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NEW IDENTITY FOR COSINE FUNCTION AND GENERALIZED BERNOULLI NUMBERS

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Abstract: For a prime number $p \geq 3$ we let $G(p) = \langle \overline{x} \rangle$ denote the group of reduced residue classes modulo p and we let $\widehat{G}(p) = \{\chi_0, \chi_1, \ldots, \chi_{p-2}\}$ denote the group of Dirichlet characters modulo p. Let l and ν_l be integers such that $\gcd(p, l) = 1$ and $\bar{l} = \overline{x}^{\nu_l}$. The main purpose of this paper is to present an explicit formula for the sum:

$$\sum_{a=0}^{p-2} |B_1(\chi_a)|^2 \cos\left(\frac{2\pi a \nu_l}{p-1}\right),$$

where $B_m(\chi)$ $(m \geq 0)$ are the generalized Bernoulli numbers associated with χ.

Keywords: cosine sum, generalized Bernoulli number, character sum, reduced residue classe, Dirichlet L-function.

1 Introduction and Main Result

Let $p \geq 3$ be a prime. Throughout this paper, we let $G(p) = \langle \overline{x} \rangle = \left\{\overline{1}, \overline{x}, \ldots, \overline{x}^{p-2}\right\}$ denote the cyclic group of reduced residue classes modulo p, and we let $\widehat{G}(p) = \langle \chi_1 \rangle = \{\chi_0, \chi_1, \ldots, \chi_{p-2}\}$ denote the cyclic group of Dirichlet characters modulo p.

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Let χ be a Dirichlet character modulo $k \geq 3$. Then the generalized Bernoulli numbers $B_m(\chi)$ $(m = 0, 1, 2, \ldots)$ are defined by:

$$\sum_{j=1}^{k} \chi(j) \frac{ze^{jz}}{e^{kz} - 1} = \sum_{m=0}^{\infty} \frac{B_m(\chi)}{m!} z^m, \quad |z| < \frac{2\pi}{k}.$$

In particular,

$$B_1(\chi) = \frac{1}{k} \sum_{a=1}^{k} a\chi(a).$$

It is well known (see e.g., [2, Proposition 4.5]) that:

$$\text{If } \chi(-1) = +1 \ (\chi \text{ is even}) \text{ and } m \equiv 1 \pmod 2, \text{ then } B_m(\chi) = 0, \tag{1}$$

$$\text{If } \chi(-1) = -1 \ (\chi \text{ is odd}) \text{ and } m \equiv 0 \pmod 2, \text{ then } B_m(\chi) = 0. \tag{2}$$

Generalized Bernoulli numbers have piqued the interest of many mathematicians who have delved into the study of sums and products related to them. This area of research explores various properties and mathematical relationships involving these numbers. The investigation of these sums and products has led to valuable insights in number theory and other mathematical disciplines. For instance, Chen and Eie [3, Proposition 7] provided a closed expression for the sums of products of generalized Bernoulli numbers. In another research, the author and Derbal [8, Theorem 3.8] provided explicit formulas for sums related to generalized Bernoulli numbers associated with primitive Dirichlet characters.

In a related context, numerous research papers have concentrated on studying finite -or not- sums involving the cosine function. Countless research papers have delved into investigating the properties, convergence, and closed-form expressions of these summations. The study of such cosine-related series has proven to be valuable in diverse fields, including calculus, complex analysis, number theory, and mathematical physics. For example Rees and Stanojevic [9] provided both necessary and sufficient conditions for the integrability of specific cosine sums. In a separate paper, Merca [7] derived formulas for various power sums of cosine functions and also obtained several combinatorial identities related to them.

In this paper, and for the first time to the best of our knowledge, we will focus on studying sums related to generalized Bernoulli numbers related to Dirichlet characters modulo a prime p and the cosine function simultaneously.

Let $p \geq 3$ be a prime. Let $m \geq 1$, l, and ν_l be integers such that $\gcd(p, l) = 1$ and $\overline{l} = \overline{x}^{\nu_l} \in G(p)$. Set

$$T(p, m, l) := \sum_{a=0}^{p-2} |B_m(\chi)|^2 \cos\left(\frac{2\pi a \nu_l}{p-1}\right).$$

According to Formulas (1) and (2), it easy to see that the sum is restricted over those even $a \leq p-2$ if m and χ are even, and restricted over those odd $a \leq p-2$ if m and χ are odd. It is the main purpose of this paper is to present an explicit formula for $T(p,1,l)$. Our main formula is the following:

Theorem 1. *Let $p \geq 3$ be a prime. Let l and ν_l be integers such that $\gcd(p,l)=1$ and $\overline{l}=\overline{x}^{\nu_l} \in G(p)$. Then*

$$T(p,1,l)=\frac{p-1}{12lp}\Big(p^2-3(l+S(l,p))p+l^2+1\Big), \tag{3}$$

where $S(l,p)$, which depends only on $p \mod l$, is defined by:

$$S(l,p)=\sum_{a=1}^{l-1}\cot\left(\frac{\pi a}{l}\right)\cot\left(\frac{\pi a p}{l}\right).$$

Example 1. *Let $p \geq 3$ be a prime. Then*

$$T(p,1,1)=\sum_{\substack{\chi \mod p\\ \chi(-1)=-1}}|B_1(\chi)|^2=\frac{(p-1)^2(p-2)}{12p}.$$

Example 2. *Let $p \geq 3$ be a prime. Then*

$$T(p,1,2)=\frac{(p-1)^2(p-5)}{24p}.$$

Example 3. *Let $p \geq 3$ be a prime. Then*

$$T(p,1,3)=\begin{cases}\dfrac{(p-1)^2(p-10)}{36p}, & \textit{if } p\equiv 1 \pmod 3;\\[2mm] \dfrac{(p-1)(p^2-7p+10)}{36p}, & \textit{if } p\equiv 2 \pmod 3.\end{cases}$$

Example 4. *Let $p \geq 3$ be a prime. Then*

$$T(p,1,4)=\begin{cases}\dfrac{(p-1)^2(p-17)}{48p}, & \textit{if } p\equiv 1 \pmod 4;\\[2mm] \dfrac{(p-1)(p^2-6p+17)}{48p}, & \textit{if } p\equiv 3 \pmod 4.\end{cases}$$

2 Proof of Theorem 1

Let $k \geq 3$ and l be integers with $\gcd(l,k)=1$. Let χ be a Dirichlet character modulo k and let $L(s,\chi)=\sum_{n=1}^{+\infty}\frac{\chi(n)}{n^s}$, $(\Re(s)>1)$ be the Dirichlet L-function corresponding to χ. Set

$$M(k,l,m,n):=\frac{2}{\varphi(k)}\sum_{\substack{\chi \mod k\\ \chi(-1)=(-1)^m=(-1)^n}}\chi(l)L(m,\chi)L(n,\overline{\chi}),$$

where φ is the Euler totient function

Many researchers have been interested in evaluating and giving the mean values of $M(k,l,m,n)$ for different values of k, l, m, and n (see e.g., [4, 5, 6, 10]).

Let $p \geq 3$ be a prime and let m be an odd integer. The following theorem gives an explicit formula for $M(p,l,m,m)$.

Theorem 2. *Let $p \geq 3$ be a prime and let m be an odd integer. Let l and ν_l be integers such that $\gcd(p,l) = 1$ and $\bar{l} = \overline{x}^{\nu_l} \in G(p)$. Then*

$$M(p,l,m,m) = \frac{(2\pi)^{2m}}{2(p-1)(m!)^2 p^{2m-1}} \sum_{a=0}^{p-2} |B_m(\chi_a)|^2 \cos\left(\frac{2\pi a \nu_l}{p-1}\right). \tag{4}$$

In order to prove Theorem 2 we need the following lemmas.

Lemma 1. *Let $p \geq 3$ be a prime and let $\chi_j, \chi_{j'} \in \widehat{G}(p)$ $(0 \leq j, j' \leq p-2)$. Then*

(1) *The character χ_j is defined by:*

$$\chi_j(n) = [\chi_1(n)]^j = \begin{cases} \exp\left(i\frac{2j\nu\pi}{p-1}\right), & \text{if } \overline{n} = \overline{x}^{\nu} \in G(p); \\ 0, & \text{otherwise.} \end{cases}$$

(2) *If j is odd, then χ_j is odd character.*
(3) *If $j + j' = p-1$, then χ_j is conjugate to $\chi_{j'}$.*

Proof. For the first and the second item (see e.g., [1]).
Now, let us prove the third item. Suppose that $j + j' = p-1$ and let $n \in \mathbb{Z}$ such that $\gcd(n,p) = 1$ and $\overline{n} = \overline{x}^{\nu} \in G(p)$. Then

$$\begin{aligned} \chi_j(n)\chi_{j'}(n) &= \exp\left(i\frac{2j\nu\pi}{p-1}\right)\exp\left(i\frac{2j'\nu\pi}{p-1}\right) \\ &= \exp\left(i\frac{2j\nu\pi}{p-1}\right)\exp\left(i\frac{2(p-1-j)\nu\pi}{p-1}\right) \\ &= \exp(i2\nu\pi) = 1, \end{aligned}$$

which means that $\overline{\chi_j} = \chi_{p-1-j} = \chi_{j'}$.
This completes the proof of the lemma. □

Lemma 2. *Let χ be a primitive character modulo $k \geq 3$ and let $\tau(\chi) = \sum_{a=1}^{k} \chi(a)\exp(\frac{2\pi i a}{k})$ be the Gaussian sum associated with χ. Then*

(1) $|\tau(\chi)| = \sqrt{k}$.
(2) $\tau(\overline{\chi}) = \chi(-1)\overline{\tau(\chi)}$.

Proof. (1) For the first item, see e.g., [1, Theorem 8.15].
(2) According to [1, Theorem 8.15] we know that:

$$\chi(n)\tau(\overline{\chi}) = \sum_{a=1}^{k} \overline{\chi}(a)\exp\left(\frac{2\pi i a n}{k}\right), \text{ for every integer } n.$$

It follows by taking $n = -1$ that:

$$\begin{aligned}\chi(-1)\tau(\overline{\chi}) &= \sum_{a=1}^{k} \overline{\chi}(a) \exp\left(\frac{-2\pi i a}{k}\right) \\ &= \overline{\sum_{a=1}^{k} \chi(a) \exp\left(\frac{2\pi i a}{k}\right)} \\ &= \overline{\tau(\chi)}.\end{aligned}$$

The proof is complete. □

As a consequence of Lemma 2, if χ is an odd primitive character modulo $k \geq 3$, then we have

$$\tau(\chi)\tau(\overline{\chi}) = \chi(-1)|\tau(\chi)|^2 = -k. \tag{5}$$

Now we prove Theorem 2.

Proof of Theorem 2. From [2, Theorem 9.6], if χ is a primitive character modulo p with $\chi(-1) = (-1)^m$ $(m \geq 1)$, then

$$L(m, \chi) = (-1)^{m-1} \frac{\tau(\chi)}{2m!} \left(\frac{2\pi i}{p}\right)^m B_m(\overline{\chi}). \tag{6}$$

Let $m \geq 1$ be an odd integer. We apply (6) and taking into consideration (5), we get

$$M(p, l, m, m) = \frac{(2\pi)^{2m}}{2(p-1)(m!)^2 p^{2m-1}} \times \sum_{\substack{\chi \mod p \\ \chi(-1)=-1}} \chi(l) B_m(\overline{\chi}) B_m(\chi). \tag{7}$$

By definition $B_m(\overline{\chi}) = \overline{B_m(\chi)}$, this fact and Lemma 1 allow us to write

$$\sum_{\substack{\chi \mod p \\ \chi(-1)=-1}} \chi(l) B_m(\overline{\chi}) B_m(\chi) = \sum_{a=0}^{p-2} |B_m(\chi_a)|^2 \Re(\chi_a(l)). \tag{8}$$

If $\overline{l} = \overline{x}^{\nu_l} \in G(p)$, then

$$\chi_a(l) = \exp\left(i \frac{2a\nu_l \pi}{p-1}\right) \text{ and } \Re(\chi_a(l)) = \cos\left(\frac{2a\nu_l \pi}{p-1}\right). \tag{9}$$

Finally, from (7), (8), and (9) we get Formula (4). This proves the Theorem 2. □

Now, we are ready to prove Theorem 1.

Proof of Theorem 1. On one hand, according to [6, Theorem 1] we have

$$M(p, l, 1, 1) = \frac{\pi^2 (p^2 - 3(l + S(l, p))p + l^2 + 1)}{6lp^2}, \tag{10}$$

where $S(l,p)$, which depends only on $p \mod l$, is defined by:

$$S(l,p)=\sum_{a=1}^{l-1}\cot\left(\frac{\pi a}{l}\right)\cot\left(\frac{\pi ap}{l}\right).$$

On the other hand, it follows by taking $m=1$ in Theorem 2 that:

$$M(p,l,1,1)=\frac{2\pi^2}{p(p-1)}\sum_{a=0}^{p-2}|B_1(\chi_a)|^2\cos\left(\frac{2\pi a\nu_l}{p-1}\right)=T(p,1,l). \tag{11}$$

Consequently, one can show that Formulas (10) and (11) imply Formula (3). This completes the proof. □

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