

A new unknotting operation for classical and welded knots

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Abstract

An unknotting operation is a local change on knot diagrams. Any knot diagram can be transformed into a trivial unknot diagram by a series of unknotting operations plus some Reidemeister moves. Unknotting operations have significant importance in knot theory; they are used to study the complexity of knots and invariants of knots. Unknotting operations are also important in biology, chemistry, and physics to study the sophisticated entanglements of strings, organic compounds, and DNA. In this paper, we introduced a new unknotting operation called the diagonal move for classical and welded knots. We show that the crossing change, Δ -move, \sharp -move, Γ -move, n -gon move, pass move, and 4-move can be realized by a sequence of diagonal moves. We also define the distance induced by the diagonal move and study its properties.

Keywords: Diagonal move, unknotting operations, distance, knots and links

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1. Introduction

For two link diagrams that are identical except inside small disks D and D' containing a few crossings, a local move is a transformation taking D to

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D' . An unknotting operation is a local move such that any knot diagram can be transformed into a diagram of the trivial knot by a finite sequence of these operations plus some Reidemeister moves. Unknotting operations are useful in studying knots and links. On the other hand, many knot invariants are extensively studied to distinguish different knots. Unknotting operations are inevitable in studying knot invariants. They also play an essential role in the simplification of sophisticated entangles of strings, organic compounds, and DNA in physics, chemistry, and biology. Many local moves are known as unknotting operations for knots; some well-known unknotting operations are shown in Fig.1.

The crossing change is a fundamental unknotting operation for any knot diagram. To show that it is an unknotting operation we first choose a base point and an orientation for the given knot diagram. A base point is an arbitrary point on a knot diagram, distinct from crossing points. Now we travel along the knot from the base point according to the orientation of the knot; we meet every crossing two times during the complete journey. When we meet a crossing, if we first pass the crossing from the lower arc, we apply a crossing change here, if we first pass the crossing from the upper arc, we skip it. Finally, we get back to the initial point. After making all of these adjustments, the resulting diagram is called a descending diagram. A descending diagram is the unknot. This is also true in the projective space [1]. More generally, a link diagram can be unlinked with appropriate crossing changes and making each component descending as above.

H. Murakami and Y. Nakanishi introduced the Δ -move in [2], and they proved that every knot can be deformed into a trivial knot by Δ -moves. The \sharp -move was introduced by Hitoshi Murakami in [3]; it is also proved that by some \sharp -moves, one can deform any knot into a trivial knot. Shibuya introduced the Γ -move and proved it is an unknotting operation [4]. The crossing change can be realized by a Γ -move. Taizo Kanenobu in [5] studied two types of Γ -moves; he showed that the two types of Γ -moves are equivalent to each other; one move is realized by the other move and vice-versa. Hoste *et al.* in [6] introduced the $H(n)$ -move for $n \geq 2$, they proved that the $H(n)$ -move is an unknotting operation. Therefore, any two knots can be transformed into each other by a finite sequence of $H(n)$ -moves. We proved that the $H(n)$ -move is an unknotting operation for virtual and welded knots and links [7]. Y. Nakanishi showed that a Δ -move can be realized by a finite sequence of 3-gon moves [8]. So any knot can be transformed into a trivial knot by a finite sequence of 3-gon moves. H. Aida in [9] generalizes the notion

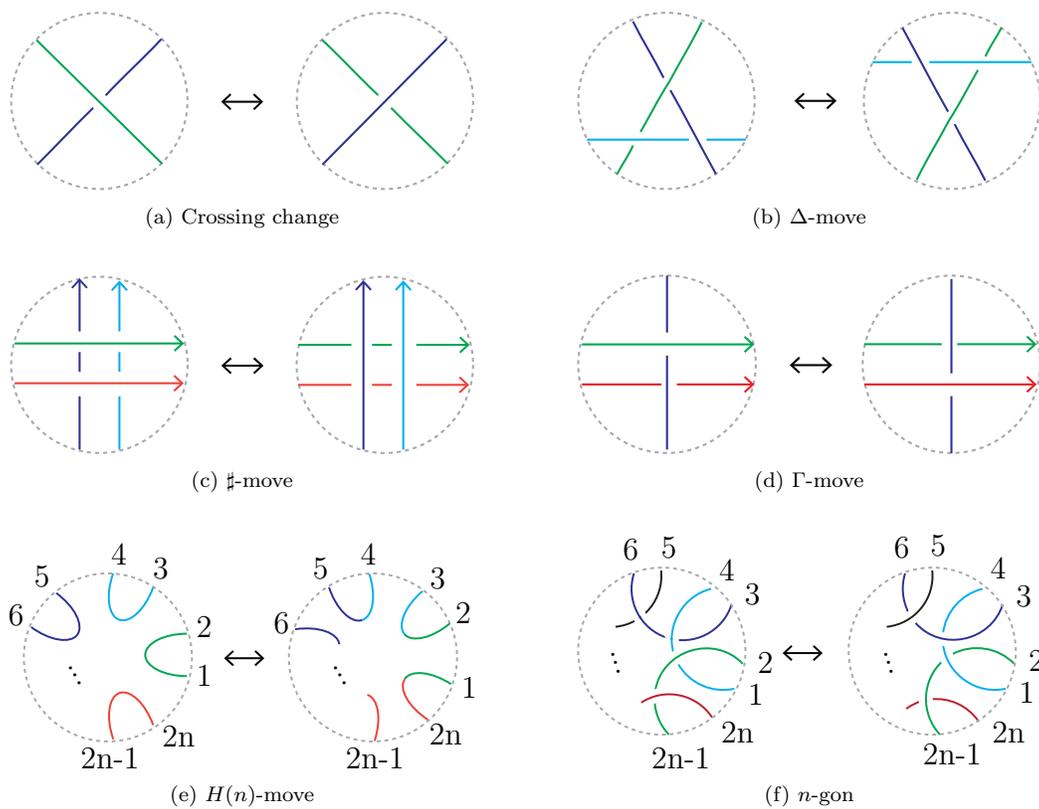


Figure 1: Some well-known unknotting operations

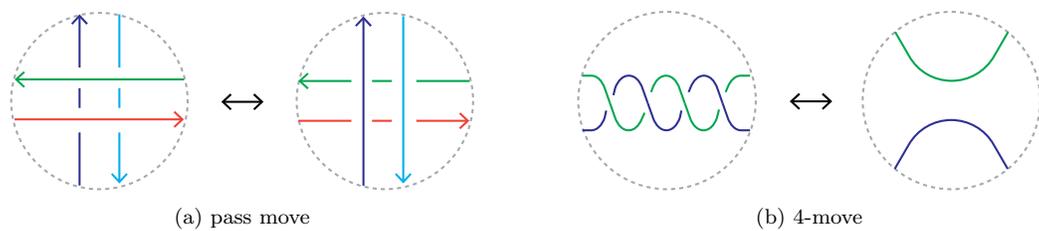


Figure 2: Illustration of pass move and 4-move

of 3-gon moves to n -gon moves; it is proved that given any knot K , there exists an integer n such that K can be transformed into a trivial knot by one n -gon move.

The pass move and 4-move are shown in Fig.2. We can define two knots to be pass equivalent if they are related by a finite sequence of pass moves. In [10] Kauffman studied the pass move and showed that any knot can be deformed into either the trivial knot or the trefoil knot by a finite sequence of pass moves. The pass move preserves the Arf invariant, two knots are pass equivalent if and only if they have the same Arf invariant. The trefoil and the unknot are not pass move equivalents, therefore, the pass move is not an unknotting operation for knots. On the other hand, in [11] Dabkowski *et al.* proved that all knots up to 12 crossings reduce to the trivial knot by 4-moves. They also showed that links of 2 components with at most 11 crossings reduce to either the trivial link or the Hopf link using a finite number of 4-moves. Jozef H. Przytycki in [12] shows that every alternating link of two components up to 12 crossings can be reduced to the trivial link or the Hopf link by 4-moves and Reidemeister moves (See also [13, 14]). We study the Gordian complex of knots using 4-moves [15]. This study provides new insights into the structure of knot diagrams and the relationships between different knot types. Whether the 4-move is an unknotting operation is an open question.

This paper is organized as follows: In Section 2, we defined the diagonal move, and proved that the diagonal move is an unknotting operation for classical knots and links. We also studied the relation of diagonal move with other local moves. We showed that the crossing change, Δ -move, \sharp -move, Γ -move, n -gon move, pass move, and 4-move can be realized by a sequence of diagonal moves. In Section 3, we discussed welded knot theory. Every welded knot diagram can be deformed into a trivial knot diagram by a finite sequence of diagonal moves plus some welded Reidemeister moves. In Section 4, we studied the distance from K to K' for an unknotting operation, and we discuss the relation of distance from K to K' for those well-known unknotting operations.

2. Diagonal move and other local moves for classical knots and links

First, we will explain the diagonal move; for simplicity, we denote it by D-move. Suppose that we have four arcs a , b , c and d and four crossings 1, 2,

3, and 4 as illustrated in Fig.3. There are two diagonal crossing pairs, $\{1,3\}$ and $\{2,4\}$. A diagonal move is a local transformation on a knot diagram that changes only diagonal crossings. There are many possible diagonal moves, however, in this paper we only study the two diagonal moves as shown in Fig.4.

Theorem 2.1. *Every knot diagram can be deformed into a trivial knot diagram by a finite sequence of diagonal moves and Reidemeister moves.*

Let an oriented knot diagram K has n crossings $c_1, c_2, c_3 \dots, c_n$. We choose an arbitrary base point, distinct from crossings and endpoints of arcs. Now we start our journey from the base point along the orientation of the knot. A crossing change can be realized by a diagonal move, as shown in Fig.5. When we meet a crossing for the first time, and the crossing is under crossing, we change it into over crossing by a diagonal move. When we again arrive at the same crossing for a second time, we leave it unchanged and finally, we return to the initial point. When we have done all these changes, the diagram is descending. A descending diagram is the unknot, therefore, any knot diagram can be deformed into a trivial knot diagram by a finite sequence of diagonal moves and Reidemeister moves.

Similarly, a link diagram, $L = l_1 \cup l_2 \cup \dots \cup l_m$ with m components, can be unlinked by diagonal move. Individually the components of L can be knotted or unknotted. Each link component l_i , has an orientation and a base point. We start from the base point of l_i ; when we meet a crossing for the first time, and the crossing is under crossing, we change it into over crossing by a diagonal move. When we arrive at a crossing for a second time, we skip it. Finally, we return to the initial point, and the l_i is unlinked. The unlinked component is descending; therefore, it is also the unknot.

We allow the diagonal move for all possible orientation of arcs a, b, c and d. The diagonal move is very closely related to the \sharp -move and the pass move; therefore, it is essential to study the relation of diagonal move with \sharp -move, pass move and other local moves of knots and links.

Theorem 2.2. *The following moves on link diagrams can be realized by diagonal moves:*

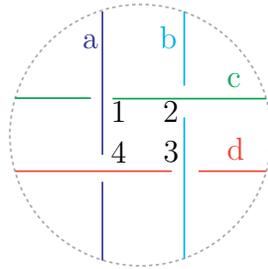


Figure 3: The diagonal move illustration

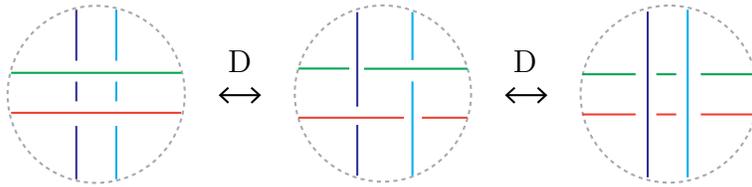


Figure 4: The diagonal move

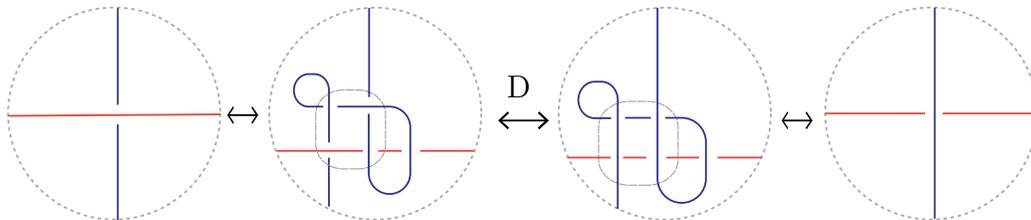


Figure 5: A crossing change can be realized by a diagonal move

- | | |
|---|--|
| 1. <i>The Δ-move</i> | 5. <i>The Γ-move</i> |
| 2. <i>The \sharp-move</i> | 6. <i>The n-gon move</i> |
| 3. <i>The $\overline{\sharp}$-move</i> | 7. <i>The 4-move</i> |
| 4. <i>The pass move</i> | |

A Δ -move can be realized by a finite sequence of diagonal moves, as demonstrated in Fig.6. The diagonal move allows for all possible orientations of arcs; therefore, it is elementary to show that the \sharp -move and pass move can be obtained by a finite sequence of diagonal moves, see Fig.7 and Fig.8. In [16] Zhang and Yang introduced a local move similar to the pass move and \sharp -move; they call it $\overline{\sharp}$ -move. They introduced the two types of pass moves and the two types of $\overline{\sharp}$ -moves. They also showed that the two types of pass moves can be obtained from each other; similarly, the two types of $\overline{\sharp}$ -moves can be obtained from each other. The $\overline{\sharp}$ -move is equivalent to the pass move, that is, a $\overline{\sharp}$ -move and a pass move can be accomplished by a sequence of each other. Since the pass move is not an unknotting operation, so the $\overline{\sharp}$ -move is not an unknotting operation for knots and links. A crossing change can be realized by a diagonal move, therefore, we can conclude that all the \sharp -move, pass move, and $\overline{\sharp}$ -move can be obtained by a finite sequence of diagonal moves and Reidemeister moves.

A Γ -move is realized by a finite sequence of diagonal moves, as shown in Fig.9. We proved that a diagonal move can realize a crossing change; therefore, a finite sequence of diagonal moves can change all the crossing of n -gon move. A 4-move can be realized by a finite sequence of diagonal moves, as shown in Fig.10.

A local move X is called a generalization of another local move Y if Y can be realized by a finite number of X local moves. We show that the crossing change, Δ -move, \sharp -move, $\overline{\sharp}$ -move Γ -move, n -gon move, pass move, and 4-move can be realized by a sequence of diagonal moves therefore, we say the diagonal move is a generalization of all these moves. The crossing change, Γ -move, and 4-move can be realized by a single diagonal move, while two diagonal moves can realize the Δ -move, \sharp -move, and pass move. The n -gon move can be realized by $\leq n$ diagonal moves. All the unknotting operations shown in Fig.1 and Fig.2 can be realized by a finite sequence of diagonal moves except $H(n)$ -move. The diagonal move is a combination of arcs and crossings, while the $H(n)$ -move involves only arcs; therefore, the author did not find any clue to generalize diagonal move for $H(n)$ -move.

We define the diagonal unknotting number $u_D(K)$ of a knot diagram K

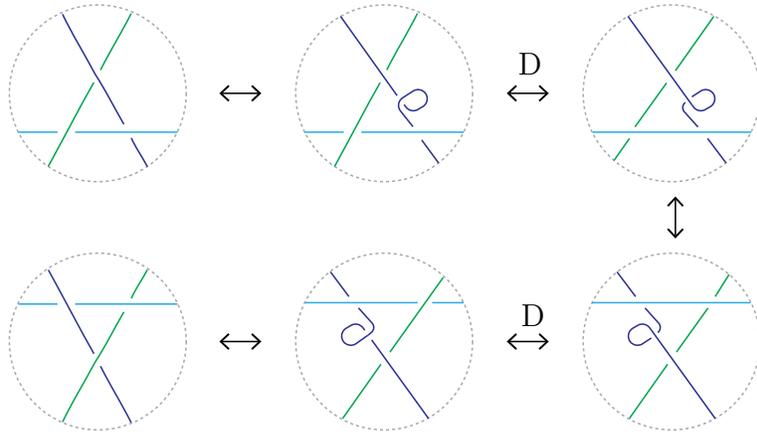


Figure 6: A Δ -move is realized by two diagonal moves

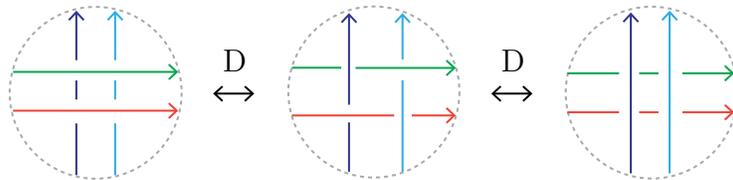


Figure 7: A \dagger -move is realized by two diagonal moves

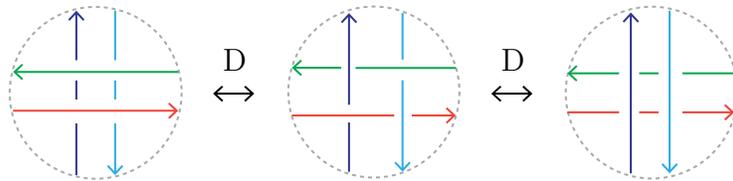


Figure 8: A pass move is realized by two diagonal moves

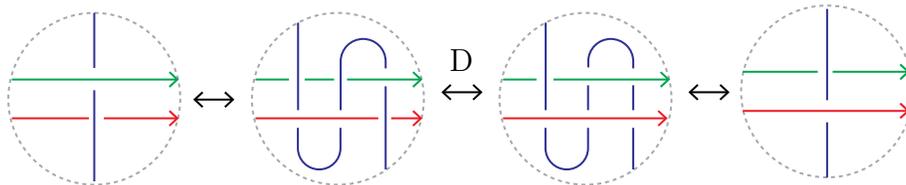


Figure 9: A Γ -move is realized by a diagonal move

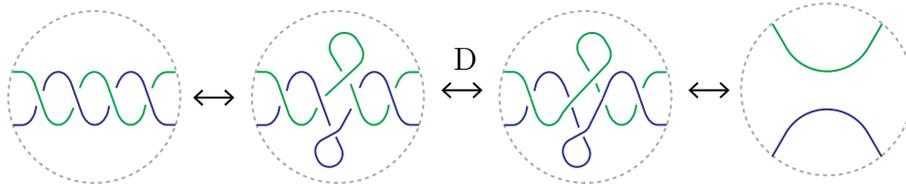


Figure 10: A 4-move is realized by a diagonal move

to be the minimum number of diagonal moves necessary to obtain a diagram of the trivial knot from K . Two knots are diagonal-equivalent if one can be obtained from the other by a combination of Reidemeister moves and diagonal moves.

Corollary 2.1. *Every two knot diagrams K and K' are diagonal-equivalent.*

3. diagonal unknotting operations for welded knots and links

Welded knot theory was introduced in [17] as a generalization of a classical knot theory. A welded knot diagram is a knot diagram that may have welded crossings as well as classical crossings. A welded knot is an equivalence class of welded knot diagrams under the three kinds of classical Reidemeister moves together with the five types of welded Reidemeister moves (for more details see [18]). Similar to classical knots, invariants and local moves also play important roles in welded knot theory. Some unknotting operations of classical knots are extended to welded knots. For example, Shin Satoh proved that the crossing changes, Δ -moves, and \sharp -moves are unknotting operations for welded knots [18]. In classical knot theory, it is known that any classical knot can be deformed into the trivial knot or the trefoil knot by a finite sequence of pass moves. The trefoil and the unknot are not pass move equivalents, therefore, the pass move is not an unknotting operation for classical knots. However, in [19] it is proved that the pass move is an unknotting operation for welded knots. If we combine the results of [18] and [19] then we have the following theorem.

Theorem 3.1. *The following local moves are unknotting operations for welded knots:*

- | | |
|--|--|
| 1. <i>The crossing change</i> | 3. <i>The \sharp-move</i> |
| 2. <i>The Δ-move</i> | 4. <i>The pass move</i> |

An orientated and based welded knot diagram D is called a descending diagram if walking along D from the base point following the orientation, we meet the over-crossing first and the under-crossing later at every classical crossing. Any descending welded knot diagram D is related to the trivial diagram by a finite sequence of welded Reidemeister moves. The crossing change, Δ -move, and \sharp -move are unknotting operations for both classical and welded knots. The crossing change, Δ -move, \sharp -move, and pass move can be realized by a finite sequence of diagonal moves. Therefore, it is obvious that a diagonal move is an unknotting operation for welded knots.

Corollary 3.1. *Every welded knot diagram can be deformed into a trivial knot diagram by a finite sequence of diagonal moves.*

So the diagonal move is an unknotting operation for welded knots, and we have the following corollary.

Corollary 3.2. *Every two welded knot diagrams K and K' are diagonal-equivalent.*

4. Unknotting operations and distance

It is a natural question to ask how far apart two distinct knots are. Distance on knots is used to answer this question. Unknotting operations play an important role in defining distance on knots. Given an unknotting operation on two distinct knots, K and K' , a diagram of K can be transformed into a diagram of K' by a finite sequence of this unknotting operation up to some Reidemeister moves. For two knots K and K' , the distance from K to K' for an unknotting operation is the minimum of times this unknotting operation needs to be used to transform a diagram of K into a diagram of K' , where the minimum is taken over all diagrams of K and K' . If a knot diagram of K is transformed into the unknot by a finite sequence of this unknotting operation, then the distance is called the unknotting number of K . The distance on knots defines a metric on the space of knots.

The crossing change, Δ -move, \sharp -move, Γ -move, n -gon move, and diagonal move are unknotting operations for both classical and welded knots and

links. For two knots K and K' , the distance from K to K' defined by the crossing change, Δ -move, \sharp -move, Γ -move, and diagonal move are denoted by $d_X(K, K')$, $d_\Delta(K, K')$, $d_\sharp(K, K')$, $d_\Gamma(K, K')$, and $d_D(K, K')$ respectively. In Theorem 2.2 we proved that the crossing change, Γ -move and 4-move can be realized by a single diagonal move, while two diagonal moves can realize the Δ -move, \sharp -move, and pass move. Therefore, we have the following theorem.

Theorem 4.1. *For any two knots K and K'*

$$d_D(K, K') \leq d_X(K, K') \leq 2d_D(K, K')$$

A crossing change can be realized by only one diagonal move, as shown in Fig.5. Since a diagonal move can be replaced by two crossing changes, therefore, we conclude that $d_D(K, K') \leq d_X(K, K') \leq 2d_D(K, K')$.

Lemma 4.1.

$$d_D(K, K') \leq d_\Gamma(K, K'), 2d_X(K, K') \leq d_\Gamma(K, K')$$

A Γ -move can be realized by only one diagonal move, as shown in Fig.9. However, a Γ -move can be realized by two crossing changes. Therefore, $d_D(K, K') \leq d_\Gamma(K, K'), 2d_X(K, K') \leq d_\Gamma(K, K')$.

Lemma 4.2.

$$2d_D(K, K') \leq 2d_X(K, K') \leq d_\Delta(K, K')$$

A Δ -move can be realized by two crossing changes. Furthermore, Theorem 4.1 proves $d_D(K, K') \leq d_X(K, K')$. So we have $2d_D(K, K') \leq 2d_X(K, K') \leq d_\Delta(K, K')$.

Lemma 4.3.

$$2d_D(K, K') \leq d_\sharp(K, K')$$

A \sharp -move can be realized by two diagonal moves, as shown in Fig.7. Therefore, $2d_D(K, K') \leq d_\sharp(K, K')$.

Examples

1. Suppose 9_1 and 5_1 are the standard knot diagrams in [20], then $d_D(9_1, 5_1) = 1 \leq d_X(9_1, 5_1) = 2$.
2. Suppose 5_1 is the standard knot diagram and 0 is the unknot, then $d_D(5_1, 0) = 1 \leq d_X(5_1, 0) = 2$. In this example, 0 is the unknot, therefore the distance is also known as unknotting number, therefore we have $u_D(5_1, 0) = 1 \leq u_X(5_1, 0) = 2$.
3. From example 1 and example 2 we can obtain the following relation $u_D(9_1, 0) = 2 \leq u_X(9_1, 0) = 4$.

Declarations

Conflict of interest

The authors declare that they have no conflict of interest. The contents of this manuscript have not been published previously.

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