AFFINITY OF SEVERAL CLASSICAL AND DYNAMIC INEQUALITIES WITH COMPREHENSIVE APPROACH ON MEASURE CHAINS

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Abstract. The present research work proposes some generalized dynamic variants of reverses of Callebaut's inequality and Hölder's inequality on time scales (measure chains). Time scales (measure chains) results are the generalized forms of their discrete, continuous, and quantum versions. Further, we establish several fractional generalized dynamic variants of reverses of Callebaut's inequality and Hölder's inequality on time scales (measure chains).

1. Introduction

Here, we present some classical inequalities. The following results concerning reverses of Callebaut's inequality and Hölder's inequality are adopted from [11].

Let ψ_τ, $\omega_\tau > 0$ and $\chi_\tau \geq 0$ for any $\tau \in \{1, \ldots, n\}$ with $\sum\limits_{\tau=1}^{n} \chi_\tau = 1$.

If there exist the constants $m, M > 0$ such that

$$0 < m \leq \frac{\psi_\tau}{\omega_\tau} \leq M < +\infty,$$

for any $\tau \in \{1, \ldots, n\}$ and $\delta \in [0, 1]$, then

$$0 \leq \sum_{\tau=1}^{n} \chi_\tau \psi_\tau^2 \sum_{\tau=1}^{n} \chi_\tau \omega_\tau^2 - \sum_{\tau=1}^{n} \chi_\tau \psi_\tau^{2(1-\delta)} \omega_\tau^{2\delta} \sum_{\tau=1}^{n} \chi_\tau \psi_\tau^{2\delta} \omega_\tau^{2(1-\delta)}$$
$$\leq 2\delta(1-\delta)(M^2 - m^2) \ln\left(\frac{M}{m}\right) \left(\sum_{\tau=1}^{n} \chi_\tau \omega_\tau^2\right)^2 \quad (1)$$

and the inequality

$$\sum_{\tau=1}^{n} \chi_\tau \psi_\tau^2 \sum_{\tau=1}^{n} \chi_\tau \omega_\tau^2 \leq \exp\left[4\delta(1-\delta)\left(K\left(\left(\frac{M}{m}\right)^2\right) - 1\right)\right] \sum_{\tau=1}^{n} \chi_\tau \psi_\tau^{2(1-\delta)} \omega_\tau^{2\delta} \sum_{\tau=1}^{n} \chi_\tau \psi_\tau^{2\delta} \omega_\tau^{2(1-\delta)}. \quad (2)$$

If there exist the constants m_1, M_1, m_2, M_2 such that

$$0 < m_1 \leq \psi_\tau \leq M_1 < +\infty, \; 0 < m_2 \leq \omega_\tau \leq M_2 < +\infty,$$

for any $\tau \in \{1, \ldots, n\}$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by putting

$$M_{p,q} := \max\left\{\left(\frac{M_1}{m_1}\right)^p, \left(\frac{M_2}{m_2}\right)^q\right\},$$

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we have

$$0 \leq 1 - \frac{\sum_{\tau=1}^{n} \chi_\tau \psi_\tau \omega_\tau}{\left(\sum_{\tau=1}^{n} \chi_\tau \psi_\tau^p\right)^{\frac{1}{p}} \left(\sum_{\tau=1}^{n} \chi_\tau \omega_\tau^q\right)^{\frac{1}{q}}} \leq \frac{2}{pq} \left(\frac{M_{p,q}^2 - 1}{M_{p,q}}\right) \ln(M_{p,q}). \tag{3}$$

We will unify and extend (1), (2) and (3) in the calculus of measure chains by applying the diamond-alpha integral. We will also unify and extend (1), (2) and (3) in the fractional calculus of time scales.

2. Preliminaries

The calculus concerning measure chains was introduced by the work of Hilger [12]. A time scale is considered an arbitrary and nonempty domain of functions. This domain is the closed subset of the reals. Here, $\mathbb{T}$ denotes a time scale, and $[a,b]_{\mathbb{T}} = [a,b] \cap \mathbb{T}$, while $a, b \in \mathbb{T}$ with $a < b$. The major purpose of the calculus of measure chains is to refine results in reconciled, harmonized, generalized, and comprehensive forms. This theory is utilized to generate dynamic inequalities in hybrid forms, see [1–7, 13, 14]. The basic study concerning theory of measure chains has attracted the attention of several readers. For further study, see [7, 8].

First, we shortly adopt the diamond-α derivative and integral, see [3, 15].

Let $\Phi(\lambda)$ be Δ-differentiable and ∇-differentiable on $\mathbb{T}$. The $\diamond_\alpha$-derivative $\Phi^{\diamond_\alpha}(\lambda)$ for $\lambda \in \mathbb{T}$ is defined by

$$\Phi^{\diamond_\alpha}(\lambda) = \alpha \Phi^{\Delta}(\lambda) + (1-\alpha)\Phi^{\nabla}(\lambda), \ \alpha \in [0,1].$$

We note that $\diamond_\alpha$-derivative takes the form of Δ-derivative for $\alpha = 1$, or it becomes the ∇-derivative for $\alpha = 0$.

Let $\Phi : \mathbb{T} \to \mathbb{R}$, $\xi, \omega \in \mathbb{T}$. Then $\int_\xi^\omega \Phi(\lambda) \diamond_\alpha \lambda$ is given as

$$\int_\xi^\omega \Phi(\lambda) \diamond_\alpha \lambda = \alpha \int_\xi^\omega \Phi(\lambda) \Delta\lambda + (1-\alpha) \int_\xi^\omega \Phi(\lambda) \nabla\lambda, \ \alpha \in [0,1],$$

with the assumption of existence of delta and nabla integrals of Φ on $\mathbb{T}$.

The next definition is about Riemann–Liouville time scale Δ-fractional integral, see [4, 6].

Let $\alpha \geq 1$ and $\Phi \in C_{rd}$. Then

$$\mathcal{I}_\xi^\alpha \Phi(\kappa) = \int_\xi^\kappa h_{\alpha-1}(\kappa, \sigma(\gamma)) \Phi(\gamma) \Delta\gamma, \tag{4}$$

on $[\xi, \kappa)_{\mathbb{T}}$, see [9] and $h_\alpha : \mathbb{T}^2 \to \mathbb{R}$, are rd-continuous (coordinate wise) functions for $\alpha \geq 0$, satisfying $h_0(\kappa, \zeta) = 1$,

$$h_{\alpha+1}(\kappa, \zeta) = \int_\zeta^\kappa h_\alpha(\gamma, \zeta) \Delta\gamma, \ \forall \zeta, \kappa \in \mathbb{T}. \tag{5}$$

Notice that

$$\mathcal{I}_\xi^1 \Phi(\kappa) = \int_\xi^\kappa \Phi(\gamma) \Delta\gamma,$$

having absolute continuity in $\kappa \in [\xi, \omega]_{\mathbb{T}}$, see [9].

The next definition is about Riemann–Liouville time scale ∇-fractional integral, see [5, 6].

Let $\alpha \geq 1$ and $\Phi \in C_{ld}$. Then

$$\mathcal{J}_\xi^\alpha \Phi(\kappa) = \int_\xi^\kappa \hat{h}_{\alpha-1}(\kappa, \rho(\gamma)) \Phi(\gamma) \nabla\gamma, \tag{6}$$

on $(\xi,\kappa]_{\mathbb{T}}$, see [9] and $\hat{h}_\alpha : \mathbb{T}^2 \to \mathbb{R}$, are ld-continuous (coordinate wise) functions for $\alpha \geq 0$, satisfying $\hat{h}_0(\kappa,\zeta) = 1$,

$$\hat{h}_{\alpha+1}(\kappa,\zeta) = \int_\zeta^\kappa \hat{h}_\alpha(\gamma,\zeta)\nabla\gamma,\ \forall\zeta,\kappa \in \mathbb{T}. \tag{7}$$

Notice that

$$\mathcal{J}_\xi^1\Phi(\kappa) = \int_\xi^\kappa \Phi(\gamma)\nabla\gamma,$$

having absolute continuity in $\kappa \in [\xi,\omega]_{\mathbb{T}}$, see [9].

The famous Young's inequality for scalars says that if $\digamma, \beth > 0$ and $\delta \in [0,1]$, then

$$\digamma^{1-\delta}\beth^\delta \leq (1-\delta)\digamma + \delta\beth \tag{8}$$

with equality if and only if $\digamma = \beth$.

The upcoming results are given in [10].

$$0 \leq (1-\delta)\digamma + \delta\beth - \digamma^{1-\delta}\beth^\delta \leq \delta(1-\delta)(\digamma - \beth)(\ln\digamma - \ln\beth) \tag{9}$$

and

$$1 \leq \frac{(1-\delta)\digamma + \delta\beth}{\digamma^{1-\delta}\beth^\delta} \leq \exp\left[4\delta(1-\delta)\left(K\left(\frac{\digamma}{\beth}\right) - 1\right)\right], \tag{10}$$

where $\digamma, \beth > 0$, $\delta \in [0,1]$.

We also consider *Kantorovich's ratio* defined by

$$K(h) := \frac{(h+1)^2}{4h},\ h > 0.$$

The function K is decreasing on $(0,1)$ and increasing on $[1,+\infty)$, $K(h) \geq 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

We need the upcoming results [11].

If $\digamma, \beth \in [m,M] \subset (0,+\infty)$, then by (8) we have

$$0 \leq (1-\delta)\digamma + \delta\beth - \digamma^{1-\delta}\beth^\delta \leq \delta(1-\delta)(M-m)\ln\left(\frac{M}{m}\right) \tag{11}$$

and (9) yields

$$1 \leq \frac{(1-\delta)\digamma + \delta\beth}{\digamma^{1-\delta}\beth^\delta} \leq \exp\left[4\delta(1-\delta)\left(K\left(\frac{M}{m}\right) - 1\right)\right], \tag{12}$$

for any $\delta \in [0,1]$.

In the coming work, we assume the existence and finiteness of integrals.

3. Main Results

The first result concerning generalized reverse time scale Callebaut's dynamic inequality by applying the $\diamond_\alpha$-integral is given.

Theorem 1. *Let $\chi, \psi, \omega \in C([a,b]_{\mathbb{T}}, \mathbb{R})$, neither $\psi \equiv 0$ nor $\omega \equiv 0$, such that $0 < m \leq \frac{|\psi(\zeta)|}{|\omega(\zeta)|} \leq M < +\infty$, $\forall\zeta \in [a,b]_{\mathbb{T}}$ for some constants $m, M > 0$. If $\delta \in [0,1]$, then*

$$\begin{aligned} 0 \leq &\int_a^b |\chi(\zeta)||\psi(\zeta)|^2 \diamond_\alpha \zeta \int_a^b |\chi(\zeta)||\omega(\zeta)|^2 \diamond_\alpha \zeta \\ &- \int_a^b |\chi(\zeta)||\psi(\zeta)|^{2(1-\delta)}|\omega(\zeta)|^{2\delta} \diamond_\alpha \zeta \int_a^b |\chi(\zeta)||\psi(\zeta)|^{2\delta}|\omega(\zeta)|^{2(1-\delta)} \diamond_\alpha \zeta \\ &\leq 2\delta(1-\delta)(M^2-m^2)\ln\left(\frac{M}{m}\right)\left(\int_a^b |\chi(\zeta)||\omega(\zeta)|^2 \diamond_\alpha \zeta\right)^2. \end{aligned} \tag{13}$$

Proof. We observe that

$$m^2 \leq \frac{|\psi(\zeta)|^2}{|\omega(\zeta)|^2}, \frac{|\psi(\eta)|^2}{|\omega(\eta)|^2} \leq M^2, \ \forall \zeta, \eta \in [a,b]_{\mathbb{T}}.$$

If $F(\zeta) = \frac{|\psi(\zeta)|^2}{|\omega(\zeta)|^2}$, $\beth(\eta) = \frac{|\psi(\eta)|^2}{|\omega(\eta)|^2}$, $\forall \zeta, \eta \in [a,b]_{\mathbb{T}}$, then inequalities (8) and (11) take the forms

$$0 \leq (1-\delta)\left(\frac{|\psi(\zeta)|}{|\omega(\zeta)|}\right)^2 + \delta\left(\frac{|\psi(\eta)|}{|\omega(\eta)|}\right)^2 - \left(\frac{|\psi(\zeta)|}{|\omega(\zeta)|}\right)^{2(1-\delta)}\left(\frac{|\psi(\eta)|}{|\omega(\eta)|}\right)^{2\delta} \leq 2\delta(1-\delta)(M^2-m^2)\ln\left(\frac{M}{m}\right). \quad (14)$$

By multiplying (14) with $|\chi(\zeta)\chi(\eta)||\omega(\zeta)|^2|\omega(\eta)|^2$, we get

$$\begin{aligned} 0 \leq\ & (1-\delta)|\chi(\zeta)||\psi(\zeta)|^2|\chi(\eta)||\omega(\eta)|^2 + \delta|\chi(\zeta)||\omega(\zeta)|^2|\chi(\eta)||\psi(\eta)|^2 \\ & - |\chi(\zeta)||\psi(\zeta)|^{2(1-\delta)}|\omega(\zeta)|^{2\delta}|\chi(\eta)||\psi(\eta)|^{2\delta}|\omega(\eta)|^{2(1-\delta)} \\ & \leq 2\delta(1-\delta)(M^2-m^2)\ln\left(\frac{M}{m}\right)|\chi(\zeta)||\omega(\zeta)|^2|\chi(\eta)||\omega(\eta)|^2, \quad (15) \end{aligned}$$

$\forall \zeta, \eta \in [a,b]_{\mathbb{T}}$. By integrating (15) over ζ, we have

$$\begin{aligned} 0 \leq\ & (1-\delta)|\chi(\eta)||\omega(\eta)|^2 \int_a^b |\chi(\zeta)||\psi(\zeta)|^2 \diamond_\alpha \zeta + \delta|\chi(\eta)||\psi(\eta)|^2 \int_a^b |\chi(\zeta)||\omega(\zeta)|^2 \diamond_\alpha \zeta \\ & - |\chi(\eta)||\psi(\eta)|^{2\delta}|\omega(\eta)|^{2(1-\delta)} \int_a^b |\chi(\zeta)||\psi(\zeta)|^{2(1-\delta)}|\omega(\zeta)|^{2\delta} \diamond_\alpha \zeta \\ & \leq 2\delta(1-\delta)(M^2-m^2)\ln\left(\frac{M}{m}\right)|\chi(\eta)||\omega(\eta)|^2 \int_a^b |\chi(\zeta)||\omega(\zeta)|^2 \diamond_\alpha \zeta. \quad (16) \end{aligned}$$

Again, by integrating (16) over η, we obtain the desired inequality (13). □

Remark 1. *We have the following case for* $\mathbb{T} = \mathbb{Z}$*:*
Let $\alpha = 1$, $a = 1$, $b = n+1$, $\chi(\tau) = \chi_\tau \geq 0$, $\psi(\tau) = \psi_\tau > 0$, $\omega(\tau) = \omega_\tau > 0$ *for* $\tau \in \{1, \ldots, n\}$. *Then,* (13) *coincides with* (1).

The following reverse of Cauchy–Bunyakovsky–Schwarz's inequality holds:

Corollary 1. *Let* $\chi, \psi, \omega \in C([a,b]_{\mathbb{T}}, \mathbb{R})$, *neither* $\psi \equiv 0$ *nor* $\omega \equiv 0$, *such that* $0 < m \leq \frac{|\omega(\zeta)|}{|\psi(\zeta)|} \leq M < +\infty$, $\forall \zeta \in [a,b]_{\mathbb{T}}$ *for some constants* $m, M > 0$. *Then*

$$0 \leq \int_a^b |\chi(\zeta)||\psi(\zeta)|^2 \diamond_\alpha \zeta \int_a^b |\chi(\zeta)||\omega(\zeta)|^2 \diamond_\alpha \zeta - \left(\int_a^b |\chi(\zeta)||\psi(\zeta)||\omega(\zeta)| \diamond_\alpha \zeta\right)^2 \leq \frac{1}{2}(M^2-m^2)\ln\left(\frac{M}{m}\right)\left(\int_a^b |\chi(\zeta)||\omega(\zeta)|^2 \diamond_\alpha \zeta\right)^2. \quad (17)$$

Proof. Take $\delta = \frac{1}{2}$ in inequality (13) and the proof holds. □

The second result concerning generalized reverse Callebaut's dynamic inequality on time scales by applying the $\diamond_\alpha$-integral is given.

Theorem 2. *Let $\chi, \psi, \omega \in C([a,b]_{\mathbb{T}}, \mathbb{R})$, neither $\psi \equiv 0$ nor $\omega \equiv 0$, such that $0 < m \leq \frac{|\psi(\zeta)|}{|\omega(\zeta)|} \leq M < +\infty$, $\forall \zeta \in [a,b]_{\mathbb{T}}$ for some constants $m, M > 0$. If $\delta \in [0,1]$, then we have the following inequality*

$$\begin{aligned}\int_a^b |\chi(\zeta)||\psi(\zeta)|^2 \diamond_\alpha \zeta \int_a^b |\chi(\zeta)||\omega(\zeta)|^2 \diamond_\alpha \zeta \\ \leq \exp\left[4\delta(1-\delta)\left(K\left(\left(\frac{M}{m}\right)^2\right)-1\right)\right] \int_a^b |\chi(\zeta)||\psi(\zeta)|^{2(1-\delta)}|\omega(\zeta)|^{2\delta} \diamond_\alpha \zeta \\ \times \int_a^b |\chi(\zeta)||\psi(\zeta)|^{2\delta}|\omega(\zeta)|^{2(1-\delta)} \diamond_\alpha \zeta. \quad (18)\end{aligned}$$

Proof. We observe that

$$m^2 \leq \frac{|\psi(\zeta)|^2}{|\omega(\zeta)|^2}, \frac{|\psi(\eta)|^2}{|\omega(\eta)|^2} \leq M^2, \ \forall \zeta, \eta \in [a,b]_{\mathbb{T}}.$$

If $F(\zeta) = \frac{|\psi(\zeta)|^2}{|\omega(\zeta)|^2}$, $\beth(\eta) = \frac{|\psi(\eta)|^2}{|\omega(\eta)|^2}$, $\forall \zeta, \eta \in [a,b]_{\mathbb{T}}$, then inequality (12) take the forms

$$\begin{aligned}(1-\delta)\left(\frac{|\psi(\zeta)|}{|\omega(\zeta)|}\right)^2 + \delta\left(\frac{|\psi(\eta)|}{|\omega(\eta)|}\right)^2 \\ \leq \exp\left[4\delta(1-\delta)\left(K\left(\left(\frac{M}{m}\right)^2\right)-1\right)\right]\left(\frac{|\psi(\zeta)|}{|\omega(\zeta)|}\right)^{2(1-\delta)}\left(\frac{|\psi(\eta)|}{|\omega(\eta)|}\right)^{2\delta}. \quad (19)\end{aligned}$$

By multiplying (19) with $|\chi(\zeta)\chi(\eta)||\omega(\zeta)|^2|\omega(\eta)|^2$, we get

$$\begin{aligned}(1-\delta)|\chi(\zeta)||\psi(\zeta)|^2|\chi(\eta)||\omega(\eta)|^2 + \delta|\chi(\zeta)||\omega(\zeta)|^2|\chi(\eta)||\psi(\eta)|^2 \\ \leq \exp\left[4\delta(1-\delta)\left(K\left(\left(\frac{M}{m}\right)^2\right)-1\right)\right]\chi(\zeta)||\psi(\zeta)|^{2(1-\delta)}|\omega(\zeta)|^{2\delta}|\chi(\eta)||\psi(\eta)|^{2\delta}|\omega(\eta)|^{2(1-\delta)}, \quad (20)\end{aligned}$$

$\forall \zeta, \eta \in [a,b]_{\mathbb{T}}$. By integrating (20) over ζ, we have

$$\begin{aligned}(1-\delta)|\chi(\eta)||\omega(\eta)|^2 \int_a^b |\chi(\zeta)||\psi(\zeta)|^2 \diamond_\alpha \zeta + \delta|\chi(\eta)||\psi(\eta)|^2 \int_a^b |\chi(\zeta)||\omega(\zeta)|^2 \diamond_\alpha \zeta \\ \leq \exp\left[4\delta(1-\delta)\left(K\left(\left(\frac{M}{m}\right)^2\right)-1\right)\right] \\ \times \chi(\eta)||\psi(\eta)|^{2\delta}|\omega(\eta)|^{2(1-\delta)} \int_a^b |\chi(\zeta)||\psi(\zeta)|^{2(1-\delta)}|\omega(\zeta)|^{2\delta} \diamond_\alpha \zeta. \quad (21)\end{aligned}$$

Again, by integrating (21) over η, we obtain the desired inequality (18). □

Remark 2. *We have the following case for $\mathbb{T} = \mathbb{Z}$:*
Let $\alpha = 1$, $a = 1$, $b = n+1$, $\chi(\tau) = \chi_\tau \geq 0$, $\psi(\tau) = \psi_\tau > 0$, $\omega(\tau) = \omega_\tau > 0$ for $\tau \in \{1, \ldots, n\}$. Then, (18) *coincides with* (2).

The following reverse of Cauchy–Bunyakovsky–Schwarz's inequality holds:

Corollary 2. *Let $\chi, \psi, \omega \in C([a,b]_{\mathbb{T}}, \mathbb{R})$, neither $\psi \equiv 0$ nor $\omega \equiv 0$, such that $0 < m \leq \frac{|\omega(\zeta)|}{|\psi(\zeta)|} \leq M < +\infty$, $\forall \zeta \in [a,b]_{\mathbb{T}}$ for some constants $m, M > 0$. Then*

$$\int_a^b |\chi(\zeta)||\psi(\zeta)|^2 \diamond_\alpha \zeta \int_a^b |\chi(\zeta)||\omega(\zeta)|^2 \diamond_\alpha \zeta \\ \leq \exp\left[K\left(\left(\frac{M}{m}\right)^2\right) - 1\right]\left(\int_a^b |\chi(\zeta)||\psi(\zeta)||\omega(\zeta)| \diamond_\alpha \zeta\right)^2. \quad (22)$$

Proof. Take $\delta = \frac{1}{2}$ in inequality (18) and the proof holds. □

The third result concerning generalized reverse Hölder's dynamic inequality on time scales by applying the $\diamond_\alpha$-integral is given.

Theorem 3. *Let $\chi, \psi, \omega \in C([a,b]_{\mathbb{T}}, \mathbb{R})$, neither $\psi \equiv 0$ nor $\omega \equiv 0$ with $\int_a^b |\chi(\zeta)| \diamond_\alpha \zeta = 1$. If there exist the constants m_1, M_1, m_2, M_2 such that*

$$0 < m_1 \leq \psi(\zeta) \leq M_1 < +\infty, \ 0 < m_2 \leq \omega(\zeta) \leq M_2 < +\infty, \ \forall \zeta \in [a,b]_{\mathbb{T}}$$

and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by putting

$$M_{p,q} := \max\left\{\left(\frac{M_1}{m_1}\right)^p, \left(\frac{M_2}{m_2}\right)^q\right\}$$

we have

$$0 \leq 1 - \frac{\int_a^b |\chi(\zeta)||\psi(\zeta)\omega(\zeta)| \diamond_\alpha \zeta}{\left(\int_a^b |\chi(\zeta)||\psi(\zeta)|^p \diamond_\alpha \zeta\right)^{\frac{1}{p}} \left(\int_a^b |\chi(\zeta)||\omega(\zeta)|^q \diamond_\alpha \zeta\right)^{\frac{1}{q}}} \leq \frac{2}{pq}\left(\frac{M_{p,q}^2 - 1}{M_{p,q}}\right)\ln(M_{p,q}). \quad (23)$$

Proof. Using the given conditions, we have

$$m_1^p \leq \int_a^b |\chi(\zeta)||\psi(\zeta)|^p \diamond_\alpha \zeta \leq M_1^p \text{ and } m_2^q \leq \int_a^b |\chi(\zeta)||\omega(\zeta)|^p \diamond_\alpha \zeta \leq M_2^q.$$

Also

$$\left(\frac{m_1}{M_1}\right)^p \leq \frac{|\psi(\zeta)|^p}{\int_a^b |\chi(\zeta)||\psi(\zeta)|^p \diamond_\alpha \zeta} \leq \left(\frac{M_1}{m_1}\right)^p$$

and

$$\left(\frac{m_2}{M_2}\right)^q \leq \frac{|\omega(\zeta)|^q}{\int_a^b |\chi(\zeta)||\omega(\zeta)|^q \diamond_\alpha \zeta} \leq \left(\frac{M_2}{m_2}\right)^q$$

giving that

$$m_{p,q} \leq \frac{|\psi(\zeta)|^p}{\int_a^b |\chi(\zeta)||\psi(\zeta)|^p \diamond_\alpha \zeta}, \frac{|\omega(\zeta)|^q}{\int_a^b |\chi(\zeta)||\omega(\zeta)|^q \diamond_\alpha \zeta} \leq M_{p,q},$$

where

$$m_{p,q} := \min\left\{\left(\frac{m_1}{M_1}\right)^p, \left(\frac{m_2}{M_2}\right)^q\right\} = \min\left\{\frac{1}{\left(\frac{M_1}{m_1}\right)^p}, \frac{1}{\left(\frac{M_2}{m_2}\right)^q}\right\} = \frac{1}{\min\left\{\left(\frac{M_1}{m_1}\right)^p, \left(\frac{M_2}{m_2}\right)^q\right\}} = \frac{1}{M_{p,q}}.$$

Using the inequality (11) for $\delta = \frac{1}{q}$, $\digamma(\zeta) = \frac{|\psi(\zeta)|^p}{\int_a^b |\chi(\zeta)||\psi(\zeta)|^p \diamond_\alpha \zeta}$, $\beth(\zeta) = \frac{|\omega(\zeta)|^q}{\int_a^b |\chi(\zeta)||\omega(\zeta)|^q \diamond_\alpha \zeta}$, $m = m_{p,q}$ and $M = M_{p,q}$, we get

$$\begin{aligned} &0 \\ &\leq \frac{1}{p}\frac{|\psi(\zeta)|^p}{\int_a^b |\chi(\zeta)||\psi(\zeta)|^p \diamond_\alpha \zeta} + \frac{1}{q}\frac{|\omega(\zeta)|^q}{\int_a^b |\chi(\zeta)||\omega(\zeta)|^q \diamond_\alpha \zeta} - \frac{|\psi(\zeta)\omega(\zeta)|}{\left(\int_a^b |\chi(\zeta)||\psi(\zeta)|^p \diamond_\alpha \zeta\right)^{\frac{1}{p}} \left(\int_a^b |\chi(\zeta)||\omega(\zeta)|^q \diamond_\alpha \zeta\right)^{\frac{1}{q}}} \\ &\leq \frac{2}{pq}\left(M_{p,q} - \frac{1}{M_{p,q}}\right)\ln(M_{p,q}). \end{aligned} \tag{24}$$

By integrating (24) over ζ, we obtain the desired inequality (23). □

Remark 3. *We have the following case for $\mathbb{T} = \mathbb{Z}$:*
Let $\alpha = 1$, $a = 1$, $b = n + 1$, $\chi(\tau) = \chi_\tau \geq 0$, $\psi(\tau) = \psi_\tau > 0$, $\omega(\tau) = \omega_\tau > 0$ for $\tau \in \{1, \ldots, n\}$. Then, (23) *coincides with* (3).

4. Fractional Inequalities

The first result concerning generalized fractional reverse time scale Callebaut's dynamic inequality by applying the Δ-Riemann–Liouville integral is given to initiate this section.

Theorem 4. *Let $\chi, \psi, \omega \in C_{rd}([a, b]_\mathbb{T}, \mathbb{R})$, neither $\psi \equiv 0$ nor $\omega \equiv 0$, such that $0 < m \leq \frac{|\psi(\eta)|}{|\omega(\eta)|} \leq M < +\infty$ for $\eta \in [a, \kappa]_\mathbb{T}$, while $\forall \kappa \in [a, b]_\mathbb{T}$ for some constants $m, M > 0$. If $\delta \in [0, 1]$, then for $\alpha \geq 1$ and $h_{\alpha-1}(., .) > 0$, one has*

$$\begin{aligned} 0 \leq\ & \mathcal{I}_a^\alpha(|\chi(\kappa)||\psi(\kappa)|^2)\mathcal{I}_a^\alpha(|\chi(\kappa)||\omega(\kappa)|^2) \\ & - \mathcal{I}_a^\alpha(|\chi(\kappa)||\psi(\kappa)|^{2(1-\delta)}|\omega(\kappa)|^{2\delta})\mathcal{I}_a^\alpha(|\chi(\kappa)||\psi(\kappa)|^{2\delta}|\omega(\kappa)|^{2(1-\delta)}) \\ & \leq 2\delta(1-\delta)(M^2 - m^2)\ln\left(\frac{M}{m}\right)\left(\mathcal{I}_a^\alpha(|\chi(\kappa)||\omega(\kappa)|^2)\right)^2. \end{aligned} \tag{25}$$

Proof. We observe that

$$m^2 \leq \frac{|\psi(\eta)|^2}{|\omega(\eta)|^2}, \frac{|\psi(\lambda)|^2}{|\omega(\lambda)|^2} \leq M^2, \ \eta, \lambda \in [a, \kappa]_\mathbb{T}, \forall \kappa \in [a, b]_\mathbb{T}.$$

If $\digamma(\eta) = \frac{|\psi(\eta)|^2}{|\omega(\eta)|^2}$, $\beth(\lambda) = \frac{|\psi(\lambda)|^2}{|\omega(\lambda)|^2}$, $\eta, \lambda \in [a, \kappa]_\mathbb{T}$, $\forall \kappa \in [a, b]_\mathbb{T}$, then inequalities (8) and (11) take the forms

$$\begin{aligned} 0 \leq (1-\delta)\left(\frac{|\psi(\eta)|}{|\omega(\eta)|}\right)^2 + \delta\left(\frac{|\psi(\lambda)|}{|\omega(\lambda)|}\right)^2 - \left(\frac{|\psi(\eta)|}{|\omega(\eta)|}\right)^{2(1-\delta)}\left(\frac{|\psi(\lambda)|}{|\omega(\lambda)|}\right)^{2\delta} \\ \leq 2\delta(1-\delta)(M^2 - m^2)\ln\left(\frac{M}{m}\right). \end{aligned} \tag{26}$$

By multiplying (26) with $h_{\alpha-1}(\kappa, \sigma(\eta))h_{\alpha-1}(\kappa, \sigma(\lambda))|\chi(\eta)\chi(\lambda)||\omega(\eta)|^2|\omega(\lambda)|^2$, we get

$$\begin{aligned} 0 \leq\ & (1-\delta)h_{\alpha-1}(\kappa, \sigma(\eta))|\chi(\eta)||\psi(\eta)|^2 h_{\alpha-1}(\kappa, \sigma(\lambda))|\chi(\lambda)||\omega(\lambda)|^2 \\ & + \delta h_{\alpha-1}(\kappa, \sigma(\eta))|\chi(\eta)||\omega(\eta)|^2 h_{\alpha-1}(\kappa, \sigma(\lambda))|\chi(\lambda)||\psi(\lambda)|^2 \\ & - h_{\alpha-1}(\kappa, \sigma(\eta))|\chi(\eta)||\psi(\eta)|^{2(1-\delta)}|\omega(\eta)|^{2\delta} h_{\alpha-1}(\kappa, \sigma(\lambda))|\chi(\lambda)||\psi(\lambda)|^{2\delta}|\omega(\lambda)|^{2(1-\delta)} \\ & \leq 2\delta(1-\delta)(M^2 - m^2)\ln\left(\frac{M}{m}\right) h_{\alpha-1}(\kappa, \sigma(\eta))|\chi(\eta)||\omega(\eta)|^2 h_{\alpha-1}(\kappa, \sigma(\lambda))|\chi(\lambda)||\omega(\lambda)|^2, \end{aligned} \tag{27}$$

$\eta, \lambda \in [a,\kappa]_{\mathbb{T}}$, $\forall \kappa \in [a,b]_{\mathbb{T}}$. By integrating (6) over η, we have

$$\begin{aligned}
0 \leq (1-\delta) h_{\alpha-1}(\kappa,\sigma(\lambda))|\chi(\lambda)||\omega(\lambda)|^2 \mathcal{I}_a^\alpha(|\chi(\kappa)||\psi(\kappa)|^2) \\
+ \delta h_{\alpha-1}(\kappa,\sigma(\lambda))|\chi(\lambda)||\psi(\lambda)|^2 \mathcal{I}_a^\alpha(|\chi(\kappa)||\omega(\kappa)|^2) \\
- h_{\alpha-1}(\kappa,\sigma(\lambda))|\chi(\lambda)||\psi(\lambda)|^{2\delta}|\omega(\lambda)|^{2(1-\delta)} \mathcal{I}_a^\alpha(|\chi(\kappa)||\psi(\kappa)|^{2(1-\delta)}|\omega(\kappa)|^{2\delta}) \\
\leq 2\delta(1-\delta)(M^2-m^2)\ln\left(\frac{M}{m}\right) h_{\alpha-1}(\kappa,\sigma(\lambda))|\chi(\lambda)||\omega(\lambda)|^2 \mathcal{I}_a^\alpha(|\chi(\kappa)||\omega(\kappa)|^2). \quad (28)
\end{aligned}$$

Again, by integrating (28) over λ, we obtain the desired inequality (25). □

The next result concerning generalized fractional reverse time scale Callebaut's dynamic inequality by applying the ∇-Riemann–Liouville integral is given to initiate this section.

Theorem 5. *Let $\chi, \psi, \omega \in C_{ld}([a,b]_{\mathbb{T}}, \mathbb{R})$, neither $\psi \equiv 0$ nor $\omega \equiv 0$, such that $0 < m \leq \frac{|\psi(\eta)|}{|\omega(\eta)|} \leq M < +\infty$ for $\eta \in [a,\kappa]_{\mathbb{T}}$, while $\forall \kappa \in [a,b]_{\mathbb{T}}$ for some constants $m, M > 0$. If $\delta \in [0,1]$, then for $\alpha \geq 1$ and $\hat{h}_{\alpha-1}(.,.) > 0$, one has*

$$\begin{aligned}
0 \leq \mathcal{J}_a^\alpha(|\chi(\kappa)||\psi(\kappa)|^2)\mathcal{J}_a^\alpha(|\chi(\kappa)||\omega(\kappa)|^2) \\
- \mathcal{J}_a^\alpha(|\chi(\kappa)||\psi(\kappa)|^{2(1-\delta)}|\omega(\kappa)|^{2\delta})\mathcal{J}_a^\alpha(|\chi(\kappa)||\psi(\kappa)|^{2\delta}|\omega(\kappa)|^{2(1-\delta)}) \\
\leq 2\delta(1-\delta)(M^2-m^2)\ln\left(\frac{M}{m}\right)\left(\mathcal{J}_a^\alpha(|\chi(\kappa)||\omega(\kappa)|^2)\right)^2. \quad (29)
\end{aligned}$$

Proof. Similar to that of proof of the Theorem 4. □

Remark 4. *We have the following case for $\mathbb{T} = \mathbb{Z}$:*
Let $\alpha = 1$, $a = 1$, $\kappa = b = n+1$, $\chi(\tau) = \chi_\tau \geq 0$, $\psi(\tau) = \psi_\tau > 0$, $\omega(\tau) = \omega_\tau > 0$ for $\tau \in \{1, \ldots, n\}$. Then, (25) coincides with (1).

The second result concerning generalized fractional reverse Callebaut's dynamic inequality on time scales by applying the Δ-Riemann–Liouville integral is given.

Theorem 6. *Let $\chi, \psi, \omega \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R})$, neither $\psi \equiv 0$ nor $\omega \equiv 0$, such that $0 < m \leq \frac{|\psi(\eta)|}{|\omega(\eta)|} \leq M < +\infty$ for $\eta \in [a,\kappa]_{\mathbb{T}}$, while $\forall \kappa \in [a,b]_{\mathbb{T}}$ for some constants $m, M > 0$. If $\delta \in [0,1]$, then for $\alpha \geq 1$ and $h_{\alpha-1}(.,.) > 0$, we have the following inequality*

$$\begin{aligned}
\mathcal{I}_a^\alpha(|\chi(\kappa)||\psi(\kappa)|^2)\mathcal{I}_a^\alpha(|\chi(\kappa)||\omega(\kappa)|^2) \\
\leq \exp\left[4\delta(1-\delta)\left(K\left(\left(\frac{M}{m}\right)^2\right)-1\right)\right]\mathcal{I}_a^\alpha(|\chi(\kappa)||\psi(\kappa)|^{2(1-\delta)}|\omega(\kappa)|^{2\delta}) \\
\times \mathcal{I}_a^\alpha(|\chi(\kappa)||\psi(\kappa)|^{2\delta}|\omega(\kappa)|^{2(1-\delta)}). \quad (30)
\end{aligned}$$

Proof. Similar to that of proof of the Theorem 2. □

The next result concerning generalized fractional reverse Callebaut's dynamic inequality on time scales by applying the ∇-Riemann–Liouville integral is given.

Theorem 7. *Let $\chi, \psi, \omega \in C_{ld}([a,b]_{\mathbb{T}}, \mathbb{R})$, neither $\psi \equiv 0$ nor $\omega \equiv 0$, such that $0 < m \leq \frac{|\psi(\eta)|}{|\omega(\eta)|} \leq M < +\infty$ for $\eta \in [a,\kappa]_{\mathbb{T}}$, while $\forall \kappa \in [a,b]_{\mathbb{T}}$ for some constants $m, M > 0$. If $\delta \in [0,1]$, then for*

$\alpha \geq 1$ and $\hat{h}_{\alpha-1}(.,.) > 0$, we have the following inequality

$$\mathcal{J}_a^{\alpha}(|\chi(\kappa)||\psi(\kappa)|^2)\mathcal{J}_a^{\alpha}(|\chi(\kappa)||\omega(\kappa)|^2) \leq \exp\left[4\delta(1-\delta)\left(K\left(\left(\frac{M}{m}\right)^2\right)-1\right)\right]\mathcal{J}_a^{\alpha}(|\chi(\kappa)||\psi(\kappa)|^{2(1-\delta)}|\omega(\kappa)|^{2\delta}) \times \mathcal{J}_a^{\alpha}(|\chi(\kappa)||\psi(\kappa)|^{2\delta}|\omega(\kappa)|^{2(1-\delta)}). \quad (31)$$

Proof. Similar to that of proof of the Theorem 2. □

The third result concerning generalized fractional reverse Hölder's dynamic inequality on time scales by applying the Δ-Riemann–Liouville integral is given.

Theorem 8. *Let $\chi, \psi, \omega \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R})$, neither $\psi \equiv 0$ nor $\omega \equiv 0$. If there exist the constants m_1, M_1, m_2, M_2 such that*

$$0 < m_1 \leq \psi(\eta) \leq M_1 < +\infty, \; 0 < m_2 \leq \omega(\eta) \leq M_2 < +\infty, \; \eta \in [a,\kappa]_{\mathbb{T}}, \forall \kappa \in [a,b]_{\mathbb{T}}$$

and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by putting

$$M_{p,q} := \max\left\{\left(\frac{M_1}{m_1}\right)^p, \left(\frac{M_2}{m_2}\right)^q\right\}$$

and for $\alpha \geq 1$, $h_{\alpha-1}(.,.) > 0$, we have the following inequality

$$0 \leq \mathcal{I}_a^{\alpha}(|\chi(\kappa)|)\left[1 - \frac{\mathcal{I}_a^{\alpha}(|\chi(\kappa)||\psi(\kappa)\omega(\kappa)|)}{\left(\mathcal{I}_a^{\alpha}(|\chi(\kappa)||\psi(\kappa)|^p)\right)^{\frac{1}{p}}\left(\mathcal{I}_a^{\alpha}(|\chi(\kappa)||\omega(\kappa)|^q)\right)^{\frac{1}{q}}}\right] \leq \frac{2}{pq}\left(\frac{M_{p,q}^2 - 1}{M_{p,q}}\right)\ln(M_{p,q}). \quad (32)$$

Proof. Similar to that of proof of the Theorem 3. □

The next result concerning generalized fractional reverse Hölder's dynamic inequality on time scales by applying the ∇-Riemann–Liouville integral is given.

Theorem 9. *Let $\chi, \psi, \omega \in C_{ld}([a,b]_{\mathbb{T}}, \mathbb{R})$, neither $\psi \equiv 0$ nor $\omega \equiv 0$. If there exist the constants m_1, M_1, m_2, M_2 such that*

$$0 < m_1 \leq \psi(\eta) \leq M_1 < +\infty, \; 0 < m_2 \leq \omega(\eta) \leq M_2 < +\infty, \; \eta \in [a,\kappa]_{\mathbb{T}}, \forall \kappa \in [a,b]_{\mathbb{T}}$$

and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by putting

$$M_{p,q} := \max\left\{\left(\frac{M_1}{m_1}\right)^p, \left(\frac{M_2}{m_2}\right)^q\right\}$$

and for $\alpha \geq 1$, $\hat{h}_{\alpha-1}(.,.) > 0$, we have the following inequality

$$0 \leq \mathcal{J}_a^{\alpha}(|\chi(\kappa)|)\left[1 - \frac{\mathcal{J}_a^{\alpha}(|\chi(\kappa)||\psi(\kappa)\omega(\kappa)|)}{\left(\mathcal{J}_a^{\alpha}(|\chi(\kappa)||\psi(\kappa)|^p)\right)^{\frac{1}{p}}\left(\mathcal{J}_a^{\alpha}(|\chi(\kappa)||\omega(\kappa)|^q)\right)^{\frac{1}{q}}}\right] \leq \frac{2}{pq}\left(\frac{M_{p,q}^2 - 1}{M_{p,q}}\right)\ln(M_{p,q}). \quad (33)$$

Proof. Similar to that of proof of the Theorem 3. □

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