TOWARD PSEUDO-CONJUGATION ACTIONS OF GROUPS

M. BAHRAMI-TAGHANAKI AND A. R. MOGHADDAMFAR

ABSTRACT. We continue our study of a specific group action which is called pseudo-conjugation action (PCA). This article aims to explore the properties of PCAs and various algebraic structures associated with them.

1. Introduction, Basic Definitions and Notation

An action of a finite group[1] G on itself, which is neither trivial nor conjugation, is called *pseudo-conjugation* if, for all $x \in G$, $x^x = x$. We denote by PCA the class of all groups that admit a pseudo-conjugation action. More recently, these groups have been studied in [1, 2].

In the sequel, we assume that the group G is a PCA-group. For any two elements $x, y \in G$, we have $x^{-1}x^y = (y^{-1})^x y$ (see [1, Proposition 2.3]) and we write $[\![x, y]\!]$ to denote this element, that is $[\![x, y]\!] = x^{-1}x^y = (y^{-1})^x y$. Note that $[\![x, y]\!] = 1$ if and only if $x^y = x$ if and only if $y^x = y$. In this situation, we say that x and y *stabilize* each other.

Also, we recall that the *orbit* of $x \in G$, denoted by x^G, is $x^G = \{x^g \mid g \in G\} \subseteq G$. Let $\kappa(G)$ denotes the number of distinct orbits of G. The *stabilizer* of $x \in G$, denoted by G_x, is the subgroup $G_x = \{g \in G \mid [\![x, g]\!] = 1\}$ of G. The Orbit-Stabilizer theorem says that the cardinality of the orbit x^G is equal to the index of the stabilizer G_x in G, i.e., $|x^G| = |G : G_x|$. The *fixed point set* of $T \subseteq G$ is defined to be

$$\mathsf{Fix}(T) = \{x \in G \mid [\![x, t]\!] = 1, \forall t \in T\}.$$

A set $T \subseteq G$ is said to be *stabilizing* if $\mathsf{Fix}(T) = T$. Notice that $\mathsf{Fix}(G) \neq G$, since a pseudo-conjugation action is always nontrivial; indeed $\mathsf{Fix}(G)$ is a proper normal subgroup of G which is properly contained in every stabilizer G_x of G (see [2, Facts 2 and 3]).

Let SS denote the class of all PCA-groups G in which, for every $z \in G \backslash \mathsf{Fix}(G)$, the stabilizer G_z is stabilizing, that is, for all $x, y \in G_z$, $[\![x, y]\!] = 1$. If G is an SS-group and $x, y \in G \backslash \mathsf{Fix}(G)$, then G_x can never be properly contained in G_y. Furthermore, we know from [1, Corollary 7.3] that G being an SS-group is equivalent to saying for all x and y in $G \backslash \mathsf{Fix}(G)$ either $G_x = G_y$ or $G_x \cap G_y = \mathsf{Fix}(G)$.

In this paper, we focus our attention on particular PCA-groups. We will study the various families of PCA-groups, such as $\mathsf{SS}_{\min}$-groups, F-groups, and $\mathrm{F}_{\min}$-groups, which will be defined later. We also associate a simple graph with each PCA-group,

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[1]Only *finite* groups will be considered in this investigation; so we shall use the term 'group' always in the sense of 'finite group'.

which is called a stabilizing graph. Among other results, an upper bound for the clique number of this graph is found based on the cardinalities of its orbits.

For the most part, the notation and terminology are fairly standard and introduced as needed. Undefined terms may be found in [6, 7]. Moreover, for a class $\mathfrak{X}$ of groups, we refer to the groups which belong to $\mathfrak{X}$ as $\mathfrak{X}$-groups. Let $\Gamma = (V_\Gamma, E_\Gamma)$ be a graph with vertex set V_Γ and edge set E_Γ. The degree $d_\Gamma(v)$ of a vertex $v \in V_\Gamma$ is the number of edges incident with x. An independent set in Γ is a collection of vertices no two of which are joined by an edge in Γ. The maximum cardinality of an independent set in Γ is called the *independence number* of Γ and is denoted by $\alpha(\Gamma)$. A *complete set* of Γ is a subset of pairwise adjacent vertices. A *clique* is a maximal complete set. The maximum cardinality of a clique in Γ is called the *clique number* of Γ and is denoted by $\omega(\Gamma)$.

2. Minimal and Maximal Stabilizers

We begin this section with the following definition.

Definition 2.1. Let G be a PCA-group. The stabilizer G_x for $x \in G\backslash\mathsf{Fix}(G)$ is called a *minimal stabilizer* (*in* G) if, $G_y \leqslant G_x$ implies $G_y = G_x$. Let $\mathsf{SS}_{\min}$ denote the set of groups G in PCA such that $G_x \cap G_y = \mathsf{Fix}(G)$ for any two distinct minimal stabilizers G_x and G_y.

Note that any PCA-group G must have at least two minimal stabilizers. Otherwise if G_x is the unique minimal stabilizer in G, then for all $y \in G$, we have that $G_x \leqslant G_y$. Therefore $x \in \bigcap_{y \in G} G_y = \mathsf{Fix}(G)$, a contradiction.

Proposition 2.2. *Let G be an $\mathsf{SS}_{\min}$-group, then each minimal stabilizer is stabilizing.*

Proof. Consider a minimal stabilizer G_z for $z \in G\backslash\mathsf{Fix}(G)$ and suppose that x is an arbitrary element in $G_z\backslash\mathsf{Fix}(G)$. We must show that $[\![x, y]\!] = 1$ for all $y \in G_z$. If this is false, choose $y \in G_z\backslash\mathsf{Fix}(G)$ with $[\![x, y]\!] \neq 1$, and consider a minimal stabilizer $G_t \leqslant G_x$. Since $y \in G_z\backslash G_x$, the minimal stabilizers G_t and G_z are distinct, and so $G_t \cap G_z = \mathsf{Fix}(G)$. Moreover, since $t \in G_t \leqslant G_x$, we conclude that $x \in G_t$. Therefore, $x \in G_t \cap G_z$, or equivalently, $x \in \mathsf{Fix}(G)$, and hence $[\![x, y]\!] = 1$, which is a contradiction. □

Let G be a non-cyclic group. A *covering* of G is a collection $\mathscr{C} = \{H_1, \ldots, H_n\}$ of proper subgroups of G with $n \geqslant 3$ and $H_1 \cup \cdots \cup H_n = G$. The covering $\mathscr{C}$ is called a *partition* of G if $H_i \cap H_j = 1$ whenever $i \neq j$.

Proposition 2.3. *Let G be an $\mathsf{SS}_{\min}$-group. Then*

$$\mathscr{P} = \{G_t/\mathsf{Fix}(G) \mid G_t \text{ a minimal stabilizer in } G\}$$

forms a partition of $G/\mathsf{Fix}(G)$ consisting of stabilizing subgroups.

Proof. It is clear that the set $\mathscr{P}$ is closed under pseudo-conjugation action and by Proposition 2.2 every subgroup in $\mathscr{P}$ is stabilizing. Thus to prove that $\mathscr{P}$ is a partition, we need to show that every element in $G/\mathsf{Fix}(G)$ lies in a unique subgroup in $\mathscr{P}$.

Take $G_x/\mathsf{Fix}(G)$ and $G_y/\mathsf{Fix}(G)$ distinct in $\mathscr{P}$. Then

$$G_x/\mathsf{Fix}(G) \cap G_y/\mathsf{Fix}(G) = (G_x \cap G_y)/\mathsf{Fix}(G) = 1.$$

Thus it is enough to show that any $x\mathsf{Fix}(G)$ lies in some $G_t/\mathsf{Fix}(G)$, where G_t is a minimal stabilizer in G. Consider the stabilizer G_x which contains some minimal stabilizer G_t in G. Then, since $t \in G_t \leqslant G_x$, $x \in G_t$ and so $x\mathsf{Fix}(G) \in G_t/\mathsf{Fix}(G)$. □

Note that we can also consider the maximal stabilizers.

Definition 2.4. Let G be a PCA-group. The stabilizer G_x for $x \in G\backslash\mathsf{Fix}(G)$ is called a *maximal stabilizer* if $G_x < G_y$ implies $G_y = G$.

3. F-groups

Definition 3.1. A PCA-group G is an F-group if for any $x \in G\backslash\mathsf{Fix}(G)$, $G_x < G_y$ implies $y \in \mathsf{Fix}(G)$. In other words, a PCA-group G is an F-group if and only if for any element $x \in G\backslash\mathsf{Fix}(G)$, G_x is both a maximal and minimal stabilizer in G.

The following corollary follows from Proposition 2.2.

Corollary 3.2. *A group G is an SS-group if and only if G is an F-group and an $\mathsf{SS}_{\min}$-group.*

Recall that by definition, for each $x \in G$, we have

$$\mathrm{Fix}(G_x) = \{y \in G \mid [\![y, t]\!] = 1, \forall t \in G_x\}.$$

It is easy to check that $\mathrm{Fix}(G_x)$ is a subgroup of G_x, and in particular, we have

$$\mathrm{Fix}(G) \leqslant \mathrm{Fix}(G_x) \leqslant G_x \leqslant G.$$

The subsequent result provides a condition that is equivalent to a group being an F-group.

Proposition 3.3. *A group G is an F-group if and only if for any pair of elements $x, y \in G\backslash\mathsf{Fix}(G)$ such that $G_x \neq G_y$, then $\mathsf{Fix}(G_x) \cap \mathsf{Fix}(G_y) = \mathsf{Fix}(G)$.*

Proof. Suppose that G is an F-group and let $G_x \neq G_y$ for x and y in $G\backslash\mathsf{Fix}(G)$. If $z \in \mathsf{Fix}(G_x) \cap \mathsf{Fix}(G_y)$, then $\langle G_x, G_y\rangle \leqslant G_z$. Hence as in an F-group every stabilizer is both maximal and minimal, it follows that $G_z = G$, or equivalently, $z \in \mathsf{Fix}(G)$. Therefore $\mathsf{Fix}(G_x) \cap \mathsf{Fix}(G_y) = \mathsf{Fix}(G)$.

Conversely, assume that $G_x < G_y$ for $x \in G\backslash\mathsf{Fix}(G)$. Then $\mathsf{Fix}(G_y) \leqslant \mathsf{Fix}(G_x)$ and therefore $\mathsf{Fix}(G_y) = \mathsf{Fix}(G)$, which implies that $y \in \mathsf{Fix}(G)$. □

We are now providing a definition as follows.

Definition 3.4. Let $\mathrm{F}_{\min}$ denote the set of PCA-groups G such that

$$\mathsf{Fix}(G_x) \cap \mathsf{Fix}(G_y) = \mathsf{Fix}(G),$$

for any two distinct minimal stabilizers G_x and G_y.

Proposition 3.5. *Let G be an $\mathrm{F}_{\min}$-group. Then*

$$\mathscr{P} = \{\mathsf{Fix}(G_t)/\mathsf{Fix}(G) \mid G_t \text{ a minimal stabilizer in } G\},$$

forms a non-trivial partition of $G/\mathsf{Fix}(G)$ consisting of stabilizing subgroups.

Proof. It is clear that the set $\mathscr{P}$ is closed under pseudo-conjugation action and every subgroup in $\mathscr{P}$ is stabilizing. Thus to show that $\mathscr{P}$ is a partition we need to show that every element in $G/\mathsf{Fix}(G)$ lies in a unique subgroup in $\mathscr{P}$.

Take $\mathsf{Fix}(G_x)/\mathsf{Fix}(G)$ and $\mathsf{Fix}(G_y)/\mathsf{Fix}(G)$ distinct in $\mathscr{P}$. Then

$$\mathsf{Fix}(G_x)/\mathsf{Fix}(G) \cap \mathsf{Fix}(G_y)/\mathsf{Fix}(G) = (\mathsf{Fix}(G_x) \cap \mathsf{Fix}(G_y))/\mathsf{Fix}(G) = 1.$$

Thus it is enough to show that any $x\mathsf{Fix}(G)$ lies in some $\mathsf{Fix}(G_t)/\mathsf{Fix}(G)$, where G_t is a minimal stabilizer in G. Consider the stabilizer G_x which contains some minimal stabilizer G_t. Then $x \in \mathsf{Fix}(G_x) \leqslant \mathsf{Fix}(G_t)$ and so $x\mathsf{Fix}(G) \in \mathsf{Fix}(G_t)/\mathsf{Fix}(G)$. □

4. Stabilizing Graphs

Given a PCA-group G and a subset $\varnothing \neq X \subseteq G$, the stabilizing graph $\Delta(X)$ is constructed as follows [2]. The vertex set of $\Delta(X)$ is X, and two distinct vertices x, y of $\Delta(X)$ are joined by an edge if and only if $[\![x, y]\!] = 1$. We will write $\alpha(X)$ to denote the independence number of $\Delta(X)$. In fact, $\alpha(X)$ denotes the maximum cardinality of any set of pairwise non-stabilizing elements of X. We will also write $\omega(X)$ to denote the clique number of $\Delta(X)$.

If $F = \mathsf{Fix}(G)$ is non-trivial, and $G = \bigsqcup x_iF$ is a coset decomposition of G, then $G = \bigcup\langle x_iF\rangle$ is a covering of G by stabilizing subgroups. Let $s(G)$ be the minimum number of stabilizing subgroups in any such collection whose union equals G. The pigeon-hole principle and our previous discussion give

$$\alpha(G) \leqslant s(G) \leqslant |G : \mathsf{Fix}(G)|.$$

Before continuing, we need the following auxiliary lemma which gives a lower bound on the independence number of a graph in terms of the cardinalities of its vertex and edge sets.

Lemma 4.1. [4, Corollary 2.1] *Let Γ be a simple graph on n vertices and m edges. Then*

$$\alpha(\Gamma) \geqslant n^2/(n + 2m),$$

with equality if and only if Γ is a disjoint union of cliques of the same cardinality.

The next result gives a lower bound to independence number of $\Delta(G)$.

Proposition 4.2. $\alpha(G) \geqslant |G|/\kappa(G)$

Proof. Let $\Gamma = \Delta(G)$. Reasoning as in [2, Proposition 3.1], we have

$$2|E_\Gamma| = \sum_{x\in G} d_\Gamma(x) = \sum_{x\in G}(|G_x| - 1) = \left(\sum_{i=1}^{\kappa(G)} |x^G||G_x|\right) - |G| = |G|(\kappa(G) - 1).$$

It follows by Lemma 4.1, therefore, that

$$\alpha(G) \geqslant |G|^2/(|G| + 2|E_\Gamma|) = |G|/\kappa(G),$$

as required. □

The proof of the next result, which is based on a direct group-theoretic approach, is exactly the same as the one given by I. M. Isaacs in [4, Theorem] for conjugation actions. Define a function $f(n)$ inductively by

$$f(1) = 1 \text{ and } f(n) = n + \binom{n}{2} f(n-1).$$

Theorem 4.3 (I. M. Isaacs)**.** *With the notation above, we have $s(G) \leqslant f(\alpha(G))$.*

Proof. Put $\alpha = \alpha(G)$. If $x, y \in G$ with $[\![x, y]\!] \neq 1$ and $c_1, c_2, \ldots, c_\alpha \in G_x \cap G_y$, then two of the elements $x, c_1y, c_2y, \ldots, c_\alpha y$ must stabilize. Since

$$[\![x, c_iy]\!] = [\![x, y]\!][\![x, c_i]\!]^y = [\![x, y]\!] \neq 1,$$

two of the latter must stabilize and thus two of the c_i must stabilize. Hence, we deduce that $\alpha(G_x \cap G_y) < \alpha(G)$, whenever $[\![x, y]\!] \neq 1$.

Let $x_1, x_2, \dots, x_\alpha$ be pairwise non-stabilizing elements and let $B_{jk} = G_{x_j} \cap G_{x_k}$ for $j \neq k$. It follows from the result of the previous paragraph that $\alpha(B_{jk}) < \alpha(G)$, so working by induction on α (and using the fact that f is monotone) we conclude that B_{jk} is the union of at most $f(\alpha - 1)$ stabilizing subgroups.

Let $A_j = G_{x_j} \backslash \bigcup_{k \neq j} B_{jk}$. Since

$$G = \bigcup_{1 \leqslant j \leqslant \alpha} G_{x_j} = \bigcup_{1 \leqslant j \leqslant \alpha} \langle A_j \rangle \cup \bigcup_{j,k} B_{jk},$$

once we show that each $\langle A_j \rangle$ is stabilizing, we have proved that

$$s(G) \leqslant \alpha + \binom{\alpha}{2} f(\alpha - 1) = f(\alpha(G)).$$

To show each $\langle A_j \rangle$ is stabilizing, let $u, v \in A_j$. Then $x_1, \dots, x_{j-1}, u, x_{j+1}, \dots, x_\alpha$, are α pairwise non-stabilizing elements, so v must stabilize with one of them. Then v stabilizes with u, and $\langle A_j \rangle$ is stabilizing. □

The next lemma shows that a clique in the stabilizing graph $\Delta(G)$ corresponds to a maximal stabilizing subgroup of G and vice versa. We omit its straightforward proof.

Lemma 4.4. *Let G be a* PCA*-group and $C \subseteq G$ is a subset. Then C is a clique in $\Delta(G)$ if and only if C is a maximal stabilizing subgroup of G.*

Recall that a graph Γ is k-colourable if the vertex set of Γ can be partitioned into at most k independent sets. Note that if Γ is k-colourable, then each clique has cardinality $\leqslant k$, and in particular, $\omega(\Gamma) \leqslant k$.

Proposition 4.5 (Berge, Corollary 1, p. 336)**.** *Let Γ be a simple graph. If, for some integer $k \geqslant 1$, the number of vertices of degree $\geqslant k$ is $\leqslant k$, then Γ is k-colourable.*

Theorem 4.6. *Let G be a* PCA*-group and let the orbits of G are indexed according to cardinality: $1 \leqslant |x_1^G| \leqslant |x_2^G| \leqslant \cdots$. Suppose that m is the smallest integer with $|x_1^G| + |x_2^G| + \cdots + |x_m^G| \geqslant |G_{x_m}|$. Then $\omega(G) \leqslant |x_1^G| + |x_2^G| + \cdots + |x_m^G|$.*

Proof. It is enough to show that $|C| \leqslant |x_1^G| + |x_2^G| + \cdots + |x_m^G|$, for each clique C in $\Delta(G)$. First, observe that, if G_{x_l} is a largest stabilizer in G, then we have $|x_1^G| = |x_2^G| = \ldots = |x_{l-1}^G| = 1$, $|x_l^G| \geqslant 2$, and $|G_{x_l}| \geqslant |G_{x_{l+1}}| \geqslant |G_{x_{l+2}}| \geqslant \cdots$. Of course, $F = \mathrm{Fix}(G)$ is contained in $C \cap G_{x_l}$, and in addition, $|C| \leqslant |G_{x_l}|$. To see this suppose there is an element $x \in C \backslash G_{x_l}$. Since $C \subseteq G_x$ and $x \notin F$, it follows that $|C| \leqslant |G_x| \leqslant |G_{x_l}|$.

On the other hand, since the orbits partition G, $m \geqslant l$. If $m = l$, then

$$|C| \leqslant |G_{x_l}| \leqslant |x_1^G| + |x_2^G| + \cdots + |x_m^G|,$$

and we are done in this case. So assume $m \geqslant l+1$. We now consider the stabilizing graph $\Delta(G \backslash F)$, and claim that $\Delta(G \backslash F)$ is $\sum_{i=1}^m |x_i^G|$-colourable. For this, in view of Proposition 4.5, we show that the number of vertices of degree $\geqslant \sum_{i=l}^m |x_i^G|$, is $< \sum_{i=l}^m |x_i^G|$. Clearly each vertex $y \in G \backslash F$ has degree $|G_y| - |\mathrm{Fix}(G)| - 1$ in $\Delta(G \backslash F)$. If $|G_y| - |F| - 1 \geqslant \sum_{i=l}^m |x_i^G|$, then

$$|G_y| - 1 \geqslant |F| + \sum_{i=l}^m |x_i^G| = |x_1^G| + |x_2^G| + \cdots + |x_m^G| \geqslant |G_{x_m}|.$$

Thus $|G_y| > |G_{x_m}|$ which implies that $|y^G| < |x_m^G|$. Hence y has already been counted among $\bigcup_{i=l}^{m-1} x_i^G$, and we have shown that $\sum_{i=l}^{m} |x_i^G|$ is an upper bound to the number of vertices of degree $\sum_{i=l}^{m} |x_i^G|$, as required.

Therefore, $\Delta(G)$ is $|F|+\sum_{i=l}^{m} |x_i^G|$-colourable, that is $\sum_{i=1}^{m} |x_i^G|$-colourable. Thus $|C| \leqslant \sum_{i=1}^{m} |x_i^G|$ as we mentioned previously, and the theorem is proved. □

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Faculty of Mathematics, K. N. Toosi University of Technology, P. O. Box 16765–3381, Tehran, IRAN.

E-mail address: bahrami.mahsa@email.kntu.ac.ir

Faculty of Mathematics, K. N. Toosi University of Technology, P. O. Box 16765–3381, Tehran, IRAN.

E-mail address: moghadam@kntu.ac.ir