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EQUILIBRIUM PROBLEM FOR A KIRCHHOFF-LOVE PLATE CONTACTING WITH THE LATERAL SURFACE ALONG A STRIP OF A GIVEN WIDTH

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Abstract: A new model of a Kirchhoff-Love plate is justified, which may come into contact by its lateral surface with a non-deformable obstacle along a strip of a given width. The non-deformable obstacle restricts displacements of the plate along the outer lateral surface. The obstacle is specified by a cylindrical surface, the generatrices of which are perpendicular to the midplane of the plate. A problem is formulated in variational form. A set of admissible displacements is determined in a suitable Sobolev space in the framework of a clamping condition and a non-penetration condition of the Signorini type. The non-penetration condition is given as a system of two inequalities. The existence and uniqueness of a solution to the problem is proven. An equivalent differential formulation and optimality conditions are found under the assumption of additional regularity of the solution to the variational problem. A qualitative connection has been established between the

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proposed model and a previously studied problem in which the plate is in contact over the entire lateral surface.

Keywords: contact problem, limit passage, variational inequality, nonpenetration condition.

1 Introduction

Boundary value problems of the theory of elasticity with inequality type conditions describing the equilibrium of bodies are successfully studied on the basis of variational inequalities [1, 2, 3]. Within the framework of nonlinear problems with non-penetration conditions, mathematical models are often studied in which elastic bodies may come into mechanical contact interaction with non-deformable obstacles [4, 5, 6, 7], or with another deformable body [8, 9, 10, 11]. For studies of the regularity of solutions to obstacle problems we refer to [12, 13, 14]. Asymptotic analysis for problems of solid mechanics with inequality type constraints can be found, for example, in [15, 16, 17]. In the case where a body has a crack (or cracks), the interaction of opposite crack faces can also be described using models subject to unilateral constraints [18, 19, 20, 21, 22, 23, 24], etc. For problems of this type, numerical methods are proposed, for example, in [25, 26]. Within the framework of the theory of elasticity, a qualitative connection between nonlinear problems describing contact interaction with obstacles and problems of the crack theory has been established for a number of mathematical models [27, 28, 29, 30]. Note that cases of simultaneous possible contact of the plate along the front surface and lateral edge are also of interest [31, 32]. We can mention the works for pointwise contact problems [33, 34], where minimization problems over nonconvex sets are investigated.

In this work, we consider a special configuration of a non-deformable obstacle in contact with a strip on the lateral cylindrical surface of a plate. In this case, the obstacle in the initial state does not come into contact with the points of the plate along the entire width, as, for example, in the works [29, 30, 35], but along a strip of a given width. It is shown that when the parameter of the width of the contact zone tends to the value of the plate thickness, we get as a limiting problem the previously known problem studied in [28]. Thus, the presented mathematical model generalizes the previously known problem describing the contact of a Kirchhoff-Love plate with a non-deformable obstacle.

2 The Variational Problem

Let $\Omega \subset \mathbf{R}^2$ be a bounded with a smooth boundary Γ, which consists of two continuous curves $\Gamma = \overline{\Gamma}_0 \cup \overline{\Gamma}_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, $\mathrm{mes}(\Gamma_0) > 0$, $\mathrm{mes}(\Gamma_1) > 0$. Denote by $\nu = (\nu_1, \nu_2)$ the external unit normal vector to Γ. For simplicity, suppose the plate has a uniform thickness $2h$. Let us assign a three-dimensional

Cartesian space $\{x_1, x_2, z\}$ with the set $\{\Omega\} \times \{0\} \subset \mathbf{R}^3$ corresponding to the middle plane of the plate.

Denote by $\chi = \chi(x) = (W, w)$ the displacement vector of the mid-surface points ($x \in \Omega$), by $W = (w_1, w_2)$ the displacements in the plane $\{x_1, x_2\}$, and by w the displacements along the axis z (deflections). The strain and integrated stress tensors are denoted by $\varepsilon_{ij} = \varepsilon_{ij}(W)$, $\sigma_{ij} = \sigma_{ij}(W)$, respectively [5]:

$$\varepsilon_{ij}(W) = \frac{1}{2}(\frac{\partial w_j}{\partial x_i} + \frac{\partial w_i}{\partial x_j}), \quad \sigma_{ij}(W) = a_{ijkl}\varepsilon_{kl}(W), \quad i, j = 1, 2,$$

where $\{a_{ijkl}\}$ is the given elasticity tensor, assumed to be symmetric and positive definite:

$$a_{ijkl} = a_{klij} = a_{jikl}, \quad i, j, k, l = 1, 2, \quad a_{ijkl} \in L^\infty(\Omega),$$
$$a_{ijkl}\xi_{ij}\xi_{kl} \geq c_0|\xi|^2, \quad \forall \xi, \quad \xi_{ij} = \xi_{ji}, \quad i, j = 1, 2, \quad c_0 = const > 0.$$

A summation convention over repeated indices is used in the sequel. Next we denote the bending moments by formulas [5]

$$m_{ij}(w) = -d_{ijkl}w,_{kl}\,, \quad i, j = 1, 2, \quad (w,_{kl} = \frac{\partial^2 w}{\partial x_k \partial x_l})$$

where tensor $\{d_{ijkl}\}$ has the same symmetry, boundedness, and positive definiteness characteristics as tensor $\{a_{ijkl}\}$. Let $B(\cdot\,,\cdot)$ be a bilinear form defined by the equality

$$B(\chi, \bar{\chi}) = \int\limits_\Omega \big\{\sigma_{ij}(W)\,\varepsilon_{ij}(\bar{W}) - m_{ij}(w)\bar{w},_{ij}\big\}dx, \tag{1}$$

where $\chi = (W, w)$, $\bar{\chi} = (\bar{W}, \bar{w})$.

Introduce the Sobolev spaces

$$H^{1,0}_{\Gamma_0}(\Omega) = \left\{v \in H^1(\Omega) \;\middle|\; v = 0 \text{ on } \Gamma_0\right\},$$
$$H^{2,0}_{\Gamma_0}(\Omega) = \left\{v \in H^2(\Omega) \;\middle|\; v = \tfrac{\partial v}{\partial \nu} = 0 \text{ on } \Gamma_0\right\},$$
$$H(\Omega) = H^{1,0}_{\Gamma_0}(\Omega)^2 \times H^{2,0}_{\Gamma_0}(\Omega).$$

It is well known that the standard expression for a potential energy functional of a Kirchhoff–Love plate has the following representation

$$\Pi(\chi) = \frac{1}{2}B(\chi, \chi) - \int\limits_\Omega F\,\chi dx, \qquad \chi = (W, w),$$

where vector $F = (f_1, f_2, f_3) \in L_2(\Omega)^3$ describes the body forces [5]. Note that the following inequality providing coercivity of functional $\Pi(\chi)$

$$B(\chi, \chi) \geq c\|\chi\|^2 \quad \forall\, \chi \in H(\Omega), \quad (\|\chi\| = \|\chi\|_{H(\Omega)}) \tag{2}$$

with a constant $c > 0$ independent of χ, holds for the bilinear form $B(\cdot, \cdot)$ [5].

Let us start with the description of a non-deformable obstacle. The obstacle has a special shape such that the plate in the initial state is in contact along a strip of the width l, where $l \in \mathbf{R}$ is a fixed number such that $0 < l \leq 2h$. Namely, we specify the obstacle by the following set:

$$\{(x_1, x_2, z) \,|\, (x_1, x_2) \in \Gamma_1, \quad z \in (-\infty, -h + l]\}.$$

Obviously, for $l = 2h$ we obtain full contact along the lateral surface of the plate, studied in [28].

In order to introduce boundary conditions of the Signorini type, we recall the well-known relations of the Kirchhoff-Love model for displacements of points $(x, z) \in \Omega \times [-h, h]$

$$W^z(x, z) = W(x) - z\nabla w, \quad |z| \leq h, \quad w^z(x, z) = w(x). \tag{3}$$

Taking into account (3) and arguing as in [5, 28], we impose the following condition for displacements on Γ_1 describing the non-penetration of plate points into a non-deformable obstacle. We require the following relations to be satisfied

$$W\nu - z\frac{\partial w}{\partial \nu} \leq 0 \quad \text{on} \quad \Gamma_1, \quad z \in [-h, -h + l], \tag{4}$$

where $W\nu = w_i\nu_i$. The inequality (4), due to linearity, can be equivalently represented as a system of two inequalities

$$W\nu + h\frac{\partial w}{\partial \nu} \leq 0, \quad W\nu + (h - l)\frac{\partial w}{\partial \nu} \leq 0 \quad \text{on} \quad \Gamma_1. \tag{5}$$

Now we can introduce the following set of admissible functions

$$K_l = \{\chi = (W, w) \in H(\Omega) \,|\, \chi \text{ satisying } (4)\}.$$

Let us formulate a variational statement of an equilibrium problem. It is required to find a function $\xi = (U, u) \in K_l$, such that

$$\Pi(\xi) = \inf_{\chi \in K_l} \Pi(\chi). \tag{6}$$

Theorem 1. *The problem* (6) *has a unique solution.*

Proof. We will apply the well known Weierstrass theorem in order to show solution existence of the minimization problem [36]. The energy functional is coercive and weakly lower semicontinuous on $H(\Omega)$ [5]. It is easy to see that the set K_l has the convexity and closedness properties. These properties of the set of admissible displacements ensure that the set K_l is weakly closed. Consequently, for the minimization problem (6) all conditions of the Weierstrass theorem are satisfied both for the functional $\Pi(\chi)$ and for the set of admissible functions K_l. This means that problem (6) has at least one solution. The functional is convex and differentiable, and as a consequence the problem (6) is equivalent to the following variational inequality

$$\xi \in K_l, \quad B(\xi, \chi - \xi) \geq \int\limits_\Omega F(\chi - \xi)dx \quad \forall\, \chi \in K_l. \tag{7}$$

Assuming that there are two different solutions ξ_1 and ξ_2, we extract two inequalities from the variational inequality

$$B(\xi_1, \xi_2 - \xi_1) \geq \int_\Omega F(\xi_2 - \xi_1)dx,$$

$$B(\xi_2, \xi_1 - \xi_2) \geq \int_\Omega F(\xi_1 - \xi_2)dx.$$

Adding the last two inequalities we get that

$$B(\xi_2 - \xi_1, \xi_2 - \xi_1) \leq 0.$$

This means, in view of (2), that $\xi_1 = \xi_2$, and also entails the uniqueness of the solution to the problem (6). □

3 The Differential Statement

Let l be a fixed number such that $0 < l \leq 2h$. Let us assume that the solution $\xi = (U, u) \in K_l$ and elasticity tensors $\{a_{ijkl}\}$, $\{d_{ijkl}\}$ are sufficiently smooth. Namely, in addition to the prescribed properties of the solution, it is sufficient to require that $\xi \in H^2(\Omega)^2 \times H^4(\Omega)$. Our aim is to find from the variational inequality the equilibrium equations fulfilled in Ω and optimality conditions satisfied on Γ_1. We will apply the following Green's formulas (8) for the functions $\chi = (W, w) \in K_l$ [5],

$$\int_\Omega \sigma_{ij}(U)\, \varepsilon_{ij}(W)dx = -\int_\Omega \sigma_{ij,j}(U)w_i\, dx + \\ + \int_\Gamma \Big(\sigma_\nu(U)W\nu + \sigma_\tau(U)W\tau \Big)\, d\Gamma, \tag{8}$$

$$\int_\Omega m_{ij}(u)w_{,ij}\, dx = \int_\Omega m_{ij,ij}(u)\, w\, dx + \int_\Gamma \left(t^\nu(u)w - m_\nu(u)\frac{\partial w}{\partial \nu} \right) d\Gamma, \tag{9}$$

where

$$\sigma_\nu(U) = \sigma_{ij}(U)\nu_i\nu_j, \quad m_\nu(u) = -m_{ij}\nu_i\nu_j,$$
$$\sigma_\tau(U) = (\sigma_\tau^1(U), \sigma_\tau^2(U)) = (\sigma_{1j}(U)\nu_j, \sigma_{2j}(U)\nu_j) - \sigma_\nu(U)\nu,$$
$$t^\nu(u) = -m_{ij,k}\tau_k\tau_j\nu_i - m_{ij,j}\nu_i, \quad \tau = (-\nu_2, \nu_1),$$
$$W\nu = w_i\nu_i, \quad W\tau = (W_\tau^1, W_\tau^2), \quad w_i = (W\nu)\nu_i + W_\tau^i, \quad i = 1, 2.$$

Along with the variational statement (6), one can deal with the corresponding differential statement. Namely, the following theorem holds.

Theorem 2. *Supposing the solution $\xi = (U, u)$ as well as elasticity tensors $\{a_{ijkl}\}$, $\{d_{ijkl}\}$ to be sufficiently smooth, the variational problem (6) is equivalent to the following boundary value problem*

$$-m_{ij,ij}(u) = f_3 \quad in \quad \Omega, \tag{10}$$

$$-\sigma_{ij,j}(U) = f_i \quad in \quad \Omega, \quad i = 1, 2, \tag{11}$$

$$\sigma_\nu(U) - \frac{1}{h} m_\nu(u) \leq 0, \quad -\sigma_\nu(U)(h-l) + m_\nu(u) \leq 0 \quad on \quad \Gamma_1, \tag{12}$$

$$\sigma_\nu(U) \leq 0,\ U\nu + h\frac{\partial u}{\partial \nu} \leq 0,\ U\nu + (h-l)\frac{\partial u}{\partial \nu} \leq 0 \quad on \quad \Gamma_1, \tag{13}$$

$$\sigma_\tau(U) = (0,0), \quad t^\nu(u) = 0, \quad \sigma_\nu(U)U\nu + m_\nu(u)\frac{\partial u}{\partial \nu} = 0 \quad on \quad \Gamma_1, \tag{14}$$

$$U = (0,0), \quad u = \frac{\partial u}{\partial \nu} = 0 \quad on \quad \Gamma_0. \tag{15}$$

Proof. Substituting $\bar{\chi} = \xi \pm \tilde{\chi}$, where $\tilde{\chi} \in C_0^\infty(\Omega)^3$, as a test function in (7), we obtain the following relation

$$\int\limits_\Omega (\sigma_{ij}(U)\, \varepsilon_{ij}(\tilde{W}) - m_{ij}(u)\tilde{w}_{,ij})dx = \int\limits_\Omega F\tilde{\chi}dx,$$

that is, the equilibrium equations

$$- m_{ij,ij}(u) = f_3 \quad \text{in} \quad \Omega, \tag{16}$$

$$- \sigma_{ij,j}(U) = f_i \quad \text{in} \quad \Omega, \quad i = 1,2, \tag{17}$$

hold in terms of distribution.

Applying Green's formulas to (7) and using (16), (17), one can show that

$$\int\limits_\Gamma \Bigg(\sigma_\nu(U)(W-U)\nu + \sigma_\tau(U)(W-U)\tau -$$

$$t^\nu(u)(w-u) + m_\nu(u)\left(\frac{\partial w}{\partial \nu} - \frac{\partial u}{\partial \nu}\right)\Bigg) d\Gamma \geq 0 \quad \forall\, \chi = (W,w) \in K_l. \tag{18}$$

Since K_l is a convex cone in $H(\Omega)$, one can substitute $\chi = \lambda\xi$ with $\lambda \geq 0$ in (2) and deduce

$$\int\limits_\Gamma \left(\sigma_\nu(U)U\nu + \sigma_\tau(U)U\tau - t^\nu(u)u + m_\nu(u)\frac{\partial u}{\partial \nu}\right) d\Gamma = 0, \tag{19}$$

$$\int\limits_\Gamma \left(\sigma_\nu(U)W\nu + \sigma_\tau(U)W\tau - t^\nu(u)w + m_\nu(u)\frac{\partial w}{\partial \nu}\right) d\Gamma \geq 0, \tag{20}$$

for all $\chi = (W,w) \in K_l$. Since the function $\chi = (W,w) \in K_l$ satisfies zero boundary conditions on Γ_0, we can rewrite (20) as follows

$$\int\limits_{\Gamma_1} \left(\sigma_\nu(U)W\nu + \sigma_\tau(U)W\tau - t^\nu(u)w + m_\nu(u)\frac{\partial w}{\partial \nu}\right) d\Gamma \geq 0. \tag{21}$$

Since the inequalities (4) does not depend on $W\tau$, due to arbitrariness of $W\tau$ on Γ_1, we infer that

$$\sigma_\tau(U) = (0,0) \quad \text{on} \quad \Gamma_1.$$

Therefore, we can reduce (21) in the following form

$$\int\limits_{\Gamma_1} \left(\sigma_\nu(U)W\nu - t^\nu(u)w + m_\nu(u)\frac{\partial w}{\partial \nu} \right) d\Gamma \geq 0 \quad \forall\, \chi = (W, w) \in K_l. \tag{22}$$

By choosing functions $\chi = (W, w)$ such that $W = (0, 0)$, $\frac{\partial w}{\partial \nu} = 0$ on Γ_1 for (22), we get

$$t^\nu(u) = 0 \quad \text{on} \quad \Gamma_1.$$

Now we can substitute test functions with the property $w = 0$, $W\nu + h\frac{\partial w}{\partial \nu} = 0$ and $\frac{\partial w}{\partial \nu} \geq 0$ on Γ_1

As a result, we have

$$\int\limits_{\Gamma_1} \left(\sigma_\nu(U)W\nu - \frac{1}{h} m_\nu(u)W\nu \right) d\Gamma \geq 0. \tag{23}$$

From here, since the value $W\nu \geq 0$ can be arbitrary, we get

$$\sigma_\nu(U) - \frac{1}{h} m_\nu(u) \leq 0 \quad \text{on} \quad \Gamma_1.$$

Now, substituting into (22) test functions satisfying $W\nu + (h - l)\frac{\partial w}{\partial \nu} = 0$, $\frac{\partial w}{\partial \nu} \leq 0$ on Γ_1, we find

$$\int\limits_{\Gamma_1} \left(-\sigma_\nu(U)(h - l)\frac{\partial w}{\partial \nu} + m_\nu(u)\frac{\partial w}{\partial \nu} \right) d\Gamma \geq 0.$$

Whence it follows that

$$-\sigma_\nu(U)(h - l) + m_\nu(u) \leq 0 \quad \text{on} \quad \Gamma_1. \tag{24}$$

Substituting further into (22) $\eta = (W, w)$ such that $w = 0$, $W\nu \leq 0$, $\frac{\partial w}{\partial \nu} = 0$, it is not difficult to establish the inequality

$$\int\limits_{\Gamma_1} \sigma_\nu(U)W\nu \, d\Gamma \geq 0,$$

which means that

$$\sigma_\nu(U) \leq 0 \quad \text{on} \quad \Gamma_1. \tag{25}$$

Note that due to $\xi = (U, u) \in K_l$ and the following revealed relations

$$t^\nu(u) = 0, \quad \sigma_\nu(U) \leq 0 \quad \text{on} \quad \Gamma_1,$$

$$\sigma_\nu(U) - \frac{1}{h} m_\nu(u) \leq 0, \quad -\sigma_\nu(U)(h - l) + m_\nu(u) \leq 0 \quad \text{on} \quad \Gamma_1,$$

the expression $\sigma_\nu(U)W\nu + m_\nu(u)\frac{\partial w}{\partial \nu}$ is non-negative on Γ_1. Indeed, for the subset Γ_1^+ of Γ_1, where $\frac{\partial w}{\partial \nu} \geq 0$ a.e. on Γ_1^+, we have

$$\sigma_\nu(U)W\nu + m_\nu(u)\frac{\partial w}{\partial \nu} =$$

$$= \sigma_\nu(U)\left(W\nu + h\frac{\partial w}{\partial \nu}\right) + (m_\nu(u) - h\sigma_\nu(U))\frac{\partial w}{\partial \nu} \geq 0, \tag{26}$$

and for the subset Γ_1^- of Γ_1, where $\frac{\partial w}{\partial \nu} \leq 0$ a.e. on Γ_1^-, we get

$$\sigma_\nu(U)W\nu + m_\nu(u)\frac{\partial w}{\partial \nu} =$$

$$= \sigma_\nu(U)\left(W\nu + (h-l)\frac{\partial w}{\partial \nu}\right) + (-\sigma_\nu(U)(h-l) + m_\nu(u))\frac{\partial w}{\partial \nu} \geq 0. \tag{27}$$

Now we recall the relation (19). Since the integrand of (19) is non-negative a.e. on Γ. Therefore, we get

$$\sigma_\nu(U)U\nu + m_\nu(u)\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_1.$$

Conversely, in order to obtain from (10)–(15) the variational inequality (7) we multiply (10) by $(u - w)$ and each equality of (11) by corresponding $(u_i - w_i)$, $i = 1, 2$, where $W = (w_1, w_2)$, w such that $\chi = (W, w) \in K_l$. Then after integrating over Ω and summing, we get

$$-\int\limits_\Omega (\sigma_{ij,j}(U)(U - W) + m_{ij,ij}(u)(w - u))dx = \int\limits_\Omega F(\chi - \xi)dx.$$

At this point, recalling the Green formulas, we get

$$\int\limits_\Omega \Big(\sigma_{ij}(U)\,\varepsilon_{ij}(W - U) - m_{ij}(u)(w - u)_{,ij}\Big)dx -$$
$$- \int\limits_\Gamma \Big(\sigma_\nu(U)(W\nu - U\nu) + \sigma_\tau(U)(W\tau - U\tau)\Big)\, d\Gamma +$$
$$+ \int\limits_\Gamma \Big(t^\nu(u)(w - u) - m_\nu(u)(\frac{\partial w}{\partial \nu} - \frac{\partial u}{\partial \nu})\Big)\, d\Gamma = \int\limits_\Omega F(\chi - \xi)dx. \tag{28}$$

Taking into account that $\sigma_\tau(U) = (0, 0)$ on Γ_1, and zero boundary conditions for ξ, χ on Γ_0, we can represent the sum of integrals over Γ in the left side of (28) as follows

$$I = \int\limits_{\Gamma_1} \Big(t^\nu(u)(w - u) - m_\nu(u)(\frac{\partial w}{\partial \nu} - \frac{\partial u}{\partial \nu}) - \sigma_\nu(U)(W\nu - U\nu)\Big)\, d\Gamma. \tag{29}$$

Then bearing in mind the equalities $\sigma_\tau(U) = (0, 0)$, $t^\nu(u) = 0$ on Γ_1, we can rewrite (29) as the following sum

$$I = \int\limits_{\Gamma_1} \Big(-m_\nu(u)(\frac{\partial w}{\partial \nu} - \frac{\partial u}{\partial \nu}) - \sigma_\nu(U)(W\nu - U\nu)\Big)\, d\Gamma. \tag{30}$$

As we can see the integrand in the last integral is nonpositive because of $\chi \in K_l$ and relations (12)–(15). It remains to note that since $I \leq 0$, the equality (28) yields the variational inequality (7). The theorem is proved. □

Remark 1. *We can note that in the framework of the theorem 1 instead of the fixed number l the existence and uniqueness can be proved for some function $l \in L^2(\Gamma_1)$ satisfying $0 < l < 2h$ a.e. on Γ_1. Furthermore, bearing in mind reasonings of the proof to the theorem 2 it can be seen that same result is true for some continuous $l \in C(\overline{\Gamma}_1)$ such that $0 < l(x) < 2h$ for all $x \in \Gamma_1$.*

4 Passage to the limit as $l \to 2h$

Passages to the limit as parameters characterizing the sizes or relative positions of structural elements, or the distances between objects inside solid bodies have been studied, for example, in [37, 38, 39, 40, 41]. The dependence of solutions on perturbation of the geometry of objects within the framework of nonlinear models of solid mechanics is also of scientific interest, see [42, 43] etc. In this section we will show that the previously studied problem in [28], which corresponds to the value $l = 2h$, is a limit problem for a family of problems with different values $l \in (0, 2h]$. Namely, we consider the sequence of functions $\{l_n\} \subset L_2(\Gamma_1)$, satisfying the properties $0 \le l_n \le 2h$ and $l_n \to 2h$ in the space $L_2(\Gamma_1)$.

Let us consider a family of variational problems with different sets of admissible displacements K_{l_n}, $n \in \mathbf{N}$:

$$\xi_n \in K_{l_n}, \quad B(\xi_n, \chi - \xi_n) \ge \int\limits_{\Omega} F(\chi - \xi_n)dx \quad \forall\, \chi \in K_{l_n}. \tag{31}$$

Substituting the test function $\chi = (0, 0, 0)$ into (31) we obtain the inequality

$$B(\xi_n, \xi_n) \le \int\limits_{\Omega} F\xi_n dx.$$

Hence, we get the following uniform estimate

$$\|\xi_n\| \le C,$$

where $C > 0$ does not depend on $n \in \mathbf{N}$. The reflexivity of the space allows us to extract a subsequence ξ_{n_k} that weakly converges in $H(\Omega)$ to some function $\tilde{\xi}$. As the next step we show that $\tilde{\xi} = (\tilde{U}, \tilde{u}) \in K_{2h}$. Since $l_n \to 2h$ converges strongly in $L_2(\Gamma_1)$, we can exract a subsequence l_{n_k} that converges almost everywhere on Γ_1 to $2h$. Extracting a subsequence again if necessary, we assume that $\{\xi_{n_k}\}$ converges on Γ_1 almost everywhere. Based on the properties of these convergent subsequences, we can pass to the limit as $k \to \infty$ in the following inequalities:

$$U_{n_k}\nu + h\frac{\partial u_{n_k}}{\partial \nu} \le 0, \quad U_{n_k}\nu + (h - l_{n_k})\frac{\partial u_{n_k}}{\partial \nu} \le 0 \quad \text{a.e. on} \quad \Gamma_1.$$

As limiting relations we obtain

$$\tilde{U}\nu + h\frac{\partial \tilde{u}}{\partial \nu} \le 0, \quad \tilde{U}\nu - h\frac{\partial \tilde{u}}{\partial \nu} \le 0 \quad \text{a.e. on} \quad \Gamma_1.$$

That is $\tilde{\xi} \in K_{2h}$. Let $\tilde{\eta} \in K_{2h}$ be an arbitrary test function, it is obvious that in this case $\tilde{\eta} \in K_{l_n}$, for all $n \in \mathbf{N}$. Therefore, passing to the limit as $n \to \infty$ in the inequalities

$$\xi_n \in K_{l_n}, \quad B(\xi_n, \tilde{\eta} - \xi_n) \geq \int_\Omega F(\tilde{\eta} - \xi_n) dx$$

with the fixed function $\tilde{\eta}$, we get

$$B(\tilde{\xi}, \tilde{\eta} - \tilde{\xi}) \geq \int_\Omega F(\tilde{\eta} - \tilde{\xi}) dx. \tag{32}$$

Thus, due to the uniqueness of the solution to the variational inequality, it follows from (32) that $\tilde{\xi}$ is a solution to the problem (6) corresponding to the value $l = 2h$ for K_{2h}.

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