields of order greater than n, as required.

(2) $\Longrightarrow$ (1). Let $GF(p^l)$ be contained in $\mathbb{F}$. If $p = 2$, the conclusion follows from [4, Corollary 3.2]. Take C_1 to be an $n \times n$ companion matrix over $GF(p^l)$. Then, as $p^l > n$, it follows by application of the main result from [4] that we may decompose $C_1 = D_1 + N_1$, where D_1 is a diagonalizable matrix over $GF(p^l)$ with no multiple eigenvalues, and N_1 is a nilpotent matrix of nilpotence index 2 over $GF(p^l)$. Now, Lemma 3.4 helps us to have that D_1 is a non-derogatory potent matrix, and, as trace $N = 0$, we can get that trace $C_1 =$ trace D_1. Therefore, any element in $GF(p^l)$ can be the trace of a non-derogatory potent matrix. Hence, each element in $GF(p^l)$ can be the trace of a potent companion matrix.

Furthermore, since $\mathbb{F}$ is a union of finite fields, it follows at once that every element in $\mathbb{F}$ can be the trace of an $n \times n$ potent companion matrix over $\mathbb{F}$. Thus, letting C be an arbitrary $n \times n$ companion matrix over $\mathbb{F}$, we then deduce that trace C is a trace of a potent $n \times n$ companion matrix. Therefore, applying Proposition 3.2(2), one infers that any $n \times n$ companion can be written as a sum of a potent matrix and a nilpotent matrix of order 2 over $\mathbb{F}$. Then all 1×1 companion matrices can be written as a sum of a potent matrix and a nilpotent matrix of order 2 over $\mathbb{F}$. Now take k a natural number, $k \geq 1$ and assume that all companion matrices of order less than or equal to k can be written as a sum of a potent matrix and a nilpotent matrix of order 2 over $\mathbb{F}$. Take A to be a $(k+1) \times (k+1)$ matrix. If A is not similar to a companion matrix then the companion matrices in the rational canonical form are of order less than or equal to k, By the induction hypothesis they have the desired decomposition, which means that the matrix A, which is similar to its rational canonical form has our decomposition. Now, if A is similar to a companion matrix, then it has the desired decomposition since we have proved that for any $n \geq 1$ any $n \times n$ companion matrix can be written as a sum of a potent matrix and a nilpotent matrix of order 2 over $\mathbb{F}$. Therefore, due to a simple induction, the conclusion is achieved. □

As an explicit example for 2×2 matrices over non-algebraically closed fields, possibly over a finite field; e.g., the field $\mathbb{F}_2$, we can give the decompositions:

- For companion matrices with the trace 1, we have:

$$\begin{pmatrix} 0 & a \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}.$$

- For companion matrices with the trace 0, we have:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

and, respectively,

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We now come to the case when the potent and square-zero matrices commute each other.

3.2. **The commuting case.** We will attack here the commuting case of weakly periodic with nilpotence index 2 companion matrices, that is, the existing potent and square-zero nilpotent matrices commute each other. To that aim, we first need the following two technical conventions.

Lemma 3.8. *Let n be a positive integer and $\mathbb{F}$ an algebraically closed field. If, for every $n \times n$ companion matrix C over $\mathbb{F}$, there exist an integer $t > 1$, a potent matrix P such that $P = P^t$ and a square-zero nilpotent N such that $C = P + N$ with $PN = NP$, then the following two points are true:*

1. *If C is invertible, then χ_C divides $(X^{t-1} - 1)^2$.*
2. *If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are non-zero elements of $\mathbb{F}$, then there exists an integer $t > 1$ such that $\lambda_i^{t-1} = 1$ for every $i \in \{1, 2, \ldots, n\}$.*

Proof. (1) Since $C = P + N$ and $PN = NP$, it follows that $CP = PC$ and $CN = NC$. But we have $(C - N)^t = C - N$. Since $CN = NC$, we can apply Newton's binomial formula to argue that there exists an $n \times n$ matrix M over $\mathbb{F}$ such that

$$C^t - tCN + MN^2 = C - N.$$

But as $N^2 = 0$, we obtain that

$$C(C^{t-1} - tN) = C - N.$$

Multiplying with $C^{t-1} + tN$, we get that

$$C(C^{2t-2} - t^2N^2) = C^t + tCN - C^{t-1}N + tN^2,$$

and using that $N^2 = 0$, we write the equality

$$C^{2t-1} = C^t + (tC - C^{t-1})N.$$

But C is invertible, and so

$$C^{2t-2} - C^{t-1} = (tI_n - C^{t-2})N.$$

Now, bearing in mind that $CN = NC$ and $N^2 = 0$, we infer

$$((C^{t-1})^2 - C^{t-1})^2 = 0,$$

and since C is invertible, we conclude

$$(C^{t-1} - 1)^2 = 0.$$

Therefore, the minimal polynomial of C divides $(X^{t-1} - 1)^2$. But the minimal polynomial of C is the characteristic polynomial, say χ_C, of C. Summarizing all the information so far, χ_C divides $(X^{t-1} - 1)^2$, as promised.

(2) Just take C to be the companion matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and apply the preceding point (1) along with the fact that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are from $\mathbb{F}$. □

Lemma 3.9. *Let n be a positive integer and $\mathbb{F}$ an algebraically closed field. If, for every $n \times n$ companion matrix C over $\mathbb{F}$, there exist an integer $t > 1$, a potent matrix P such that $P = P^t$ and a square zero nilpotent N such that $C = P + N$ with $PN = NP$, then the following two conditions are valid:*

1. *If C is invertible, then there exists a polynomial $q \in \mathbb{F}[X]$ such that χ_C divides $(q(X) - X)^2$.*
2. *If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are non-zero elements of $\mathbb{F}$, then there exists a polynomial $q \in \mathbb{F}[X]$ such that $q(\lambda_i) = \lambda_i$ for every $i \in \{1, 2, \ldots, n\}$.*

Proof. (1) Since $C = P + N$ and $PN = NP$, it follows that $CP = PC$ and $CN = NC$. It is well known that all matrices that commute with a companion matrix C can be interpreted just as polynomials in C over $\mathbb{F}$. So, there exists $q \in \mathbb{F}[X]$ with $C = q(C) + N$. Now, since $N^2 = 0$, it follows that $(q(C) - C)^2 = 0$. Therefore, the minimal polynomial of C obviously divides $(q(X) - X)^2$. But the minimal polynomial of C is the characteristic polynomial, say χ_C, of C. Summarizing all the information thus far, χ_C divides $(q(X) - X)^2$, as asked for.

(2) Just take C to be the companion matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and next employ the preceding condition (1) together with the fact that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are from $\mathbb{F}$. □

The following claim from number theory is a well-known folklore fact, but nevertheless we state it here only for the sake of completeness and the reader's convenience.

Lemma 3.10. *Let a, b, c be three positive integers. Then,* $\gcd(bc, a)$ *divides* $\gcd(b, a) \cdot \gcd(c, a)$.

Proof. Let we set $f = \gcd(a, bc)$, $f_1 = \gcd(a, b)$ and $f_2 = \gcd(a, c)$. Then, there exist integers s_1, s_2, t_1, t_2 such that

$$f_1 = s_1 a + t_1 b$$

and

$$f_2 = s_2 a + t_2 c.$$

Thus,

$$f_1 f_2 = s_1 s_2 a^2 + s_1 t_2 ac + t_1 s_2 ba + t_1 t_2 bc.$$

But f divides a and f divides bc, so f divides a^2, f divides ac, f divides ba and f divides bc. Hence, f divides $f_1 f_2$ whence $\gcd(bc, a)$ divides $\gcd(b, a) \cdot \gcd(c, a)$, and the claim sustained. □

The next comments are needed to explain the complicated situation in the commuting case.

Remark 3.11. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be distinct non-zero elements of $\mathbb{F}$. By Lemma 3.9(2), there exists $q \in \mathbb{F}[X]$ such that $q(\lambda_i) = \lambda_i$ for every $i \in \{1, 2, \ldots, n\}$. Take $GF(p^{l_1})$ such that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are in $GF(p^{l_1})$, and such that $q \in GF(p^{l_2})[X]$ with $l = \max(l_1, l_2)$. Then, $\lambda_1, \lambda_2, \ldots, \lambda_n$ are in $GF(p^l)$, while $q \in GF(p^l)[X]$.

Furthermore, since $\lambda_1, \lambda_2, \ldots, \lambda_n$ are in $\mathbb{F}$, it follows from Lemma 3.8(2) that there exists a non-negative integer $t > 1$ such that $\lambda_i^{t-1} = 1$ for every $i \in \{1, 2, \ldots, n\}$. Now, since $\lambda_1, \lambda_2, \ldots, \lambda_n$ are in $GF(p^l)$, we can infer that

$$\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \subseteq \{h, h^2, \ldots, h^d = 1\},$$

where $h = g^{\frac{p^l - 1}{d}}$ with g a generator of the multiplicative group of $GF(p^l)$, and $d = \gcd(p^l - 1, t - 1)$. Therefore, $n \leq d$ and there exist

$$\{j_1, j_2, \ldots, j_n\} \subseteq \{1, 2, \ldots, d\}$$

such that $\lambda_i = h^{j_i}$. Finally, we extract that $q(h^{j_i}) = h^{j_i}$ for every $i \in \{1, 2, \ldots, n\}$ with $h^d = 1$, as expected.

The next example illustrates some concrete aspects of our calculating manipulations.

Example 3.12. If in the previous Remark 3.11 we take $\lambda_1 = g^i$ and $\lambda_2 = g^{i+1}$ with $i \in \{1, 2, \dots, p^l - 2\}$, then one inspects that

$$\{g^i, g^{i+1}\} \subseteq \{g^{\frac{p^l-1}{d}}, g^{2\frac{p^l-1}{d}}, \dots, g^{d\frac{p^l-1}{d}}\}.$$

Therefore,

$$\{i, i+1\} \subseteq \{\frac{p^l-1}{d}, 2\frac{p^l-1}{d}, \dots, d\frac{p^l-1}{d}\}.$$

Hence, the ordinary fraction $\frac{p^l-1}{d}$ is a common divisor of both i and $i+1$, so it is necessarily 1. We thus have now that $p^l - 1 = d = \gcd(p^l - 1, t - 1)$. In conclusion, $p^l - 1$ divides $t - 1$, as desired to demonstrate.

We are now ready to establish our second main result treating the commuting case.

Theorem 3.13. *Suppose $n \geq 2$ is an integer and $\mathbb{F}$ is an algebraically closed field. Then, there is a $n \times n$ matrix over $\mathbb{F}$ that is not the sum of a potent matrix and a square-zero matrix over $\mathbb{F}$ which matrices commute each other.*

Proof. Suppose $n \geq 1$ is an integer, and assume the contrary that all $n \times n$ matrices over $\mathbb{F}$ are the sum of a potent matrix and a square-zero matrix over $\mathbb{F}$ which commute. So, thanks to Theorem 3.7, we have that $\mathbb{F}$ is an infinitely countable field of odd characteristic, which is a countable union of its finite subfields of order greater than n, with odd characteristic. Thus, the second part in the statement of Proposition 3.5 applies to write that $\mathbb{F} = \cup_{i=1}^{\infty} GF(p^{l_i})$, where l_i divides l_{i+1} for every positive integer i. Therefore, there exists the sequence of integers greater than 1, say $(c_i)_{i\geq 1}$ such that $l_{i+1} = l_i \cdot c_i$ and $(l_i)_{i\geq 1}$ is a strictly increasing infinite sequence of positive integers.

Take g_1 to be the generator of the multiplicative group of $GF(p^{l_1})$ and put

$$\lambda_1 = g_1^{[\frac{p^{l_1}-1}{n}]}, \lambda_2 = g_1^{2\cdot[\frac{p^{l_1}-1}{n}]}, \dots, \lambda_n = g_1^{n\cdot[\frac{p^{l_1}-1}{n}]}.$$

Consequently, Lemma 3.8 allows us to have the existence of a positive integer $t > 1$ such that

$$\lambda_1^{t-1} = \lambda_2^{t-1} = \cdots = \lambda_n^{t-1} = 1.$$

Let C be the companion matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Note that the integer t above has the property that $C = P + N$ with $P^t = P$, $N^2 = 0$ and $PN = NP$. If, eventually, there are more than one such decompositions of C, then we can take t to be the minimum of such integers t's. So, t can be chosen to be a fixed uniquely determined integer having the mentioned property.

Furthermore, since we are working in $GF(p^{l_1})$, Remark 3.11 leads us to the relation

$$\{g_1^{[\frac{p^{l_1}-1}{n}]}, g_1^{2\cdot[\frac{p^{l_1}-1}{n}]}, \dots, g_1^{n\cdot[\frac{p^{l_1}-1}{n}]}\} \subseteq \{g_1^{\frac{p^{l_1}-1}{d_1}}, g_1^{2\frac{p^{l_1}-1}{d_1}}, \dots, g_1^{d_1\frac{p^{l_1}-1}{d_1}}\},$$

where $d_i = \gcd(p^{l_i} - 1, t - 1)$. Take $j_i \in \{1, 2, \dots, d_i\}$ such that

$$g_1^{[\frac{p^{l_1}-1}{n}]} = g_1^{j_1\cdot\frac{p^{l_1}-1}{d_1}},$$

and since the order of g_1 in the multiplicative group of $GF(p^{l_1})$ is $p^{l_1} - 1$, it follows that

$$[\frac{p^{l_1}-1}{n}] = j_1 \cdot \frac{p^{l_1}-1}{d_1}.$$

Analogously, we obtain that

$$[\frac{p^{l_i}-1}{n}] = j_i \cdot \frac{p^{l_i}-1}{d_i},$$

for any positive integer i. Therefore,

$$j_i \cdot \frac{p^{l_i}-1}{d_i} = j_{i+1} \cdot \frac{p^{l_{i+1}}-1}{d_{i+1}},$$

and so

$$\frac{j_i}{j_{i+1}} = \frac{d_i}{d_{i+1}} \cdot \frac{(p^{l_i})^{c_i}-1}{p^{l_i}-1}.$$

Take $s_i = (p^{l_i})^{c_i-1} + (p^{l_i})^{c_i-2} + \cdots + p^{l_i} + 1$. It thus follows that

$$\frac{j_i}{j_{i+1}} = \frac{\gcd(p^{l_i}-1, t-1)}{\gcd((p^{l_i})^{c_i}-1, t-1)} \cdot s_i.$$

Also, by Lemma 3.10 we have that there exists a strictly positive integer k_i such that

$$\gcd((p^{l_i})^{c_i}-1, t-1) = \frac{\gcd(p^{l_i}-1, t-1) \cdot \gcd(s_i, t-1))}{k_i}.$$

Now, we obtain that

$$\frac{j_i}{j_{i+1}} = k_i \cdot \frac{s_i}{\gcd(s_i, t-1)}.$$

However, since $\gcd(s_i, t-1)$ divides s_i, it follows that j_{i+1} divides j_i for every positive integer $i \geq 1$. But $d_i \geq j_i \geq 1$ and thus there exists $i_0 \geq 1$ such that $j_i = j_{i+1}$ for every $i \geq i_0$. So, we obtain

$$k_i \cdot \frac{s_i}{\gcd(s_i, t-1)} = 1,$$

which forces $s_i = \gcd(s_i, t-1)$. Hence, s_i divides $t-1$ and so $s_i \leq t-1$. But $p^{l_i} < s_i$ and then $p^{l_i} < t-1$, for every $i \geq i_0$, which is in sharp contradiction with the fact that $(l_i)_{i\geq 1}$ is a strictly increasing infinite sequence of positive integers. In conclusion, one has that $\mathbb{F}$ is finite, which is an absurd. This substantiates our claim after all. □

In the spirit of the last result, we pose the following conjecture.

Conjecture: Given $n \geq 1$ is an integer and $\mathbb{F}$ is a field. Then all companion $n \times n$ matrices over $\mathbb{F}$ are the sum of a potent matrix and a square-zero matrix over $\mathbb{F}$ which matrices commute each other if, and only if, $\mathbb{F}$ is a finite (and hence potent) field and $n = 1$.

On the other side, in regard to [1] and [6], we close our work with the following question of some interest and importance for the further development of the explored subject.

Problem 3.14. *Suppose that D is a division ring and $n \geq 1$ is an integer. Does it follow that the matrix ring $\mathrm{M}_n(D)$ is weakly periodic if, and only if, D is a finite (and hence potent) field?*

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