

**EQUILIBRIUM PROBLEM FOR A KIRCHHOFF-LOVE
PLATE CONTACTING WITH THE LATERAL
SURFACE ALONG A STRIP OF A GIVEN WIDTH**N.P. LAZAREV¹, D.Y. NIKIFOROV, AND G.M. SEMENOVA

Abstract: A new model of a Kirchhoff-Love plate is justified, which may come into contact by its lateral surface with a non-deformable obstacle along a strip of a given width. The non-deformable obstacle restricts displacements of the plate along the outer lateral surface. The obstacle is specified by a cylindrical surface, the generatrices of which are perpendicular to the midplane of the plate. A problem is formulated in variational form. A set of admissible displacements is determined in a suitable Sobolev space in the framework of a clamping condition and a non-penetration condition of the Signorini type. The non-penetration condition is given as a system of two inequalities. The existence and uniqueness of a solution to the problem is proven. An equivalent differential formulation and optimality conditions are found under the assumption of additional regularity of the solution to the variational problem. A qualitative connection has been established between the

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proposed model and a previously studied problem in which the plate is in contact over the entire lateral surface.

Keywords: contact problem, limit passage, variational inequality, nonpenetration condition.

1 Introduction

Boundary value problems of the theory of elasticity with inequality type conditions describing the equilibrium of bodies are successfully studied on the basis of variational inequalities [1, 2, 3]. Within the framework of nonlinear problems with non-penetration conditions, mathematical models are often studied in which elastic bodies may come into mechanical contact interaction with non-deformable obstacles [4, 5, 6, 7], or with another deformable body [8, 9, 10, 11]. In the case where a body has a crack (or cracks), the interaction of opposite crack faces can also be described using models subject to unilateral constraints [12, 13, 14, 15, 16, 17, 18], etc. For problems of this type, numerical methods are proposed, for example, in [19, 20]. Within the framework of the theory of elasticity, a qualitative connection between nonlinear problems describing contact interaction with obstacles and problems of the crack theory has been established for a number of mathematical models [21, 22, 23, 24]. Note that cases of simultaneous possible contact of the plate along the front surface and lateral edge are also of interest [25, 26]. We can mention the works for pointwise contact problems [27, 28], where minimization problems over nonconvex sets are investigated.

In this work, we consider a special configuration of a non-deformable obstacle in contact with a strip on the lateral cylindrical surface of a plate. In this case, the obstacle in the initial state does not come into contact with the points of the plate along the entire width, as, for example, in the works [23, 24, 29], but along a strip of a given width. It is shown that when the parameter of the width of the contact zone tends to value of the plate thickness, we get as a limiting problem the previously known problem studied in [22]. Thus, the presented mathematical model generalizes the previously known problem describing the contact of a Kirchhoff-Love plate with a non-deformable obstacle.

2 The Variational Problem

Let $\Omega \subset \mathbf{R}^2$ be a bounded with a smooth boundary Γ , which consists of two continuous curves $\Gamma = \bar{\Gamma}_0 \cup \bar{\Gamma}_1$, $\text{mes}(\Gamma_0) > 0$, $\text{mes}(\Gamma_1) > 0$. Denote by $\nu = (\nu_1, \nu_2)$ the external unit normal vector to Γ . For simplicity, suppose the plate has a uniform thickness $2h$. Let us assign a three-dimensional Cartesian space $\{x_1, x_2, z\}$ with the set $\{\Omega\} \times \{0\} \subset \mathbf{R}^3$ corresponding to the middle plane of the plate.

Denote by $\chi = \chi(x) = (W, w)$ the displacement vector of the mid-surface points ($x \in \Omega$), by $W = (w_1, w_2)$ the displacements in the plane $\{x_1, x_2\}$, and by w the displacements along the axis z (deflections). The strain and integrated stress tensors are denoted by $\varepsilon_{ij} = \varepsilon_{ij}(W)$, $\sigma_{ij} = \sigma_{ij}(W)$, respectively [5]:

$$\varepsilon_{ij}(W) = \frac{1}{2} \left(\frac{\partial w_j}{\partial x_i} + \frac{\partial w_i}{\partial x_j} \right), \quad \sigma_{ij}(W) = a_{ijkl} \varepsilon_{kl}(W), \quad i, j = 1, 2,$$

where $\{a_{ijkl}\}$ is the given elasticity tensor, assumed to be symmetric and positive definite:

$$\begin{aligned} a_{ijkl} &= a_{klij} = a_{jikl}, \quad i, j, k, l = 1, 2, \quad a_{ijkl} \in L^\infty(\Omega), \\ a_{ijkl} \xi_{ij} \xi_{kl} &\geq c_0 |\xi|^2, \quad \forall \xi, \quad \xi_{ij} = \xi_{ji}, \quad i, j = 1, 2, \quad c_0 = \text{const} > 0. \end{aligned}$$

A summation convention over repeated indices is used in the sequel. Next we denote the bending moments by formulas [5]

$$m_{ij}(w) = -d_{ijkl} w_{,kl}, \quad i, j = 1, 2, \quad (w_{,kl} = \frac{\partial^2 w}{\partial x_k \partial x_l})$$

where tensor $\{d_{ijkl}\}$ has the same symmetry, boundedness, and positive definiteness characteristics as tensor $\{a_{ijkl}\}$. Let $B(\cdot, \cdot)$ be a bilinear form defined by the equality

$$B(\chi, \bar{\chi}) = \int_{\Omega} \{ \sigma_{ij}(W) \varepsilon_{ij}(\bar{W}) - m_{ij}(w) \bar{w}_{,ij} \} dx, \quad (1)$$

where $\chi = (W, w)$, $\bar{\chi} = (\bar{W}, \bar{w})$.

Introduce the Sobolev spaces

$$\begin{aligned} H_{\Gamma_0}^{1,0}(\Omega) &= \left\{ v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_0 \right\}, \\ H_{\Gamma_0}^{2,0}(\Omega) &= \left\{ v \in H^2(\Omega) \mid v = \frac{\partial v}{\partial \nu} = 0 \text{ on } \Gamma_0 \right\}, \\ H(\Omega) &= H_{\Gamma_0}^{1,0}(\Omega)^2 \times H_{\Gamma_0}^{2,0}(\Omega). \end{aligned}$$

It is well known that the standard expression for an potential energy functional of a Kirchhoff–Love plate has the following representation

$$\Pi(\chi) = \frac{1}{2} B(\chi, \chi) - \int_{\Omega} F \chi dx, \quad \chi = (W, w),$$

where vector $F = (f_1, f_2, f_3) \in L_2(\Omega)^3$ describes the body forces [5]. Note that the following inequality providing coercivity of functional $\Pi(\chi)$

$$B(\chi, \chi) \geq c \|\chi\|^2 \quad \forall \chi \in H(\Omega), \quad (\|\chi\| = \|\chi\|_{H(\Omega)}) \quad (2)$$

with a constant $c > 0$ independent of χ , holds for the bilinear form $B(\cdot, \cdot)$ [5].

Let us start with the description of the of a non-deformable obstacle. The obstacle has a special shape such that the plate in the initial state is in

contact along a strip of the width l , where $l \in \mathbb{R}$, $0 < l \leq 2h$. Namely, we specify the obstacle by the following set:

$$\{(x_1, x_2, z) \mid (x_1, x_2) \in \Gamma_1, \quad z \in (-\infty, -h + l]\}.$$

Obviously, for $l = 2h$ we obtain full contact along the lateral surface of the plate, studied in [22].

On Γ_1 we impose the following condition describing the non-penetration of plate points into a non-deformable obstacle. We require the following relations to be satisfied

$$W\nu - z \frac{\partial w}{\partial \nu} \leq 0 \quad \text{on } \Gamma_1, \quad z \in [-h, -h + l], \quad (3)$$

where $W\nu = w_i \nu_i$. The inequality (3), due to linearity, can be equivalently represented as a system of two inequalities

$$W\nu + h \frac{\partial w}{\partial \nu} \leq 0, \quad W\nu + (h - l) \frac{\partial w}{\partial \nu} \leq 0 \quad \text{on } \Gamma_1. \quad (4)$$

Now we can introduce the following set of admissible functions

$$K_l = \{\chi = (W, w) \in H(\Omega) \mid \chi \text{ satisfying (3)}\}.$$

Let us formulate a variational statement of an equilibrium problem. It is required to find a function $\xi = (U, u) \in K_l$, such that

$$\Pi(\xi) = \inf_{\chi \in K_l} \Pi(\chi). \quad (5)$$

Theorem 1. *The problem (5) has a unique solution.*

Proof. We will apply the well known Weierstrass theorem in order to show solution existence of the minimization problem [32]. The energy functional is coercive and weakly lower semicontinuous on $H(\Omega)$ [5]. It is easy to see that the set K_l has the convexity and closedness properties. These properties of the set of admissible displacements ensure that the set K_l is weakly closed. Consequently, for the minimization problem (5) all conditions of the Weierstrass theorem are satisfied both for the functional $\Pi(\chi)$ and for the set of admissible functions K_l . This means that problem (5) has at least one solution. The functional is convex and differentiable, and as a consequence the problem (5) is equivalent to the following variational inequality

$$\xi \in K_l, \quad B(\xi, \chi - \xi) \geq \int_{\Omega} F(\chi - \xi) dx \quad \forall \chi \in K_l. \quad (6)$$

Assuming that there are two different solutions ξ_1 and ξ_2 , we extract two inequalities from the variational inequality

$$B(\xi_1, \xi_2 - \xi_1) \geq \int_{\Omega} F(\xi_2 - \xi_1) dx,$$

$$B(\xi_2, \xi_1 - \xi_2) \geq \int_{\Omega} F(\xi_1 - \xi_2) dx.$$

Adding the last two inequalities we get that

$$B(\xi_2 - \xi_1, \xi_2 - \xi_1) \leq 0.$$

This means, in view of (2), that $\xi_1 = \xi_2$, and also entails the uniqueness of the solution to the problem (5). \square

3 The Differential Statement

Let l be a fixed number such that $0 < l \leq 2h$. Let us assume that the solution $\xi = (U, u) \in K_l$ is sufficiently smooth. Our aim is to find from the variational inequality the equilibrium equations fulfilled in Ω and optimality conditions satisfied on Γ_1 . We will apply the following Green's formulas (7) for the functions $\chi = (W, w) \in K_l$ [5],

$$\begin{aligned} \int_{\Omega} \sigma_{ij}(U) \varepsilon_{ij}(W) dx &= - \int_{\Omega} \sigma_{ij,j}(U) w_i dx + \\ &+ \int_{\Gamma} \left(\sigma_{\nu}(U) W \nu + \sigma_{\tau}(U) W \tau \right) d\Gamma, \end{aligned} \quad (7)$$

$$\int_{\Omega} m_{ij}(u) w_{,ij} dx = \int_{\Omega} m_{ij,ij}(u) w dx + \int_{\Gamma} \left(t^{\nu}(u) w - m_{\nu}(u) \frac{\partial w}{\partial \nu} \right) d\Gamma, \quad (8)$$

where

$$\sigma_{\nu}(U) = \sigma_{ij}(U) \nu_i \nu_j, \quad m_{\nu}(u) = -m_{ij} \nu_i \nu_j,$$

$$\sigma_{\tau}(U) = (\sigma_{\tau}^1(U), \sigma_{\tau}^2(U)) = (\sigma_{1j}(U) \nu_j, \sigma_{2j}(U) \nu_j) - \sigma_{\nu}(U) \nu,$$

$$t^{\nu}(u) = -m_{ij,k} \tau_k \tau_j \nu_i - m_{ij,j} \nu_i, \quad \tau = (-\nu_2, \nu_1),$$

$$W \nu = w_i \nu_i, \quad W \tau = (W_{\tau}^1, W_{\tau}^2), \quad w_i = (W \nu) \nu_i + W_{\tau}^i, \quad i = 1, 2.$$

Along with the variational statement (5), one can deal with the corresponding differential statement. Namely, the following theorem holds.

Theorem 2. *Supposing the solution $\xi = (U, u)$ to be sufficiently smooth, the variational problem (5) is equivalent to the following boundary value problem*

$$-m_{ij,ij}(u) = f_3 \quad \text{in } \Omega, \quad (9)$$

$$-\sigma_{ij,j}(U) = f_i \quad \text{in } \Omega, \quad i = 1, 2, \quad (10)$$

$$\sigma_{\nu}(U) \leq 0, \quad \sigma_{\nu}(U) - \frac{1}{h} m_{\nu}(u) \leq 0, \quad -\sigma_{\nu}(U)(h-l) + m_{\nu}(u) \leq 0 \quad \text{on } \Gamma_1, \quad (11)$$

$$t^{\nu}(u) = 0, \quad \sigma_{\tau}(U) = (0, 0), \quad U \nu + h \frac{\partial u}{\partial \nu} \leq 0, \quad U \nu + (h-l) \frac{\partial u}{\partial \nu} \leq 0 \quad \text{on } \Gamma_1, \quad (12)$$

$$U = u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma_0. \quad (13)$$

Proof. Substituting $\bar{\chi} = \xi \pm \tilde{\chi}$, where $\tilde{\chi} \in C_0^\infty(\Omega)^3$, as a test function in (6), we obtain the following relation

$$\int_{\Omega} (\sigma_{ij}(U) \varepsilon_{ij}(\tilde{W}) - m_{ij}(u) \tilde{w}_{,ij}) dx = \int_{\Omega} F \tilde{\chi} dx,$$

that is, the equilibrium equations

$$-m_{ij,ij}(u) = f_3 \quad \text{in } \Omega, \quad (14)$$

$$-\sigma_{ij,j}(U) = f_i \quad \text{in } \Omega, \quad i = 1, 2, \quad (15)$$

hold in terms of distribution.

Applying Green's formulas to (6) and using (14), (15), one can show that

$$\int_{\Gamma} \left(\sigma_{\nu}(U)(W - U)\nu + \sigma_{\tau}(U)(W - U)\tau - t^{\nu}(u)(w - u) + m_{\nu}(u) \left(\frac{\partial w}{\partial \nu} - \frac{\partial u}{\partial \nu} \right) \right) d\Gamma \geq 0 \quad \forall \chi = (W, w) \in K. \quad (16)$$

Since K_l is a convex cone in $H(\Omega)$, one can substitute $\chi = \lambda \xi$ with $\lambda \geq 0$ in (2) and deduce

$$\int_{\Gamma} \left(\sigma_{\nu}(U)U\nu + \sigma_{\tau}(U)\bar{U}\tau - t^{\nu}(u)u + m_{\nu}(u) \frac{\partial u}{\partial \nu} \right) d\Gamma = 0, \quad (17)$$

$$\int_{\Gamma} \left(\sigma_{\nu}(U)W\nu + \sigma_{\tau}(U)W\tau - t^{\nu}(u)w + m_{\nu}(u) \frac{\partial w}{\partial \nu} \right) d\Gamma \geq 0, \quad (18)$$

for all $\chi = (W, w) \in K_l$. Let us suppose that $\chi = (W, w) \in K_l$ and $\chi = (0, 0, 0)$ on Γ_0 . In this case we can rewrite (18) as follows

$$\int_{\Gamma_1} \left(\sigma_{\nu}(U)W\nu + \sigma_{\tau}(U)W\tau - t^{\nu}(u)w + m_{\nu}(u) \frac{\partial w}{\partial \nu} \right) d\Gamma \geq 0. \quad (19)$$

Since the value $W\tau$ is not included in the inequalities (3), due to arbitrariness of $W\tau$ on Γ_1 , we infer that

$$\sigma_{\tau}(U) = (0, 0) \quad \text{on } \Gamma_1.$$

Therefore, we can reduce (19) in the following form

$$\int_{\Gamma_1} \left(\sigma_{\nu}(U)W\nu - t^{\nu}(u)w + m_{\nu}(u) \frac{\partial w}{\partial \nu} \right) d\Gamma \geq 0 \quad \forall \chi = (W, w) \in K. \quad (20)$$

By choosing functions $\chi = (W, w)$ such that $W = 0$, $\frac{\partial w}{\partial \nu} = 0$ on Γ_1 for (20), we get

$$t^{\nu}(u) = 0 \quad \text{on } \Gamma_1.$$

Now we can substitute test functions with the property $w = 0$, $W\nu + h \frac{\partial w}{\partial \nu} = 0$ and $\frac{\partial w}{\partial \nu} \geq 0$ on Γ_1

As a result, we have

$$\int_{\Gamma_1} \left(\sigma_\nu(U)W\nu - \frac{1}{h}m_\nu(u)W\nu \right) d\Gamma \geq 0. \quad (21)$$

From here, since the value $W\nu \geq 0$ can be arbitrary, we get

$$\sigma_\nu(U) - \frac{1}{h}m_\nu(u) \leq 0 \quad \text{on } \Gamma_1.$$

Now, substituting into (20) test functions satisfying $W\nu + (h-l)\frac{\partial w}{\partial \nu} = 0$, $\frac{\partial w}{\partial \nu} \leq 0$ on Γ_1 , we find

$$\int_{\Gamma_1} \left(-\sigma_\nu(U)(h-l)\frac{\partial w}{\partial \nu} + m_\nu(u)\frac{\partial w}{\partial \nu} \right) d\Gamma \geq 0.$$

Whence it follows that

$$-\sigma_\nu(U)(h-l) + m_\nu(u) \leq 0 \quad \text{on } \Gamma_1. \quad (22)$$

Substituting further into (20) $\eta = (W, w)$ such that $w = 0$, $W\nu \leq 0$, $\frac{\partial w}{\partial \nu} = 0$, it is not difficult to establish the inequality

$$\int_{\Gamma_1} \left(\sigma_\nu(U)W\nu + m_\nu(u)\frac{\partial w}{\partial \nu} \right) d\Gamma \geq 0,$$

which means that

$$\sigma_\nu(U) \leq 0 \quad \text{on } \Gamma_1. \quad (23)$$

The last equality allow us to represent (21) in the form

$$\int_{\Gamma_1} \sigma_\nu(U) \left(W\nu + h\frac{\partial w}{\partial \nu} \right) d\Gamma \geq 0,$$

From here, it follows that

$$\sigma_\nu(U) \leq 0 \quad \text{on } \Gamma_1.$$

Note that due to $\xi = (U, u) \in K_l$ and the following revealed relations

$$\sigma_\nu(U) \leq 0, \quad \sigma_\nu(U) - \frac{1}{h}m_\nu(u) \leq 0, \quad -\sigma_\nu(U)(h-l) + m_\nu(u) \leq 0 \quad \text{on } \Gamma_1,$$

the expression $\sigma_\nu(U)W\nu + m_\nu(u)\frac{\partial w}{\partial \nu}$ is non-negative on Γ_1 . Indeed, for the case $\frac{\partial w}{\partial \nu} \geq 0$ on Γ_1 , we have

$$\sigma_\nu(U)W\nu + m_\nu(u)\frac{\partial w}{\partial \nu} = \sigma_\nu(\phi) \left(W\nu + h\frac{\partial w}{\partial \nu} \right) + (m_\nu(\phi) - h\sigma_\nu(U))\frac{\partial w}{\partial \nu} \geq 0, \quad (24)$$

and for the another case $\frac{\partial w}{\partial \nu} \leq 0$ on Γ_1 , we get

$$\sigma_\nu(U)W\nu + m_\nu(\phi)\frac{\partial w}{\partial \nu} =$$

$$= \sigma_\nu(U) \left(W\nu + (h-l) \frac{\partial w}{\partial \nu} \right) + (-\sigma_\nu(U)(h-l) + m_\nu(\phi)) \frac{\partial w}{\partial \nu} \geq 0. \quad (25)$$

Now we recall the relation (17). Since the integrand of (17) is non-negative a.e. on Γ . Therefore, we get

$$\sigma_\nu(U)U\nu + m_\nu(u) \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma_1.$$

Conversely, in order to obtain from (9)–(13) the variational inequality (6) we multiply (9) by $(u-w)$ and each equality of (10) by corresponding $(u_i - w_i)$, $i = 1, 2$, where $W = (w_1, w_2)$, w such that $\chi = (W, w) \in K_l$. Then after integrating over Ω and summing, we get

$$- \int_{\Omega} (\sigma_{ij,j}(U)(U - W) + m_{ij,ij}(u)(w - u)) dx = \int_{\Omega} F(\chi - \xi) dx.$$

At this point, recalling the Green formulas, we get

$$\begin{aligned} & \int_{\Omega} \left(\sigma_{ij}(U) \varepsilon_{ij}(W - U) - m_{ij}(u)(w - u)_{,ij} \right) dx - \\ & - \int_{\Gamma} \left(\sigma_\nu(U)(W\nu - U\nu) + \sigma_\tau(U)(W\tau - U\tau) \right) d\Gamma + \\ & + \int_{\Gamma} \left(t^\nu(u)(w - u) - m_\nu(u) \left(\frac{\partial w}{\partial \nu} - \frac{\partial u}{\partial \nu} \right) \right) d\Gamma = \int_{\Omega} F(\chi - \xi) dx. \quad (26) \end{aligned}$$

Taking into account that $\sigma_\tau(U) = (0, 0)$ on Γ_1 , $\xi = \chi = (0, 0, 0)$ on Γ_0 , we can represent the sum of integrals over Γ in the left side of (26) as follows

$$I = \int_{\Gamma_1} \left(t^\nu(u)(w - u) - m_\nu(u) \left(\frac{\partial w}{\partial \nu} - \frac{\partial u}{\partial \nu} \right) - \sigma_\nu(U)(W\nu - U\nu) \right) d\Gamma. \quad (27)$$

Then bearing in mind the equalities $\sigma_\tau(U) = (0, 0)$, $t^\nu(u) = 0$ on Γ_1 , we can rewrite (27) as the following sum

$$I = \int_{\Gamma_1} \left(-m_\nu(u) \left(\frac{\partial w}{\partial \nu} - \frac{\partial u}{\partial \nu} \right) - \sigma_\nu(U)(W\nu - U\nu) \right) d\Gamma. \quad (28)$$

As we can see the integrand in the last integral is nonpositive because of $\chi \in K_l$ and relations (11)–(13). It remains to note that since $I \leq 0$, the equality (26) yields the variational inequality (6). The theorem is proved. \square

4 Passage to the limit as $l \rightarrow 0$

Passages to the limit as parameters characterizing the sizes or relative positions of structural elements, or the distances between objects inside solid bodies have been studied, for example, in [34, 33, 35]. The dependence of solutions on perturbation of the geometry of objects within the framework of

nonlinear models of solid mechanics is also of scientific interest, see [36, 37] etc. In this section we will show that the previously studied problem in [22], which corresponds to the value $l = 2h$, is a limit problem for a family of problems with different values $l \in (0, 2h]$. Namely, we consider the sequence $\{l_n\} \subset L_2(\Gamma_1)$, satisfying the properties $0 \leq l_n \leq 2h$ and $l_n \rightarrow 2h$ in the space $L_2(\Gamma_1)$.

Let us consider a family of variational problems with different sets of admissible displacements K_{l_n} , $n \in \mathbf{N}$:

$$\xi_n \in K_{l_n}, \quad B(\xi_n, \chi - \xi_n) \geq \int_{\Omega} F(\chi - \xi_n) dx \quad \forall \chi \in K_{l_n}. \quad (29)$$

Substituting the test function $\chi = (0, 0, 0)$ into (29) we obtain the inequality

$$B(\xi_n, \xi_n) \leq \int_{\Omega} F \xi_n dx.$$

Hence, we get the following uniform estimate

$$\|\xi_n\| \leq C,$$

where $C > 0$ does not depend on $n \in \mathbf{N}$. The reflexivity of the space allows us to extract a subsequence ξ_{n_k} that weakly converges in $H(\Omega)$ to some function $\tilde{\xi}$. As the next step we show that $\tilde{\xi} = (\tilde{U}, \tilde{u}) \in K_{2h}$. Since $l_n \rightarrow 2h$ converges strongly in $L_2(\Gamma_1)$, we can extract a subsequence l_{n_k} that converges almost everywhere on Γ_1 to $2h$. Extracting a subsequence again if necessary, we assume that $\{\xi_{n_k}\}$ converges on Γ almost everywhere. Based on the properties of these convergent subsequences, we can pass to the limit as $k \rightarrow \infty$ in the following inequalities:

$$U_{n_k} \nu + h \frac{\partial u_{n_k}}{\partial \nu} \leq 0, \quad U_{n_k} \nu + (h - l_{n_k}) \frac{\partial u_{n_k}}{\partial \nu} \leq 0 \quad \text{a.e. on } \Gamma_1.$$

As limiting relations we obtain

$$\tilde{U} \nu + h \frac{\partial \tilde{u}}{\partial \nu} \leq 0, \quad \tilde{U} \nu - h \frac{\partial \tilde{u}}{\partial \nu} \leq 0 \quad \text{a.e. on } \Gamma_1.$$

That is $\tilde{\xi} \in K_{2h}$. Let $\tilde{\eta} \in K_{2h}$ be an arbitrary test function, it is obvious that in this case $\tilde{\eta} \in K_{l_n}$, for all $n \in \mathbf{N}$. Therefore, passing to the limit as $n \rightarrow \infty$ in the inequalities

$$\xi_n \in K_{l_n}, \quad B(\xi_n, \tilde{\eta} - \xi_n) \geq \int_{\Omega} F(\tilde{\eta} - \xi_n) dx \quad \forall \tilde{\eta} \in K_{l_n}$$

with the fixed function $\tilde{\eta}$, we get

$$B(\tilde{\xi}, \tilde{\eta} - \tilde{\xi}) \geq \int_{\Omega} F(\tilde{\eta} - \tilde{\xi}) dx. \quad (30)$$

Thus, due to the uniqueness of the solution to the variational inequality, it follows from (30) that $\tilde{\xi}$ is a solution to the problem (5) corresponding to the value $l = 2h$ for K_{2h} .

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