The characters of involutive automorphisms of simple Lie algebras.

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Abstract

The paper deals with the arbitrary finite-dimensional irreducible representations of simple complex Lie algebras $\mathfrak{g}$ of types B_2 and G_2 and real forms $\mathfrak{so}(1,4)$, $\mathfrak{so}(3,2)$ and G of these algebras. Involutive automorphisms θ on Cartan subalgebras of these algebras are considered. Formulae for characters value $\chi(\theta)$ are obtained. That allows to find the number of linearly idependent space-like and time-like vectors in the representation space.

Key words: simple Lie algebras, characters of representations , signature formulae, de Sitter groups

Mathematics Subject Classifications (2000): 17B10;17B20.

1 Introduction

Consider a simple complex Lie algebra $\mathfrak{g}$ and an irreducible finite-dimensional representation $\varphi : \mathfrak{g} \to \mathfrak{sl}(V)$. Denote by $\mathfrak{g}_r$ the real form of inner type for algebra $\mathfrak{g}$. Then $\varphi(\mathfrak{g}_r) \subseteq \mathfrak{su}(p,q)$, where $p - q = \delta(\mathfrak{g}_r)$ is the signature of the invariant Hermitian form on V. It is possible to find $\delta(\mathfrak{g}_r)$ using Weyl character formula for the representation φ. In the paper [1] F.I.Karpelevich derived formulae for δ in the case of classical Lie algebras. In [2] Lie algebras $\mathfrak{su}(p,q)$ are considered and convenient tables for δ in the case of small rank of $\mathfrak{g}$ are found. In [3, 4, 5] formulae for $|\delta|$ in the case of $\mathfrak{g}_r = G, FI, FII, \mathfrak{so}(\mathfrak{p},\mathfrak{q})$ where presented. Nevertheless in applications the exact tables including the sign of δ are necessary. In the paper [6] the algebras $\mathfrak{so}(4,1)$ and $\mathfrak{so}(3,2)$ are explored and formulae for δ are presented as the summation taken over all the orbits of the system of weights for the representation φ. And for δ evaluation the results of V.G.Kac from [7] on automorphisms of finite order were used. We obtain in this paper the tables of δ in terms of the marks of the highest weight of representation φ. Besides similar table for any representation of exceptional Lie algebra of type G_2 are derived. And it is not nesessary to know the system of all weights of the representation φ.

2 Definitions

Consider the compact real form $\mathfrak{g}_\tau$ of simple complex Lie algebra $\mathfrak{g}$, where τ is the conjugation of algebra $\mathfrak{g}$ with respect to $\mathfrak{g}_\tau$. Let θ be involution of algebra $\mathfrak{g}$, and $\theta(\mathfrak{g}_\tau) = \mathfrak{g}_\tau$. Consider the conjugation $\sigma = \tau \circ \theta = \theta \circ \tau$ and let

$$\mathfrak{g}_\sigma = (\theta + 1)\mathfrak{g}_\tau + \sqrt{-1} \cdot (\theta - 1)\mathfrak{g}_\tau.$$

The algebra $\mathfrak{g}_\sigma$ is the real form of algebra $\mathfrak{g}$ and $\mathfrak{k} = (\theta + 1)\mathfrak{g}_\tau$ is maximal compact subalgebra in $\mathfrak{g}_\sigma$. Consider a Cartan subalgebra $\mathfrak{t}$ of the algebra $\mathfrak{g}_\tau$ such that $\theta(\mathfrak{t}) = \mathfrak{t}$, then $\mathfrak{h} = \mathfrak{t}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}$. Denote by R the root system associated with the pare $(\mathfrak{g}, \mathfrak{h})$. For Killing form $B(,)$ of algebra $\mathfrak{g}$ consider $(,) = -\dfrac{1}{(2\pi)^2} \cdot B(,)$ a positive definite scalar product on $\mathfrak{t}$. For $\alpha \in R$, denote by H_α an element of $\mathfrak{h}$ such that $B(H_\alpha, H) = \alpha(H)$ for any $H \in \mathfrak{h}$. Denote by $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ the set of simple roots of R and let $\{H_i\}_{i=1}^r$ is a basis of $\mathfrak{t}$ such that $(H_i, \alpha_j) = \delta_{ij}, i, j = 1, \ldots, r$. Then an involution $\theta = exp(ad(H_{i_0}/2))$, $1 \le i_0 \le r$ defines the real algebra $\mathfrak{g}_\sigma$ and

$$\delta = p - q = \chi(H_{i_0}/2),$$

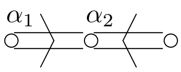
where $\chi(H)$ is the character of representation φ. Here p and q is the number of positive and negative signs in the bilinear $\mathfrak{g}_\sigma$-invariant form on V. Embed the roots R to $\mathfrak{t}$ by formula: $2\pi\sqrt{-1}H_\alpha$, $\alpha \in R$. Then

$$\chi(H_{i_0}/2) = \sum_{\lambda_i} dimV^{\lambda_i} \cdot exp(\pi\sqrt{-1}(\lambda_i, H_{i_0})). \tag{1}$$

Here $\{\lambda_i\}$ is the set of all weights of representation φ and $dimV^{\lambda_i}$ is the multiplicity of the weight λ_i.

3 The case $\mathfrak{g} = \mathfrak{so}(5, \mathbb{C})$

The roots system of the algebra $\mathfrak{g}$ is of type B_2. The extended Dynkin diagram for B_2 is



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And $\alpha_1 = (1; -1)$, $\alpha_2 = (0; 1)$ are the simple roots in a standard basis $(\overrightarrow{\varepsilon}_1, \overrightarrow{\varepsilon}_2)$ from [8] (table II). Furthermore the positive roots are $(1, 0), (0, 1), (1, -1), (1, 1)$. Suppose ρ is half the sum of the positive roots of algebra $\mathfrak{g}$, $\rho = \frac{1}{2}(3, 1) = \omega_1 + \omega_2$, where $\omega_1 = (1, 0)$, $\omega_2 = \frac{1}{2}(1, 1)$ are the basis representations of algebra $\mathfrak{g}$. This means that $\dfrac{2\,(\omega_i, \alpha_i)}{(\alpha_i, \alpha_i)} = \delta_{ij}$, $i, j = 1, 2$ and hence $H_1 = \dfrac{2\omega_1}{(\alpha_1, \alpha_1)} = \omega_1$ and $H_2 = \dfrac{2\omega_2}{(\alpha_2, \alpha_2)} = 2\omega_2$ is the basis of subalgebra $\mathfrak{t}$ such that $(H_i, \alpha_j) = \delta_{ij}$. Then $\theta = exp(ad(H_1/2))$ defines the algebra $\mathfrak{g}_\sigma = \mathfrak{so}(3, 2)$ and $\theta = exp(ad(H_2/2))$ defines the algebra $\mathfrak{g}_\sigma = \mathfrak{so}(1, 4)$ Denote by$R^\vee$ be the root system dual to R ,that is

$$R^\vee = \left\{ \frac{2\alpha}{(\alpha, \alpha)} | \alpha \in R \right\}$$

and let W be a Weyl group of R, $P(R^\vee)$ be a group of weights for $R^\vee$ [8]. From Weyl character formula $A_\rho(H)\cdot\chi_\lambda(H)=A_{\lambda+\rho}(H)$ we derive that

$$\chi_\lambda\left(\frac{H_i}{2}\right)=\lim_{t\to 1}\frac{A_{\lambda+\rho}\left(t\dfrac{H_i}{2}\right)}{A_\rho\left(t\dfrac{H_i}{2}\right)},\quad i=1,2. \tag{2}$$

Here λ is the highest weight of the representation φ and

$$A_{\lambda+\rho}(H)=\sum_{s\in W}det(s)\cdot exp(2\pi\sqrt{-1}(s(\lambda+\rho),H)), \tag{3}$$

$$A_\rho(H)=(2\sqrt{-1})^l\prod_{\beta\in R,\beta>0}sin(\pi(\beta,H)), \tag{4}$$

where l is the number of positive roots.

Now as in [3] we can give the following definition. The elements T_1 and $T_2\in\mathfrak{t}$ are called equivalent if there exists $s\in W$ such that $s(T_1)-T_2\in P(R^\vee)$, and we shall write $T_1\equiv T_2(modP(R^\vee))$. If $T_1\equiv T_2$ then $|\chi_\lambda(T_1))|=|\chi_\lambda(T_2))|$ (see [3, 4, 5]).Furthermore

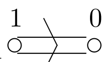
$$\frac{H_2}{2}\equiv\rho;\,\frac{H_1}{2}\equiv\frac{1}{2}(\rho+\omega_2)=\frac{1}{2}\rho^*,$$

where ρ^* is half the sum of positive roots from $R^\vee$. From (2) and (4) it follows that

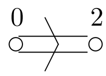
$$\chi_\lambda\left(\frac{H_2}{2}\right)=\chi_\lambda(\rho)=\lim_{t\to 1}\frac{\prod\limits_{\beta\in R,\beta>0}sin(\pi t(\beta,\lambda+\rho))}{\prod\limits_{\beta\in R,\beta>0}sin(\pi t(\beta,\rho))}, \tag{5}$$

$$\chi_\lambda\left(\frac{H_1}{2}\right)=\chi_\lambda\left(\frac{\rho^*}{2}\right)=\lim_{t\to 1}\frac{\prod\limits_{\beta^*\in R^\vee,\beta^*>0}sin\left(\left(\dfrac{\pi t}{2}\right)(\beta^*,\lambda+\rho)\right)}{\prod\limits_{\beta^*\in R^\vee,\beta^*>0}sin\left(\left(\dfrac{\pi t}{2}\right)(\beta^*,\rho)\right)}. \tag{6}$$

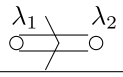
Evaluating an indeterminate form of limits (5),(6) we derive the table 1 for the value of $\chi_\lambda\left(\frac{H_1}{2}\right)$ and $\chi_\lambda\left(\frac{H_2}{2}\right)$.

From table 1 it follows that for the representation 

the character $\chi_\lambda\left(\frac{H_1}{2}\right)=1$ and hence $\varphi(\mathfrak{g}_\sigma)=\mathfrak{so}(3,2)$, $\chi_\lambda\left(\frac{H_2}{2}\right)=-3$ and hence $\varphi(\mathfrak{g}_\sigma)=\mathfrak{so}(1,4)$. Lie groups associated with these algebras are actively exploring in physics as isometry groups in de Sitter space.

Consider the adjoint representation 

. From table 1 it follows that $\chi_\lambda\left(\frac{H_1}{2}\right)=-2$ and hence $\varphi(\mathfrak{so}(3,2))\subset\mathfrak{su}(4,6)$, $\chi_\lambda\left(\frac{H_2}{2}\right)=-2$ and hence $\varphi(\mathfrak{so}(1,4))\subset\mathfrak{su}(6,4)$.That coinside with the results in [6]

Table 1. Character values for the representations 

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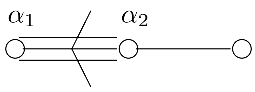
λ_1	λ_2	$\mathfrak{g}_\sigma = \mathfrak{so}(3,2)$	$\mathfrak{g}_\sigma = \mathfrak{so}(1,4)$
a	o	0	0
e	e	$(-1)^{\frac{1}{2}\lambda_2} \cdot \frac{1}{2}(\lambda_1+\lambda_2+2)$	$\frac{1}{2} \cdot (\lambda_1+1)(\lambda_1+\lambda_2+2)$
o	e	$(-1)^{\frac{1}{2}\lambda_2} \cdot \frac{1}{2}(\lambda_1+1)$	$-\frac{1}{2} \cdot (\lambda_1+1)(\lambda_1+\lambda_2+2)$

Symbol $e(o)$ in the columns λ_i denotes an even(odd) λ_i,
symbol a denotes any λ_i independent of whether it is even or odd.

4 The case $\mathfrak{g} = G_2$.

The extended Dynkin diagram for G_2 is



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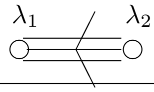
The automorphism $\theta = exp(ad(H_2/2))$ defines the real algebra G and [4, 9]

$$\frac{H_2}{2} \equiv \frac{\rho}{2}(mod P(R^\vee)).$$

Then using the same technique as in section 3 we derive the table 2 for the value of $\chi_\lambda\left(\frac{H_2}{2}\right)$.

Table 2. Character values for the representations 

λ_1	λ_2	$\mathfrak{g}_\sigma = G$
o	o	0
e	e	$\frac{1}{2} \cdot (\lambda_1+3\lambda_2+4)(\lambda_1+\lambda_2+2)$
e	o	$-\frac{1}{2} \cdot (\lambda_2+1)(2\lambda_1+3\lambda_2+5)$
o	e	$\frac{1}{2} \cdot (\lambda_2+1)(\lambda_1+2\lambda_2+3)$

Symbol $e(o)$ in the column λ_i denotes an even(odd) λ_i.

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