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Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru
ISSN 1813-3304

Vol. 20, No. 2, pp. 144–145 (2023)
https://doi.org/10.33048/semi.2023.20.???

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MSC 47H09

COMPLETE BI-COMPLEX VALUED RECTANGULAR METRIC SPACE

KRISHNA BHATTACHARJEE, RAKHAL DAS,
AND PRASANNA POOJARY

Communicated by xxx

Abstract: In this paper, we introduce a space that is a generalisation of bicomplex valued metric spaces. We used rectangular inequality instead of triangular inequality. As a result, we have obtained some new results regarding the complete bicomplex valued rectangular metric spaces. We employed the well-known Banach contraction to investigate fixed points in the bicomplex valued rectangular metric space. Furthermore, we provide sufficient conditions for a pair of contractive mappings in bicomplex-valued rectangular metric spaces to have a common fixed point. We also provide some non-trivial examples to demonstrate the validity of our proven results.

Keywords: Contraction mapping; fixed point; Common fixed point; Rectangular metric space.
Subject Classification 2020: 47H09; 46T99; 47H10; 54H25; 30G35.

Krishna B., Rakhal D., Prasanna P. Complete Bi-complex Valued Rectangular Metric Space.

The work is not supported by any external soure.
Received April, 26, 2024, Published December, xx, 202x.

1 Introduction

Segre [21] endeavored to create special algebras in a novel way. He proposed bicomplex numbers, tricomplex numbers, and other commutative generalisations of complex numbers as elements of an infinite set of algebras. Subsequently, during the 1930s, further academics made significant contributions to this field (see references [10]-[19]). Unfortunately, no advancements were achieved in this domain during the subsequent half-century. Price [19] subsequently pioneered the field of function theory and bicomplex algebra. This topic holds significant relevance in various mathematical science domains, as well as other domains within the realm of science and technology, which have recently garnered a resurgence of interest. One notable study on the fundamental functions of bicomplex numbers has been conducted by Luna-Elizarrar?s et al. [18]. Beg et al. [5] investigated the existence of common fixed points for maps on cone metric spaces represented by topological vector spaces. Azam et al. [3] extended the scope of the study to encompass complex valued metric space. Additionally, they demonstrated a common fixed point theorem for two self-contracting mappings. Rouzkard and Imdad [12] not only demonstrated another well-known common fixed point theorem that satisfies a rational inequality in complex valued metric spaces, but they also extended the findings of Azam et al. [1].

Research on the Banach contraction principle [4] is ongoing and is a well-liked and useful tool for resolving existence problems in many areas of mathematical analysis. "Let (X, d) be a complete metric space and T be a mapping of X into itself satisfying $d(Tx, Ty) \leq kd(x, y), \forall x, y \in X$, where k is a constant in $[0, 1)$. Then, T has a unique fixed point $x^* \in X$," according to the well-known Banach theorem. Branciari [8] introduced the concept of rectangular metric space (RMS) by illustrating an analogy to the Banach Contraction Principle in such space. He achieved this by substituting the sum on the right side of the triangular inequality in the definition of a metric space with a three-term expression. Several fixed point theorems have been developed for various contractions on rectangular metric space.

Certain fixed point results for rational type expressions in partially ordered complex valued metric spaces were demonstrated by Choudhury et al. [10]. The fixed point of mapping satisfying rational inequality in complex valued metric spaces was demonstrated by Bhat et al. [7]. The attempts are also visible in $\{\mathrm{cf.}[6], [9]\}$. A few common fixed point theorems were demonstrated by Choi et al. [13] in relation to two weakly compatible mappings in bicomplex valued metric spaces. Some common fixed point theorems under rational contractions for a pair of mappings in bicomplex valued rectangular metric spaces were proved by Jebril et al. [14]. These works are continued in this article. We study the bicomplex valued rectangular metric spaces in more detail and prove fixed point theorems for two contractive type mappings that satisfy a rational inequality.

Subsequently, we delineate fundamental concepts and symbols for subsequent use. We define $\mathbb{C}_0, \mathbb{C}_1$, and $\mathbb{C}_2$, respectively, to represent the set of real, complex, and bicomplex numbers.

1.1. Bicomplex Number. The bicomplex number, as stated by Segre [21], is:

$$\xi = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2,$$

where $a_1, a_2, a_3, a_4 \in \mathbb{C}_0$, and the independent units i_1, i_2 are such that $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1$, we denote the set of bicomplex numbers $\mathbb{C}_2$ as follows:

$$\mathbb{C}_2 = \{\xi : \xi = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2, a_1, a_2, a_3, a_4 \in \mathbb{C}_0\},$$

i.e.,

$$\mathbb{C}_2 = \{\xi : \xi = z_1 + i_2 z_2, z_1, z_2 \in \mathbb{C}_1\},$$

where $z_1 = a_1 + a_2 i_1 \in \mathbb{C}_1$ and $z_2 = a_3 + a_4 i_1 \in \mathbb{C}_1$.

In $\mathbb{C}_2$, there are four idempotent elements: $0, 1; e_1 = \frac{1+i_1 i_2}{2}$; and $e_2 = \frac{1-i_1 i_2}{2}$ out of which the nontrivial components e_1 and e_2, such that $e_1 + e_2 = 1$ and $e_1 e_2 = 0$. One unique way to express every bicomplex number $z_1 + i_2 z_2$ is as the sum of e_1 and e_2, specifically

$$\xi = z_1 + i_2 z_2 = (z_1 - i_1 z_2)\, e_1 + (z_1 + i_1 z_2)\, e_2.$$

The idempotent representation of bicomplex numbers and the complex coefficients $\xi_1 = (z_1 - i_1 z_2)$ are the names given to this representation of ξ. The bicomplex numbers ξ have idempotent components, denoted by and $\xi_2 = (z_1 + i_1 z_2)$.

The norm $\|\cdot\|$ of $\mathbb{C}_2$ is a positive real valued function and $\|\cdot\| : \mathbb{C}_2 \to \mathbb{C}_0^+$is defined by

$$\begin{aligned}\|\xi\| = \|z_1 + i_2 z_2\| &= \left\{|z_1|^2 + |z_2|^2\right\}^{\frac{1}{2}} \\ &= \left[\frac{|(z_1 - i_1 z_2)|^2 + |(z_1 + i_1 z_2)|^2}{2}\right]^{\frac{1}{2}} = \left(a_1^2 + a_2^2 + a_3^2 + a_4^2\right)^{\frac{1}{2}},\end{aligned}$$

where $\xi = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2 = z_1 + i_2 z_2 \in \mathbb{C}_2$.

The linear space $\mathbb{C}_2$ with respect to the defined norm is a norm linear space, also $\mathbb{C}_2$ is complete; therefore $\mathbb{C}_2$ is the Banach space.

The definition of $\precsim_{i_2}$, the partial order relation on $\mathbb{C}_2$, is:

If $\mathbb{C}_2$ is the set of bicomplex numbers, and $\xi = z_1 + i_2 z_2, \eta = w_1 + i_2 w_2 \in \mathbb{C}_2$, then $\xi \precsim_{i_2} \eta$ if and only if $z_1 \precsim w_1$ and $z_2 \precsim w_2$,

that is, $\xi \precsim_{i_2} \eta$, if any of the subsequent circumstances holds true:

(1) $z_1 = w_1, z_2 = w_2$,

(2) $z_1 \prec w_1, z_2 = w_2$,

(3) $z_1 = w_1, z_2 \prec w_2$, and
(4) $z_1 \prec w_1, z_2 \prec w_2$.

Specifically, if $\xi \precsim_{i_2} \eta$ and $\xi \neq \eta$, that is, if one of (2), (3), and (4) is satisfied, we can write $\xi \lnsim_{i_2} \eta$; if only (4) is satisfied, we will write $\xi \prec_{i_2} \eta$.

For any two bicomplex numbers $\xi, \eta \in \mathbb{C}_2$ we can verify the followings:

(i) $\xi \precsim_{i_2} \eta \Rightarrow \|\xi\| \leq \|\eta\|$,
(ii) $\|\xi + \eta\| \leq \|\xi\| + \|\eta\|$,
(iii) $\|a\xi\| = a\|\xi\|$, where a is a non negative real number,
(iv) $\|\xi\eta\| \leq \sqrt{2}\|\xi\|\|\eta\|$, and the equality holds only when at least one of ξ and η is degenerated,
(v) $\|\xi^{-1}\| = \|\xi\|^{-1}$ if ξ is a degenerated bicomplex number with $0 \prec \xi$,
(vi) $\left\|\frac{\xi}{\eta}\right\| = \frac{\|\xi\|}{\|\eta\|}$, if η is a degenerated bicomplex number.

1.2. Bicomplex valued metric space. Choi et al. [13] defined the bicomplex valued metric spaces as:

Definition 1. [13] Let $\bar{P}$ be a nonempty set. Suppose the mapping $\wp : \bar{P}\times\bar{P} \to \mathbb{C}_2$ satisfies the following conditions:

(1) $0 \precsim_{i_2} \wp(x, y)$ for all $x, y \in \bar{P}$ (positivity),
(2) $\wp(x, y) = 0$ if and only if $x = y$,
(3) $\wp(x, y) = \wp(y, x)$ for all $x, y \in \bar{P}$ (symmetry), and
(4) $\wp(x, y) \precsim_{i_2} \wp(x, z) + \wp(z, y)$ for all $x, y, z \in \bar{P}$ (triangle inequility).

Then the pair $(\bar{P}, \wp)$ is called the bicomplex valued metric spaces.

2 Main results

In this section, we have presented the notion of bicomplex valued rectangular metric space in this paper, which is a generalization of the notion of bicomplex valued metric space. A bicomplex valued rectangular metric space is defined as follows:

Definition 2. Let X be a nonempty set. Suppose the mapping $\wp : X\times X \to \mathbb{C}_2$ satisfies the following conditions:

(1) $0 \precsim_{i_2} \wp(p, q)$ for all $p, q \in \bar{P}$ (positivity),
(2) $\wp(p, q) = 0$ if and only if $p = q$,
(3) $\wp(p, q) = \wp(q, p)$ for all $p, q \in \bar{P}$ (symmetry), and
(4) $\wp(p, q) \precsim_{i_2} \wp(p, x) + \wp(x, y) + \wp(y, q)$ for all $p, q \in \bar{P}$ and all distinct points $p, q \in \bar{P} \setminus \{x, y\}$ (rectangular inequility).

Then the pair $(\bar{P}, \wp)$ is called the bicomplex valued rectangular metric space.

Example 1. Consider $\bar{P} = \left\{0, \frac{1}{4}, \frac{1}{2}, 2\right\}$, define a bicomplex valued rectangular metric $\wp : \bar{P} \times \bar{P} \to \mathbb{C}_2$ by $\wp(p, q) = (1 + i_2)\,|x - y|, \forall p, q \in \bar{P}$.

From the above definition of $\wp$ one can easily verify that $(\bar{P}, \wp)$ is a bicomplex valued rectangular metric spaces.

Definition 3. For a bicomplex valued rectangular metric spaces $(\bar{P}, \wp)$

(1) A sequence $\left\{\bar{\vartheta}_n\right\}$ in $\bar{P}$ is said to be a convergent sequence and converges to a point x if for any $0 \prec_{i_2} r \in \mathbb{C}_2$ there exists a natural number $n_0 \in \mathbb{N}$ such that $\wp\left(\bar{\vartheta}_n, x\right) \prec_{i_2} r$ for all $n > n_0$ and we write $\lim_{n\to\infty} \bar{\vartheta}_n = \vartheta$ or $\bar{\vartheta}_n \to \vartheta$ as $n \to \infty$.
(2) A sequence $\left\{\bar{\vartheta}_n\right\}$ in $\bar{P}$ is said to be a Cauchy sequence in $(\bar{P}, \wp)$ if for any $0 \prec_{i_2} r \in \mathbb{C}_2$ there exists a natural number $n_0 \in \mathbb{N}$ such that $\wp\left(\bar{\vartheta}_n, \bar{\vartheta}_{n+m}\right) \prec_{i_2} r$ for all $m, n \in \mathbb{N}$ and $n > n_0$.
(3) A bicomplex valued rectangular metric spaces $(\bar{P}, \wp)$ is said to be complete if every Cauchy sequence in $\bar{P}$ is convergent in $\bar{P}$.

Lemma 1. Let $(\bar{P}, \wp)$ be a bicomplex valued rectangular metric spaces and $\left\{\bar{\vartheta}_n\right\}$ be a sequence in $\bar{P}$. Then $\left\{\bar{\vartheta}_n\right\}$ is convergent sequence and converges to a point $\bar{\vartheta}$ if and only if $\lim_{n\to\infty} \left\|\wp\left(\bar{\vartheta}_n, \bar{\vartheta}\right)\right\| = 0$.

Lemma 2. Let $(\bar{P}, \wp)$ be a bicomplex valued rectangular metric spaces and $\left\{\bar{\vartheta}_n\right\}$ be a sequence in $\bar{P}$. Then $\left\{\bar{\vartheta}_n\right\}$ is a Cauchy sequence in $\bar{P}$ if and only if $\lim\limits_{n,m\to\infty} \left\|\wp\left(\bar{\vartheta}_n, \bar{\vartheta}_{n+m}\right)\right\| = 0$.

Theorem 1. Let $(\bar{P}, \wp)$ be a complete bicomplex valued rectangular metric space and $T : \bar{P} \to \bar{P}$ be a mapping satisfying:

$$\wp(T\bar{\vartheta}, T\bar{y}) \precsim_{i_2} \alpha\wp(\bar{\vartheta}, \bar{y}) \tag{1}$$

for all $\bar{\vartheta}, \bar{y} \in \bar{P}$, where $\alpha \in [0, 1)$. Then T has a unique fixed point.

Proof. Let T satisfy Equation (1), $\bar{\vartheta}_0 \in \bar{P}$ be an arbitrary point and define the sequence $\left\{\bar{\vartheta}_n\right\}$ by $\bar{\vartheta}_n = T^n\bar{\vartheta}_0$. From (1), we get

$$\begin{aligned} \wp(\bar{\vartheta}_n, \bar{\vartheta}_{n+1}) &= \wp(T\bar{\vartheta}_{n-1}, T\bar{\vartheta}_n) \\ &\precsim_{i_2} \alpha\wp(\bar{\vartheta}_{n-1}, \bar{\vartheta}_n). \end{aligned} \tag{2}$$

Using again Equation (1), we have

$$\wp(\bar{\vartheta}_{n-1}, \bar{\vartheta}_n) \precsim_{i_2} \alpha\wp(\bar{\vartheta}_{n-2}, \bar{\vartheta}_{n-1}),$$

and by Equation (2), we get

$$\wp(\bar{\vartheta}_n, \bar{\vartheta}_{n+1}) \precsim_{i_2} \alpha^2\wp(\bar{\vartheta}_{n-2}, \bar{\vartheta}_{n-1}).$$

If we continue this process, we obtain

$$\wp(\bar{\vartheta}_n, \bar{\vartheta}_{n+1}) \precsim_{i_2} \alpha^n\wp(\bar{\vartheta}_0, \bar{\vartheta}_1). \tag{3}$$

Using rectangular inequility and Equation (3) for all $n, m \in \mathbb{N}$ with $n < m$, we have

$$\begin{aligned}
\wp(\bar{\vartheta}_n, \bar{\vartheta}_m) &\precsim_{i_2} [\wp(\bar{\vartheta}_n, \bar{\vartheta}_{n+1}) + \wp(\bar{\vartheta}_{n+1}, \bar{\vartheta}_{n+2}) + \wp(\bar{\vartheta}_{n+2}, \bar{\vartheta}_m)] \\
&\precsim_{i_2} [\wp(\bar{\vartheta}_n, \bar{\vartheta}_{n+1}] + [\wp(\bar{\vartheta}_{n+1}, \bar{\vartheta}_{n+2}) + \wp(\bar{\vartheta}_{n+2}, \bar{\vartheta}_m)] \\
&\precsim_{i_2} [\wp(\bar{\vartheta}_n, \bar{\vartheta}_{n+1})] + [\wp(\bar{\vartheta}_{n+1}, \bar{\vartheta}_{n+2})] + [\wp(\bar{\vartheta}_{n+2}, \bar{\vartheta}_{n+3})] + \cdots + \wp(\bar{\vartheta}_{m-1}, \bar{\vartheta}_m)] \\
&\precsim_{i_2} (\alpha^n + \alpha^{n+1} + \alpha^{n+2} + \cdots + \alpha^{m-1})\wp(\bar{\vartheta}_0, \bar{\vartheta}_1) \\
&\precsim_{i_2} \alpha^n[1 + \alpha + (\alpha)^2 + (\alpha)^3 + \cdots + (\alpha)^{m-n-1}]\wp(\bar{\vartheta}_0, \bar{\vartheta}_1) \\
&\precsim_{i_2} \frac{\alpha^n}{1-\alpha}\wp(\bar{\vartheta}_0, \bar{\vartheta}_1).
\end{aligned}$$

Thus, we have

$$\left\|\wp(\bar{\vartheta}_n, \bar{\vartheta}_m)\right\| \leq \frac{\alpha^n}{1-\alpha}\left\|\wp(\bar{\vartheta}_0, \bar{\vartheta}_1)\right\|.$$

Since $\alpha \in [0, 1)$, taking limits as $n \to \infty$, then

$$\frac{\alpha^n}{1-\alpha}\left\|\wp(\bar{\vartheta}_0, \bar{\vartheta}_1)\right\| \to 0.$$

This means that

$$\left\|\wp(\bar{\vartheta}_n, \bar{\vartheta}_m)\right\| \to 0.$$

So $\left\{\bar{\vartheta}_n\right\}$ is bicomplex valued Cauchy sequence by Lemma (2). Completeness of $(\bar{P}, \wp)$ gives us that there is an element $u \in \bar{P}$ such that $\left\{\bar{\vartheta}_n\right\}$ is bicomplex valued convergent to u.

We show that u is a fixed point of T, i.e., $Tu = u$. For any $n \in \mathbb{N}$, we get

$$\begin{aligned}
\wp(u, Tu) &\precsim_{i_2} [\wp(u, \bar{\vartheta}_n) + \wp(\bar{\vartheta}_n, \bar{\vartheta}_{n+1}) + \wp(\bar{\vartheta}_{n+1}, Tu)] \\
&= [\wp(u, \bar{\vartheta}_n) + \wp(\bar{\vartheta}_n, \bar{\vartheta}_{n+1}) + \wp(T\bar{\vartheta}_n, Tu)] \\
&\precsim_{i_2} [\wp(u, \bar{\vartheta}_n) + \wp(\bar{\vartheta}_n, \bar{\vartheta}_{n+1}) + \alpha\wp(\bar{\vartheta}_n, u)].
\end{aligned}$$

Since $\bar{\vartheta}_n$ converges to u as $n \to \infty$, it follows from the last inequality that $\wp(u, Tu) = 0$, i.e., $Tu = u$.

Finally, we prove the uniqueness. Let $w \neq u$ be another fixed point of T. Using (1), we have

$$\wp(z, w) = \wp(Tz, Tw) \precsim_{i_2} \alpha\wp(z, w),$$

and

$$\|\wp(z, w)\| \leq \alpha \|\wp(z, w)\|.$$

Since $\alpha \in [0,1)$, we have $\|\wp(z,w)\| \leq 0$ which is a contradiction. Thus, we conclude that $u = w$ and so u is a unique fixed point of T. □

Example 2. In the Example (1), we consider the mapping $T : \bar{P} \to \bar{P}$ defined by

$$T(\bar{\vartheta}) = \begin{cases} 0 & \text{if } \bar{\vartheta} = 0, \\ \frac{1}{10} & \text{if } \bar{\vartheta} = \frac{1}{4}, \\ 0 & \text{if } \bar{\vartheta} = \frac{1}{2}, \\ \frac{1}{2} & \text{if } \bar{\vartheta} = 2. \end{cases}$$

Let $a = \frac{1}{2}$, then clearly $a < 1$. Also the condition (1) of the Theorem (1) is satisfied. So clearly, $\bar{\vartheta} = 0$ is the unique fixed point of T.

Theorem 2. Let $(\bar{P}, \wp)$ be a complete bicomplex valued rectangular metric space and $T : \bar{P} \to \bar{P}$ be a continuous mapping such that for some function $\phi : \bar{P} \to \mathbb{C}_0$. If for each $\bar{\vartheta} \in \bar{P}$ the following condition hold:

$$\wp(\bar{\vartheta}, T(\bar{\vartheta})) \precsim_{i_2} \phi(\bar{\vartheta}) - \phi(T(\bar{\vartheta})), \tag{4}$$

then $\left\{T^n(\bar{\vartheta})\right\}$ converges to a fixed point of T for all $\bar{\vartheta} \in \bar{P}$.

Proof. For any fixed $\bar{\vartheta} \in \bar{P}$, let $\bar{\vartheta}_n = T^n(\bar{\vartheta}), n \in \mathbb{N}$. From (4), we obtain

$$0 \precsim_{i_2} \phi(\bar{\vartheta}) - \phi(T(\bar{\vartheta})) \quad \Leftrightarrow \quad \phi(\bar{\vartheta}) \succsim_{i_2} \phi(T(\bar{\vartheta}))$$

for all $\bar{\vartheta} \in \bar{P}$ and so,

$$\phi\left(\bar{\vartheta}_{n+1}\right) = \phi\left(T^{n+1}(\bar{\vartheta})\right) = \phi\left(T\left(T^n(\bar{\vartheta})\right)\right) = \phi\left(T\left(\bar{\vartheta}_n\right)\right) \precsim_{i_2} \phi\left(\bar{\vartheta}_n\right).$$

Since we conclude that $\left\{\phi\left(T^n(\bar{\vartheta})\right)\right\}$ is monotonically decreasing and bounded below, we have $\lim_{n\to\infty} \phi\left(T^n(\bar{\vartheta})\right) = a \geq 0$. If $m, n \in \mathbb{N}$ and $m > n$, then by using rectangular inequality and Equation(4), we have

$$
\begin{aligned}
& \wp\left(\bar{\vartheta}_n, \bar{\vartheta}_m\right) \\
\precsim_{i_2} \;& \wp\left(\bar{\vartheta}_n, \bar{\vartheta}_{n+1}\right) + \wp\left(\bar{\vartheta}_{n+1}, \bar{\vartheta}_{n+2}\right) + \wp\left(\bar{\vartheta}_{n+2}, \bar{\vartheta}_m\right) \\
= \;& \wp\left(\bar{\vartheta}_n, \bar{\vartheta}_{n+1}\right) + \wp\left(\bar{\vartheta}_{n+1}, \bar{\vartheta}_{n+2}\right) + \wp\left(\bar{\vartheta}_{n+2}, \bar{\vartheta}_m\right) \\
\precsim_{i_2} \;& \wp\left(\bar{\vartheta}_n, \bar{\vartheta}_{n+1}\right) + \wp\left(\bar{\vartheta}_{n+1}, \bar{\vartheta}_{n+2}\right) + \wp\left(\bar{\vartheta}_{n+2}, \bar{\vartheta}_{n+3}\right) + \wp\left(\bar{\vartheta}_{n+3}, \bar{\vartheta}_{n+4}\right) \\
& + \wp\left(\bar{\vartheta}_{n+4}, \bar{\vartheta}_m\right) \\
& \vdots \\
\precsim_{i_2} \;& \wp\left(\bar{\vartheta}_n, \bar{\vartheta}_{n+1}\right) + \wp\left(\bar{\vartheta}_{n+1}, \bar{\vartheta}_{n+2}\right) + \cdots + \wp\left(\bar{\vartheta}_{m-3}, \bar{\vartheta}_{m-2}\right) + \wp\left(\bar{\vartheta}_{m-2}, \bar{\vartheta}_{m-1}\right) \\
& + \wp\left(\bar{\vartheta}_{m-1}, \bar{\vartheta}_m\right) \\
\precsim_{i_2} \;& \phi\left(\bar{\vartheta}_n\right) - \phi\left(\bar{\vartheta}_{n+1}\right) + \phi\left(\bar{\vartheta}_{n+1}\right) - \phi\left(\bar{\vartheta}_{n+2}\right) + \cdots + \phi\left(\bar{\vartheta}_{m-2}\right) - \phi\left(\bar{\vartheta}_{m-1}\right) \\
& + \phi\left(\bar{\vartheta}_{m-1}\right) - \phi\left(\bar{\vartheta}_m\right) \\
= \;& \phi\left(\bar{\vartheta}_n\right) - \phi\left(\bar{\vartheta}_m\right).
\end{aligned}
$$

Using the fact that $\lim_{n\to\infty} \phi\left(\bar{\vartheta}_n\right) = a$, we have,

$$
\begin{aligned}
\wp\left(\bar{\vartheta}_n, \bar{\vartheta}_m\right) &\precsim_{i_2} \phi\left(\bar{\vartheta}_n\right) - \phi\left(\bar{\vartheta}_m\right). \\
\Longrightarrow \wp\left(\bar{\vartheta}_n, \bar{\vartheta}_m\right) &\precsim_{i_2} a - a \;[\textit{taking limits as } n, m \to \infty] \\
&= 0.
\end{aligned}
$$

Thus, we have

$$\left\| \wp\left(\bar{\vartheta}_n, \bar{\vartheta}_m\right) \right\| \leq 0 \; as \; n, m \to \infty.$$

$$\Longrightarrow \left\| \wp\left(\bar{\vartheta}_n, \bar{\vartheta}_m\right) \right\| \to 0 \; as \; n, m \to \infty.$$

Thus, $\lim\limits_{m,n\to\infty} \wp\left(\bar{\vartheta}_n, \bar{\vartheta}_m\right) = 0$ and $\left\{T^n(\bar{\vartheta})\right\}$ is a Cauchy sequence in $\bar{P}$ by Lemma (2). Since $\bar{P}$ is complete bicomplex valued rectangular b-metric, there exists a point $u \in \bar{P}$ such that $\lim_{n\to\infty} T^n(\bar{\vartheta}) = u$ and from continuity of $T, u = T(u)$. □

Theorem 3. Let $(\bar{P}, \wp)$ be a complete bicomplex valued rectangular metric spaces with degenerated $1 + \wp(p, q)$ and $\|1 + \wp(p, q)\| \neq 0$ for all $p, q \in \bar{P}$ and let $S, T : \bar{P} \to \bar{P}$ be mappings satisfying the condition

$$\wp(Sp, Tq) \precsim_{i_2} a\wp(p, q) + \frac{b\wp(p, Sp)\wp(q, Tq)}{1 + \wp(p, q)}$$

for all $p, q \in \bar{P}$, where a, b are non-negative real numbers with $a+\sqrt{2}b < 1$. Then S, T has a unique common fixed point.

Proof. Let $\bar{\vartheta}_0$ be an arbitrary point in $\bar{P}$. We construct a sequence $\left\{\bar{\vartheta}_n\right\}$ such that

$$\bar{\vartheta}_{2k+1} = S\bar{\vartheta}_{2k}, \quad \bar{\vartheta}_{2k+2} = T\bar{\vartheta}_{2k+1}, \quad k = 0, 1, 2, \ldots$$

Then, we have

$$\begin{aligned}
\wp\left(\bar{\vartheta}_{2k+1}, \bar{\vartheta}_{2k+2}\right) &= \wp\left(S\bar{\vartheta}_{2k}, T\bar{\vartheta}_{2k+1}\right) \\
&\precsim_{i_2} a\wp\left(\bar{\vartheta}_{2k}, \bar{\vartheta}_{2k+1}\right) + \frac{b\wp\left(\bar{\vartheta}_{2k}, S\bar{\vartheta}_{2k}\right)\wp\left(\bar{\vartheta}_{2k+1}, T\bar{\vartheta}_{2k+1}\right)}{1+\wp\left(\bar{\vartheta}_{2k}, \bar{\vartheta}_{2k+1}\right)} \\
&\precsim_{i_2} a\wp\left(\bar{\vartheta}_{2k}, \bar{\vartheta}_{2k+1}\right) + \frac{b\wp\left(\bar{\vartheta}_{2k}, \bar{\vartheta}_{2k+1}\right)\wp\left(\bar{\vartheta}_{2k+1}, \bar{\vartheta}_{2k+2}\right)}{1+\wp\left(\bar{\vartheta}_{2k}, \bar{\vartheta}_{2k+1}\right)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left\|\wp\left(\bar{\vartheta}_{2k+1}, \bar{\vartheta}_{2k+2}\right)\right\| \leq & a\left\|\wp\left(\bar{\vartheta}_{2k}, \bar{\vartheta}_{2k+1}\right)\right\| \\
& + \sqrt{2}b\frac{\left\|\wp\left(\bar{\vartheta}_{2k}, \bar{\vartheta}_{2k+1}\right)\right\|}{\left\|1+\wp\left(\bar{\vartheta}_{2k}, \bar{\vartheta}_{2k+1}\right)\right\|}\left\|\wp\left(\bar{\vartheta}_{2k+1}, \bar{\vartheta}_{2k+2}\right)\right\|.
\end{aligned}$$

Also, $\left\|\wp\left(\bar{\vartheta}_{2k}, \bar{\vartheta}_{2k+1}\right)\right\| \leq \left\|1+\wp\left(\bar{\vartheta}_{2k}, \bar{\vartheta}_{2k+1}\right)\right\|$.
Thus,

$$\begin{aligned}
\left\|\wp\left(\bar{\vartheta}_{2k+1}, \bar{\vartheta}_{2k+2}\right)\right\| &\leq a\left\|\wp\left(\bar{\vartheta}_{2k}, \bar{\vartheta}_{2k+1}\right)\right\| + \sqrt{2}b\left\|\wp\left(\bar{\vartheta}_{2k+1}, \bar{\vartheta}_{2k+2}\right)\right\|. \\
\text{i.e., } (1-\sqrt{2}b)\left\|\wp\left(\bar{\vartheta}_{2k+1}, \bar{\vartheta}_{2k+2}\right)\right\| &\leq a\left\|\wp\left(\bar{\vartheta}_{2k}, \bar{\vartheta}_{2k+1}\right)\right\|. \\
\text{i.e., } \left\|\wp\left(\bar{\vartheta}_{2k+1}, \bar{\vartheta}_{2k+2}\right)\right\| &\leq \frac{a}{(1-\sqrt{2}b)}\left\|\wp\left(\bar{\vartheta}_{2k}, \bar{\vartheta}_{2k+1}\right)\right\|.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\wp\left(\bar{\vartheta}_{2k+2}, \bar{\vartheta}_{2k+3}\right) = \wp\left(T\bar{\vartheta}_{2k+1}, S\bar{\vartheta}_{2k+2}\right) &= \wp\left(S\bar{\vartheta}_{2k+2}, T\bar{\vartheta}_{2k+1}\right) \\
&\precsim_{i_2} a\wp\left(\bar{\vartheta}_{2k+2}, \bar{\vartheta}_{2k+1}\right) + \frac{b\wp\left(\bar{\vartheta}_{2k+2}, S\bar{\vartheta}_{2k+2}\right)\wp\left(\bar{\vartheta}_{2k+1}, T\bar{\vartheta}_{2k+1}\right)}{1+\wp\left(\bar{\vartheta}_{2k+2}, \bar{\vartheta}_{2k+1}\right)} \\
&\precsim_{i_2} a\wp\left(\bar{\vartheta}_{2k+2}, \bar{\vartheta}_{2k+1}\right) + \frac{b\wp\left(\bar{\vartheta}_{2k+2}, \bar{\vartheta}_{2k+3}\right)\wp\left(\bar{\vartheta}_{2k+1}, \bar{\vartheta}_{2k+2}\right)}{1+\wp\left(\bar{\vartheta}_{2k+2}, \bar{\vartheta}_{2k+1}\right)}.
\end{aligned}$$

Therefore we obtain that,

$$\begin{aligned}
&\left\|\wp\left(\bar{\vartheta}_{2k+2},\bar{\vartheta}_{2k+3}\right)\right\| \leq a\left\|\wp\left(\bar{\vartheta}_{2k+2},\bar{\vartheta}_{2k+1}\right)\right\| \\
&\qquad\qquad +\sqrt{2}b\frac{\left\|\wp\left(\bar{\vartheta}_{2k+1},\bar{\vartheta}_{2k+2}\right)\right\|}{\left\|1+\wp\left(\bar{\vartheta}_{2k+1},\bar{\vartheta}_{2k+2}\right)\right\|}\left\|\wp\left(\bar{\vartheta}_{2k+2},\bar{\vartheta}_{2k+3}\right)\right\| \\
&\text{i.e., } \left\|\wp\left(\bar{\vartheta}_{2k+2},\bar{\vartheta}_{2k+3}\right)\right\| \leq a\left\|\wp\left(\bar{\vartheta}_{2k+2},\bar{\vartheta}_{2k+1}\right)\right\|+\sqrt{2}b\left\|\wp\left(\bar{\vartheta}_{2k+2},\bar{\vartheta}_{2k+3}\right)\right\|, \\
&\text{as } \left\|\wp\left(\bar{\vartheta}_{2k+1},\bar{\vartheta}_{2k+2}\right)\right\| \leq \left\|1+\wp\left(\bar{\vartheta}_{2k+1},\bar{\vartheta}_{2k+2}\right)\right\| \\
&\text{i.e., } (1-\sqrt{2}b)\left\|\wp\left(\bar{\vartheta}_{2k+2},\bar{\vartheta}_{2k+3}\right)\right\| \leq a\left\|\wp\left(\bar{\vartheta}_{2k+2},\bar{\vartheta}_{2k+3}\right)\right\| \\
&\text{i.e., } \left\|\wp\left(\bar{\vartheta}_{2k+2},\bar{\vartheta}_{2k+3}\right)\right\| \leq \frac{a}{(1-\sqrt{2}b)}\left\|\wp\left(\bar{\vartheta}_{2k+2},\bar{\vartheta}_{2k+3}\right)\right\|. \qquad (5)
\end{aligned}$$

Suppose that $\alpha=\frac{a}{1-\sqrt{2}b}$. So clearly we claim that $\alpha<1$ as $a+\sqrt{2}b<1$. Now letting $2k+1=n$ and from (5) it follows that

$$\begin{aligned}
\left\|\wp\left(\bar{\vartheta}_{n+1},\bar{\vartheta}_{n+2}\right)\right\| &\leq \alpha\left\|\wp\left(\bar{\vartheta}_{n},\bar{\vartheta}_{n+1}\right)\right\| \\
&\leq \alpha^2\left\|\wp\left(\bar{\vartheta}_{n-1},\bar{\vartheta}_{n}\right)\right\| \leq \cdots \leq \alpha^{n+1}\left\|\wp\left(\bar{\vartheta}_{0},\bar{\vartheta}_{1}\right)\right\|.
\end{aligned}$$

Also for any two positive integers m,n with $m>n$ we get that

$$\wp\left(\bar{\vartheta}_{n},\bar{\vartheta}_{m}\right) \precsim_{i_2} \wp\left(\bar{\vartheta}_{n},\bar{\vartheta}_{n+1}\right)+\wp\left(\bar{\vartheta}_{n+1},\bar{\vartheta}_{n+2}\right)+\cdots+\wp\left(\bar{\vartheta}_{m-1},\bar{\vartheta}_{m}\right).$$

Therefore,

$$\begin{aligned}
\left\|\wp\left(\bar{\vartheta}_{n},\bar{\vartheta}_{m}\right)\right\| &\leq \left\|\wp\left(\bar{\vartheta}_{n},\bar{\vartheta}_{n+1}\right)\right\|+\left\|\wp\left(\bar{\vartheta}_{n+1},\bar{\vartheta}_{n+2}\right)\right\|+\cdots+\left\|\wp\left(\bar{\vartheta}_{m-1},\bar{\vartheta}_{m}\right)\right\| \\
&\leq \left[\alpha^n+\alpha^{n+1}+\cdots+\alpha^{m-1}\right]\left\|\wp\left(\bar{\vartheta}_{0},\bar{\vartheta}_{1}\right)\right\|.
\end{aligned}$$

i.e., $\left\|d\left(\bar{\vartheta}_{n},\bar{\vartheta}_{m}\right)\right\| \leq \alpha^n\left[1+\alpha+\alpha^2+\cdots+\alpha^{m-n-1}\right]\left\|\wp\left(\bar{\vartheta}_{0},\bar{\vartheta}_{1}\right)\right\|.$

Since, $0\leq\alpha<1$, then $1+\alpha+\alpha^2+\cdots+\alpha^{m-n-1}\leq\frac{1}{1-\alpha}$.
Hence,

$$\left\|\wp\left(\bar{\vartheta}_{n},\bar{\vartheta}_{m}\right)\right\| \leq \frac{\alpha^n}{1-\alpha}\left\|\wp\left(\bar{\vartheta}_{0},\bar{\vartheta}_{1}\right)\right\|.$$

Again $\frac{\alpha^n}{1-\alpha}\longrightarrow 0$ as $n\longrightarrow\infty$. Then for any $\varepsilon>0$ there exists a natural number n_0 such that $\left\|\wp\left(\bar{\vartheta}_{n},\bar{\vartheta}_{m}\right)\right\|<\varepsilon$ for all $m,n>n_0$. Hence $\left\{\bar{\vartheta}_{n}\right\}$ is a Cauchy sequence in $\bar{P}$. Also, $\bar{P}$ is a complete bicomplex valued rectangular metric space. Then there exists $\bar{u}\in\bar{P}$ such that $\lim_{n\longrightarrow\infty}\bar{\vartheta}_{n}=\bar{u}$.

Now we show that $\bar{u}=S\bar{u}$. If not then there exist $0\prec_{i_2}\xi\in\mathbb{C}_2$ such that $\wp(\bar{u},S\bar{u})=\xi$.

Therefore we have,

$$\begin{aligned}
\xi &= \wp(\bar{u}, S\bar{u}) \\
&\precsim_{i_2} \wp\left(\bar{u}, \bar{\vartheta}_{2k+1}\right) + \wp\left(\bar{\vartheta}_{2k+1}, \bar{\vartheta}_{2k+2}\right) + \wp\left(\bar{\vartheta}_{2k+2}, S\bar{u}\right) \\
&\precsim_{i_2} \wp\left(\bar{u}, \bar{\vartheta}_{2k+1}\right) + \wp\left(\bar{\vartheta}_{2k+1}, \bar{\vartheta}_{2k+2}\right) + \wp\left(T\bar{\vartheta}_{2k+1}, S\bar{u}\right) \\
&\precsim_{i_2} \wp\left(\bar{u}, \bar{\vartheta}_{2k+1}\right) + \wp\left(\bar{\vartheta}_{2k+1}, \bar{\vartheta}_{2k+2}\right) + a\wp\left(\bar{\vartheta}_{2k+1}, \bar{u}\right) + \frac{b\wp\left(\bar{\vartheta}_{2k+1}, T\bar{\vartheta}_{2k+1}\right)\wp(\bar{u}, S\bar{u})}{1 + \wp\left(\bar{\vartheta}_{2k+1}, \bar{u}\right)}
\end{aligned}$$

$$i.e., \xi \precsim_{i_2} \wp\left(\bar{u}, \bar{\vartheta}_{2k+1}\right) + \wp\left(\bar{\vartheta}_{2k+1}, \bar{\vartheta}_{2k+2}\right) + a\wp\left(\bar{\vartheta}_{2k+1}, \bar{u}\right) + \frac{b\wp\left(\bar{\vartheta}_{2k+1}, \bar{\vartheta}_{2k+2}\right)\xi}{1 + \wp\left(\bar{\vartheta}_{2k+1}, \bar{u}\right)}.$$

Hence,

$$\|\xi\| \leq \left\|\wp\left(\bar{u}, \bar{\vartheta}_{2k+1}\right)\right\| + \left\|\wp\left(\bar{\vartheta}_{2k+1}, \bar{\vartheta}_{2k+2}\right)\right\| + a\left\|\wp\left(\bar{\vartheta}_{2k+1}, \bar{u}\right)\right\| + \sqrt{2}b\frac{\left\|\wp\left(\bar{\vartheta}_{2k+1}, \bar{\vartheta}_{2k+2}\right)\right\| \|\xi\|}{\left\|1 + \wp\left(\bar{\vartheta}_{2k+1}, \bar{u}\right)\right\|}.$$

Since $\lim_{n \longrightarrow \infty} \bar{\vartheta}_n = \bar{u}$, taking limit on both sides as $n \longrightarrow \infty$ we get that $\|\xi\| \leq 0$, which is a contradiction. Therefore $\|\xi\| = 0 \Rightarrow \|\wp(\bar{u}, S\bar{u})\| = 0 \Rightarrow \bar{u} = S\bar{u}$. Similarly, we can show that $\bar{u} = T\bar{u}$. Hence S and T have a common fixed point.
Now we show that S and T have unique common fixed point. If possible suppose $\bar{u}^* \in \bar{P}$ be another common fixed point of S and T.
Then,

$$\wp\left(\bar{u}, \bar{u}^*\right) = \wp\left(S\bar{u}, T\bar{u}^*\right) \precsim_{i_2} a\wp\left(\bar{u}, \bar{u}^*\right) + \frac{b\wp(\bar{u}, S\bar{u})\wp\left(\bar{u}^*, T\bar{u}^*\right)}{1 + \wp\left(\bar{u}, \bar{u}^*\right)}.$$

$$\begin{aligned}
&\text{i.e., } \left\|\wp\left(\bar{u}, \bar{u}^*\right)\right\| \leq a\left\|\wp\left(\bar{u}, \bar{u}^*\right)\right\| + \sqrt{2}b\frac{\|\wp(\bar{u}, S\bar{u})\| \left\|\wp\left(\bar{u}^*, T\bar{u}^*\right)\right\|}{\left\|1 + \wp\left(\bar{u}, \bar{u}^*\right)\right\|}. \\
&\text{i.e., } \left\|\wp\left(\bar{u}, \bar{u}^*\right)\right\| \leq a\left\|\wp\left(\bar{u}, \bar{u}^*\right)\right\|. \\
&\text{i.e., } \left\|\wp\left(\bar{u}, \bar{u}^*\right)\right\| = 0. \\
&\qquad\quad \text{i.e., } \bar{u} = \bar{u}^*.
\end{aligned}$$

This completes the proof of the theorem. □

Corollary 1. Let $(\bar{P}, \wp)$ be a complete bicomplex valued rectangular metric spaces with degenerated $1 + \wp(p, q)$ and $\|1 + d(p, q)\| \neq 0$ for all $p, q \in \bar{P}$ and $S : \bar{P} \to \bar{P}$ be any mapping satisfying the condition

$$\wp(Sp, Sq) \precsim_{i_2} a\wp(p, q) + \frac{b\wp(p, Sp)\wp(q, Sq)}{1 + \wp(p, q)} \tag{6}$$

for all $p, q \in \bar{P}$, where a, b are non-negative real numbers with $a + \sqrt{2}b < 1$. Then S has a unique fixed point.

Proof. We can easily prove this result by applying the Theorem (3) and taking $T = S$. □

Example 3. In the Example (2), we take the mapping $T : \bar{P} \to \bar{P}$ defined by

$$T(x) = \begin{cases} 0 & \text{if } x = 0, \\ \frac{1}{10} & \text{if } x = \frac{1}{4}, \\ 0 & \text{if } x = \frac{1}{2}, \\ \frac{1}{2} & \text{if } x = 2. \end{cases}$$

And let $a = \frac{1}{2}$ and $b = \frac{1}{8}$, then clearly $a + \sqrt{2}b = \frac{1}{2} + \sqrt{2}\frac{1}{8} = 0.67675 < 1$. Also the Equation (6) of the Corollary (1) is satisfied. So clearly, $x = 0$ is the unique fixed point of T.

Corollary 2. Let $(\bar{P}, \wp)$ be a complete bicomplex valued rectangular metric spaces with degenerated $1 + \wp(p, q)$ and $\|1 + d(p, q)\| \neq 0$ for all $p, q \in \bar{P}$ and let $S : \bar{P} \to \bar{P}$ be any mapping satisfying the condition

$$\wp\left(S^n p, S^n q\right) \precsim_{i_2} a\wp(p, q) + \frac{b\wp\left(p, S^n p\right)\wp\left(q, S^n q\right)}{1 + \wp(p, q)}$$

for all $p, q \in \bar{P}$, where a, b are non-negative real numbers with $a + \sqrt{2}b < 1$. Then S has a unique fixed point.

Proof. By Corollary (1) there exists a unique point $u \in \bar{P}$ such that

$$\begin{aligned} S\bar{u} &= \bar{u}. \\ \text{i.e., } S^2\bar{u} &= S\bar{u} = \bar{u}. \\ &\vdots \\ \text{i.e., } S^n\bar{u} &= \bar{u}. \end{aligned}$$

Therefore,

$$\wp(S\bar{u}, \bar{u}) = \wp\left(SS^n\bar{u}, S^n\bar{u}\right) = \wp\left(S^nS\bar{u}, S^n\bar{u}\right) \precsim_{i_2} a\wp(S\bar{u}, \bar{u}) + \frac{b\wp\left(S\bar{u}, S^nS\bar{u}\right)\wp\left(\bar{u}, S^n\bar{u}\right)}{1 + \wp(S\bar{u}, \bar{u})}.$$

$$\text{i.e., } \wp(S\bar{u}, \bar{u}) \precsim_{i_2} a\wp(S\bar{u}, \bar{u}) + \frac{b\wp\left(S\bar{u}, S^nS\bar{u}\right)\wp(\bar{u}, \bar{u})}{1 + \wp(S\bar{u}, \bar{u})}.$$

i.e., $\wp(S\bar{u}, \bar{u}) \precsim_{i_2} a\wp(S\bar{u}, \bar{u})$.

i.e., $\|\wp(S\bar{u}, \bar{u})\| \leq a\|\wp(S\bar{u}, \bar{u})\|$.

i.e., $\|\wp(S\bar{u}, \bar{u})\| = 0$.

i.e., $\bar{u} = \bar{u}^*$.

This completes the proof of the Corollary. □

Theorem 4. Let $(\bar{P}, \wp)$ be a complete bicomplex valued rectangular metric spaces and let the mappings $S, T : \bar{P} \to \bar{P}$ satisfy the condition

$$\wp(Sp, Tq) \precsim_{i_2} \frac{\alpha[\wp(p, Sp)\wp(p, Tq) + \wp(q, Tq)\wp(q, Sp)]}{\wp(p, Tq) + \wp(q, Sp)} \tag{7}$$

for all $p, q \in \bar{P}$ and if $\|\wp(p, Tq) + \wp(q, Sp)\| \neq 0$ and $\wp(p, Tq) + \wp(q, Sp)$ is degenerated, where α be any non-negative real number with $0 \leq \alpha < 1$. Then S, T have a unique common fixed point.

Proof. Let $\bar{\vartheta}_0$ be an arbitrary point in $\bar{P}$. We consider a sequence $\left\{\bar{\vartheta}_n\right\}$ in $\bar{P}$ such that

$$\bar{\vartheta}_{2n+1} = S\bar{\vartheta}_{2n}, \text{ and } \bar{\vartheta}_{2n+2} = T\bar{\vartheta}_{2n+1} \text{ for all } n = 0, 1, 2, \ldots$$

Then, by using Equation (7) we have,

$$\begin{aligned}
& \wp\left(\bar{\vartheta}_{2n+1}, \bar{\vartheta}_{2n+2}\right) \\
& = \wp\left(S\bar{\vartheta}_{2n}, T\bar{\vartheta}_{2n+1}\right) \\
& \precsim_{i_2} \frac{\alpha\left[\wp\left(\bar{\vartheta}_{2n}, S\bar{\vartheta}_{2n}\right)\wp\left(\bar{\vartheta}_{2n}, T\bar{\vartheta}_{2n+1}\right) + \wp\left(\bar{\vartheta}_{2n+1}, T\bar{\vartheta}_{2n+1}\right)\wp\left(\bar{\vartheta}_{2n+1}, S\bar{\vartheta}_{2n}\right)\right]}{\wp\left(\bar{\vartheta}_{2n}, T\bar{\vartheta}_{2n+1}\right) + \wp\left(\bar{\vartheta}_{2n+1}, S\bar{\vartheta}_{2n}\right)} \\
& \precsim_{i_2} \frac{\alpha\left[\wp\left(\bar{\vartheta}_{2n}, \bar{\vartheta}_{2n+1}\right)\wp\left(\bar{\vartheta}_{2n}, \bar{\vartheta}_{2n+2}\right) + \wp\left(\bar{\vartheta}_{2n+1}, \bar{\vartheta}_{2n+2}\right)\wp\left(\bar{\vartheta}_{2n+1}, \bar{\vartheta}_{2n+1}\right)\right]}{\wp\left(\bar{\vartheta}_{2n}, \bar{\vartheta}_{2n+2}\right) + \wp\left(\bar{\vartheta}_{2n+1}, \bar{\vartheta}_{2n+1}\right)} \\
& \precsim_{i_2} \frac{\alpha\wp\left(\bar{\vartheta}_{2n}, \bar{\vartheta}_{2n+1}\right)\wp\left(\bar{\vartheta}_{2n}, \bar{\vartheta}_{2n+2}\right)}{\wp\left(\bar{\vartheta}_{2n}, \bar{\vartheta}_{2n+2}\right)}, \quad \textit{since } \wp\left(\bar{\vartheta}_{2n+1}, \bar{\vartheta}_{2n+1}\right) = 0 \\
& \precsim_{i_2} \alpha\wp\left(\bar{\vartheta}_{2n}, \bar{\vartheta}_{2n+1}\right) \\
\Longrightarrow \quad & \wp\left(\bar{\vartheta}_{2n+1}, \bar{\vartheta}_{2n+2}\right) \precsim_{i_2} \alpha\wp\left(\bar{\vartheta}_{2n}, \bar{\vartheta}_{2n+1}\right).
\end{aligned} \tag{8}$$

So from Equation (8) it follows that:

$$\wp\left(\bar{\vartheta}_{n+1}, \bar{\vartheta}_{n+2}\right) \precsim_{i_2} \alpha\wp\left(\bar{\vartheta}_n, \bar{\vartheta}_{n+1}\right).$$

Therefore for all $n \geq 0$ we get,

$$\wp\left(\bar{\vartheta}_{n+1}, \bar{\vartheta}_{n+2}\right) \precsim_{i_2} \alpha\wp\left(\bar{\vartheta}_n, \bar{\vartheta}_{n+1}\right) \precsim_{i_2} \alpha^2\wp\left(\bar{\vartheta}_{n-1}, \bar{\vartheta}_n\right) \precsim_{i_2} \cdots \precsim_{i_2} \alpha^{n+1}\wp\left(\bar{\vartheta}_0, \bar{\vartheta}_1\right).$$

Thus for any two positive integers m, n with $m > n$ we have,

$$\begin{aligned}\wp\left(\bar{\vartheta}_n,\bar{\vartheta}_m\right) &\precsim_{i_2} \wp\left(\bar{\vartheta}_n,\bar{\vartheta}_{n+1}\right)+\wp\left(\bar{\vartheta}_{n+1},\bar{\vartheta}_{n+2}\right)+\cdots+\wp\left(\bar{\vartheta}_{m-1},\bar{\vartheta}_m\right)\\ &\precsim_{i_2} \alpha^n\wp\left(\bar{\vartheta}_0,\bar{\vartheta}_1\right)+\alpha^{n+1}\wp\left(\bar{\vartheta}_0,\bar{\vartheta}_1\right)+\cdots+\alpha^{m-1}\wp\left(\bar{\vartheta}_0,\bar{\vartheta}_1\right)\\ &\precsim_{i_2} \left[\alpha^n+\alpha^{n+1}+\cdots+\alpha^{m-1}\right]\wp\left(\bar{\vartheta}_0,\bar{\vartheta}_1\right)\\ &\precsim_{i_2} \alpha^n\left[1+\alpha+\alpha^2+\cdots+\alpha^{m-n-1}\right]\wp\left(\bar{\vartheta}_0,\bar{\vartheta}_1\right)\\ &\precsim_{i_2} \frac{\alpha^n}{1-\alpha}\wp\left(\bar{\vartheta}_0,\bar{\vartheta}_1\right).\end{aligned}$$

Since $0\le\alpha<1$, then $1+\alpha+\alpha^2+\cdots+\alpha^{m-n-1}\le\frac{1}{1-\alpha}$.
Hence,

$$\left\|\wp\left(\bar{\vartheta}_n,\bar{\vartheta}_m\right)\right\|\le\frac{\alpha^n}{1-\alpha}\left\|\wp\left(\bar{\vartheta}_0,\bar{\vartheta}_1\right)\right\|.$$

Again $\frac{\alpha^n}{1-\alpha}\longrightarrow 0$ as $n\longrightarrow\infty$, then for any $\varepsilon>0$ there exists a positive integer n_0 such that $\left\|\wp\left(\bar{\vartheta}_n,\bar{\vartheta}_m\right)\right\|<\varepsilon$ for all $m,n>n_0$. Hence $\left\{\bar{\vartheta}_n\right\}$ is a Cauchy sequence in $\bar{P}$. Also, $\bar{P}$ is a complete bicomplex valued rectangular metric space. Therefore there exists $\bar{u}\in\bar{P}$ such that $\lim_{n\to\infty}\bar{\vartheta}_n=\bar{u}$. Now we show that $\bar{u}=S\bar{u}$. If not then there exists $0\prec_{i_2}\xi\in\mathbb{C}_2$ such that $\wp(\bar{u},S\bar{u})=\xi$.
Therefore,

$$\begin{aligned}\xi &= \wp(\bar{u},S\bar{u})\\ &\precsim_{i_2} \wp\left(\bar{u},\bar{\vartheta}_{n+1}\right)+\wp\left(\bar{\vartheta}_{n+1},\bar{\vartheta}_{n+2}\right)+\wp\left(\bar{\vartheta}_{n+2},S\bar{u}\right)\\ &\precsim_{i_2} \wp\left(\bar{u},\bar{\vartheta}_{n+1}\right)+\wp\left(\bar{\vartheta}_{n+1},\bar{\vartheta}_{n+2}\right)+\wp\left(S\bar{u},T\bar{\vartheta}_{n+1}\right)\\ &\precsim_{i_2} \wp\left(\bar{u},\bar{\vartheta}_{n+1}\right)+\wp\left(\bar{\vartheta}_{n+1},\bar{\vartheta}_{n+2}\right)+\frac{\alpha\left[\wp(\bar{u},S\bar{u})\wp\left(\bar{u},T\bar{\vartheta}_{n+1}\right)+\wp\left(\bar{\vartheta}_{n+1},T\bar{\vartheta}_{n+1}\right)\wp\left(\bar{\vartheta}_{n+1},S\bar{u}\right)\right]}{\wp\left(\bar{u},T\bar{\vartheta}_{n+1}\right)+\wp\left(\bar{\vartheta}_{n+1},S\bar{u}\right)}\\ &\precsim_{i_2} \wp\left(\bar{u},\bar{\vartheta}_{n+1}\right)+\wp\left(\bar{\vartheta}_{n+1},\bar{\vartheta}_{n+2}\right)+\frac{\alpha\left[\xi\wp\left(\bar{u},\bar{\vartheta}_{n+2}\right)+\wp\left(\bar{\vartheta}_{n+1},\bar{\vartheta}_{n+2}\right)\wp\left(\bar{\vartheta}_{n+1},S\bar{u}\right)\right]}{\wp\left(\bar{u},\bar{\vartheta}_{n+2}\right)+\wp\left(\bar{\vartheta}_{n+1},S\bar{u}\right)},\end{aligned}$$

which yields that

$$\begin{aligned}\|\xi\| \le\ & \left\|\wp\left(\bar{u},\bar{\vartheta}_{n+1}\right)\right\|+\left\|\wp\left(\bar{\vartheta}_{n+1},\bar{\vartheta}_{n+2}\right)\right\|\\ &+\sqrt{2}\alpha\frac{\left[\|\xi\|\left\|\wp\left(\bar{u},\bar{\vartheta}_{n+2}\right)\right\|+\left\|\wp\left(\bar{\vartheta}_{n+1},\bar{\vartheta}_{n+2}\right)\right\|\left\|\wp\left(\bar{\vartheta}_{n+1},S\bar{u}\right)\right\|\right]}{\left\|\wp\left(\bar{u},\bar{\vartheta}_{n+2}\right)+\wp\left(\bar{\vartheta}_{n+1},S\bar{u}\right)\right\|} \quad (9)\end{aligned}$$

Taking limit as $n\to\infty$ then $\bar{\vartheta}_n\to\bar{u}$ and from Equation (9) we get, $\|\xi\|\le 0$

which is a contradiction as $\|\xi\| \geq 0$. Therefore we conclude that $\|\xi\| = 0 \Rightarrow \|\wp(\bar{u}, S\bar{u})\| = 0 \Rightarrow \bar{u} = S\bar{u}$. Similarly, we can show that $\bar{u} = T\bar{u}$. Hence S and T have a common fixed point.
Now we show that S and T have unique common fixed point. If possible suppose that $\bar{u}^* \in \bar{P}$ be another common fixed point of S and T.
Then

$$\wp(\bar{u}, \bar{u}^*) = \wp(S\bar{u}, T\bar{u}^*) \precsim_{i_2} \frac{\alpha\left[\wp(\bar{u}, S\bar{u})\wp(\bar{u}, T\bar{u}^*) + \wp(\bar{u}^*, T\bar{u}^*)\wp(\bar{u}^*, S\bar{u})\right]}{\wp(\bar{u}, T\bar{u}^*) + \wp(\bar{u}^*, S\bar{u})}$$

$$\|\wp(\bar{u}, \bar{u}^*)\| \leq \sqrt{2}\alpha \frac{\left[\|\wp(\bar{u}, S\bar{u})\| \|\wp(\bar{u}, T\bar{u}^*)\| + \|\wp(\bar{u}^*, T\bar{u}^*)\| \|\wp(\bar{u}^*, S\bar{u})\|\right]}{\|\wp(\bar{u}, T\bar{u}^*) + \wp(\bar{u}^*, S\bar{u})\|}.$$

i.e., $\|\wp(\bar{u}, \bar{u}^*)\| \leq 0$,

which is a contradiction. Therefore we conclude that

$$\|\wp(\bar{u}, \bar{u}^*)\| = 0 \implies \bar{u} = \bar{u}^*.$$

This completes the proof of the theorem. □

3 Conclusion

This paper presents a novel concept of a complete bicomplex valued rectangular metric space, where we have updated the general background of bicomplex valued metric space and demonstrated some well-known fixed point results. The findings of our study demonstrate the singularity of a fixed point under various contraction conditions. We anticipate that these results will make a valuable contribution to future research in this particular field. If we apply the concepts described in this paper to future studies on alternative metric spaces, such as bicomplex valued control metric space and bicomplex valued cone metric space, we may find intriguing results.

Authors Contribution: All the authors have equal contributions to the preparation of this article.

Conflict of Interest: The authors declare that they have no conflict of interest.

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Krishna Bhattacharjee
Department of Mathematics,
ICFAI University,
Agartala-799210, Tripura, India
Email address: bhattacharya.ayan2013@gmail.com

Rakhal Das
Department of Mathematics,
ICFAI University,
Agartala-799210, Tripura, India
Email address: rakhaldas95@gmail.com

Prasanna Poojary
Department of Mathematics,
Manipal Institute of Technology Bengaluru,
Manipal Academy of Higher Education, Manipal, India
Email address: poojary.prasanna@manipal.edu; poojaryprasanna34@gmail.com