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AUTOMORPHISMS OF SOME CYCLIC EXTENSIONS OF FREE GROUPS OF RANK THREE

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Abstract: The description of the group of outer automorphisms of the Gersten group was obtained in 2021. In this paper, we study the possibility of extending the methods of that work to an infinite class of cyclic extensions of a free group of rank three

$$G_k = \langle a, b, c, t | a^t = a, b^t = ba^k, c^t = c \rangle.$$

We have found the generating elements of the group $Out(G_k)$ and obtained a description of the structure of this group.

Keywords: Free group, split cyclic extension, group of outer automorphisms.

1 Introduction

In the paper [1] of 2006, Bogopolski, Martino, and Ventura described the outer automorphism groups of all infinite cyclic split extensions of the free group of rank 2 as follows:

$$M_\varphi = F_2 \rtimes_\varphi \mathbb{Z},$$

assuming that automorphisms act on the right, i.e., $\varphi : a \mapsto a\varphi$; we write a^t for $t^{-1}at$ and $\hat{t}$ for the conjugation by t.

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In 1994 using the group

$$H = F_3 \rtimes \mathbb{Z} = \langle a, b, c, t \mid a^t = a, b^t = ba, c^t = ca^2 \rangle,$$

S. Gersten [2] proved that groups $Aut(F_n), n \geqslant 3$ and $Out(F_n), n \geqslant 4$ are not $CAT(0)$ groups.

The group H is a cyclic split extension of the group F_3 with the basis $\{a, b, c\}$, using automorphism $\varphi : a \mapsto a, b \mapsto ba, c \mapsto ca^2$.

In 2021 [7], a generating set of the group $Out(H)$ was found and proved that $Out(H) \cong (F_3 \times \mathbb{Z}^3) \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$.

If φ is an automorphism of a free group of rank n, then we denote the matrix of the mapping induced by φ on $F_n^{ab} \cong \mathbb{Z}^n$ by $\varphi^{ab} \in GL_2(\mathbb{Z})$.

A description of the groups $Out(M_\varphi)$ is obtained in Theorem 1.1 [1], depending on the type of matrix φ^{ab}. In particular, a uniform description is obtained for all groups $Out(M_\varphi)$ with unitriangular matrix φ^{ab}. This matrix is unitriangular For the Gersten group.

In our work, we are trying to understand whether it is possible to obtain a classification (similar to [1]) of the group of outer automorphisms of cyclic extensions of a free group of rank three depending on matrix φ^{ab}. To do this, we study a series of cyclic extensions of a free group of rank three:

$$G_k = F_3 \rtimes \mathbb{Z} = \langle a, b, c, t \mid a^t = a, b^t = ba^k, c^t = c \rangle, k \neq 0.$$

The group G_k is defined by an automorphism $\varphi_k : a \mapsto a, b \mapsto ba^k, c \mapsto c$. Matrix φ^{ab} of such an automorphism is unitriangular.

Trying to describe $Out(G_k)$ using the methods of work [1], we found that they are applicable to describe the generators of $Out(G_k)$, but there not enough ideas of work [1] to describe the structure of this group.

We succeeded in finding the generators (see section 3) of the group $Out(G_k)$ and proved the following theorem:

Theorem 1. *$Out(G_k) \cong ((\mathbb{Z}^2 \times \mathbb{Z}_k) \times N) \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$, where $N-$ is a rectangular Artinian group, the structure of which is described in Lemma 6. N does not depend on the parameter k.*

2 The Lemma about Fixed Points

The proof of Lemma 4.2 from [1] and Lemma 1 from [7] can be adapted for G_k.

Lemma 1. *Let φ be an automorphism of $F_3 = \langle a, b, c \rangle$ such that $a\varphi = a, b\varphi = ba^k, c\varphi = c$, let $k \neq 0, r \neq 0$ be an integer and let $w \in F_3$. Then the following is true:*

1. *$Fix\varphi_k = Fix\varphi_k^r = Fix\varphi_{kr} = \langle a, c, bab^{-1} \rangle$,*
2. *If $w\varphi_k^r$ is conjugate to w, then w is conjugate to an element of $Fix\ \varphi$.*

Proof. Let us prove the first statement. Note that a and c lie in $Fix\varphi$.

Represent an arbitrary word $w \in \mathbb{F}_3$ in the form

$$w = v_1(a, c)b^{\pm 1}v_2(a, c)b^{\pm 1}...v_n(a, c),$$

where $v_i(a,c)$ are words starting with $a^{\pm 1}, c^{\pm 1}$. $v_i(a,c)\varphi = v_i(a,c)$.

Understand further when $w \in Fix\varphi$, that is $w\varphi = w$. It is enough to study the following cases.

If $w = v_1(a,c)bv_2(a,c)b = w\varphi = v_1(a,c)ba^kv_2(a,c)ba^k$ then $a^kv_2(a,c) = v_2(a,c)$. Then $k=0$ that is contradiction.

If $w = v_1(a,c)b^{-1}v_2(a,c)b^{-1} = w\varphi = v_1(a,c)a^{-k}b^{-1}v_2(a,c)a^{-k}b^{-1}$ then $v_1(a,c)a^{-k} = v_1(a,c)$. Then $k=0$ that is a contradiction.

If $w = v_1(a,c)b^{-1}v_2(a,c)b = w\varphi = v_1(a,c)a^{-k}b^{-1}v_2(a,c)ba^k$ then $v_1(a,c)a^{-k} = v_1(a,c)$. Then $k=0$, which is a contradiction.

If $w = v_1(a,c)bv_2(a,c)b^{-1} = w\varphi = v_1(a,c)ba^kv_2(a,c)a^{-k}b^{-1}$, then $a^kv_2(a,c)a^{-k} = v_2(a,c)$. Then $v_2 = a^m$, therefore $bab^{-1} \in Fix\varphi$.

Thus $Fix\varphi \subset \langle a, c, bab^{-1}\rangle$. The reverse inclusion is obvious.

The first statement is proven.

Let us now prove the second statement. If w is a word from a and c, then the statement is obvious.

We can assume that w is cyclically reduced and contains the letters b or b^{-1}. By cyclically rearranging and reversing w if necessary, we can assume that w begins with b.

Represent w in the form $w = ba^mw_0$ or $w = ba^mb^{-1}w_0$, where $w_0 \in \mathbb{F}_3$ and the last subword is not equal to b^{-1}. This representation of w is stable using φ^r, which means $w\varphi_k^r$ starts with b and does not end with b^{-1}. Therefore, $w\varphi^r$ is cyclically reduced.

Since w and $w\varphi_k^r$ are conjugate, $w\varphi_k^r$ is a cyclic permutation of w. Therefore, for a suitable s we obtain $w\varphi_k^{rs} = w$. Applying the first statement, we obtain $w \in Fix\varphi_k$.

The second statement is proven.

□

3 The Generators of $Out(G_k)$

It is easy to check that the following maps extend to automorphisms of the group $G_k = \langle a, b, c, t \mid a^t = a, b^t = ba^k, c^t = c\rangle$ for some fixed $k \neq 0$.

$$\psi : \begin{cases} a \mapsto a, \\ b \mapsto tb, \\ c \mapsto c, \\ t \mapsto t, \end{cases} \quad \chi : \begin{cases} a \mapsto a, \\ b \mapsto b, \\ c \mapsto tc, \\ t \mapsto t, \end{cases} \quad \beta : \begin{cases} a \mapsto a, \\ b \mapsto ba, \\ c \mapsto c, \\ t \mapsto t, \end{cases}$$

$$\kappa : \begin{cases} a \mapsto a, \\ b \mapsto ab, \\ c \mapsto ac, \\ t \mapsto t, \end{cases} \quad \mu : \begin{cases} a \mapsto a, \\ b \mapsto b, \\ c \mapsto bab^{-1}c, \\ t \mapsto t, \end{cases} \quad \theta_2 : \begin{cases} a \mapsto a, \\ b \mapsto c^{-1}b, \\ c \mapsto c, \\ t \mapsto t, \end{cases}$$

$$\omega : \begin{cases} a \mapsto a^{-1}, \\ b \mapsto b, \\ c \mapsto c, \\ t \mapsto t^{-1}, \end{cases} \quad \theta_1 : \begin{cases} a \mapsto a, \\ b \mapsto b, \\ c \mapsto c^{-1}, \\ t \mapsto t, \end{cases} \quad \theta_3 : \begin{cases} a \mapsto a^{-1}, \\ b \mapsto b^{-1}, \\ c \mapsto b^{-1}cb, \\ t \mapsto ta^{-k}. \end{cases}$$

Let $\alpha \in Aut(G)$. Denote the coset of the subgroup $Inn(G)$ in the group $Aut(G)$ with the representative α by $[\alpha]$. Then $Out(G) = \{[\alpha] : \alpha \in Aut(G)\}$.

We will assume that the conjugation mapping acts as follows on an arbitrary element $\hat{x} : p \mapsto x^{-1}px$.

Further we will prove that

$$Out(G_k) = \langle [\psi], [\chi], [\beta], [\kappa], [\mu], [\theta_2], [\omega], [\theta_1], [\theta_3] \rangle.$$

Lemma 2. *Let ξ be an arbitrary automorphism of the group $G_k = \langle a, b, c, t | a^t = a, b^t = ba^k, c^t = c\rangle$. Then there are $\tilde{l}, \tilde{k} \in \mathbb{Z}, \varepsilon \in \{0, 1\}, x \in F_3$ such that:*

$$\xi \circ \hat{x} \circ \chi^{\tilde{l}} \circ \psi^{\tilde{k}} \circ \omega^{\varepsilon} : \begin{cases} a \mapsto v^s, \\ b \mapsto v_2, \\ c \mapsto v_3, \\ t \mapsto tv^d, \end{cases}$$

where $\hat{x}$ is a conjugation using $x \in F_3$, $v \in Fix\varphi, v_2, v_3 \in F_3, s, d \in \mathbb{Z}$.

Proof. Let $\xi \in Aut(G_k)$ be an arbitrary automorphism. Consider the action of ξ on the generators of the group G_k (we collect the powers of t from the left in view of the relations of the group)

$$\xi : \begin{cases} a \mapsto t^p w_1', \\ b \mapsto t^l w_2, \\ c \mapsto t^r w_3, \\ t \mapsto t^q w_4, \end{cases}$$

where $p, l, r, q \in \mathbb{Z}, w_1', w_2, w_3, w_4 \in F_3$.

Since ξ is an automorphism, it respects relations. Applying ξ to some of them, we get the following:

(1) $\xi(b^t) = \xi(ba^k) \Rightarrow w_4^{-1}t^{-q}t^l w_2 t^q w_4 = t^l w_2 (t^p w_1')^k, k \neq 0 \Rightarrow p = 0$, since the sum of the degrees t on the left and right must be the same.

(2) $\xi(a^t) = \xi(a) \Rightarrow w_1'\varphi^q$ is conjugate to w_1' in F_3. By the lemma about fixed elements, we find that w_1' is conjugate to an element from $Fix\varphi$.

From (2) it follows that $w_1' = xw_1x^{-1}, x \in F_3, w_1 \in Fix\varphi$. Let us further put $w_1 = v^s$ such that the root of v in F_3 is not extracted. Let us now apply the composition $\xi \circ \hat{x}$ to the generators.

$$\xi \circ \hat{x} : \begin{cases} a \mapsto v^s, \\ b \mapsto t^l v_2, \\ c \mapsto t^r v_3, \\ t \mapsto t^q v_4, \end{cases}$$

where $v \in Fix\varphi, v_2, v_3, v_4, x \in F_3$.

Since a and t commute, their images with respect to the composition of automorphisms $\xi \circ \hat{x}$ must also commute. Therefore, v^s commutes with $t^q v_4$, but since $v \in Fix\varphi$, $v = v\varphi = t^{-1}vt$. Therefore v^s commutes with v_4. Since two words commute in a free group only if they are powers of some third word and v is no longer rooted, then $v_4 = v^d$, where $d \in \mathbb{Z}$.

As a result, we obtain a system of images of generatives $(v^s, t^l v_2, t^r v_3, t^q v^d)$ of the group G_k.

Consider the relation $c^t = c$ and apply composition $\xi \circ \hat{x}$ to it:

$$v^{-d} t^{-q} t^r v_3 t^q v^d = t^r v_3.$$

Since a and t commute, then v^d commutes with t, and therefore $v_3\varphi^q = v^d v_3 v^{-d}$, that is, $v_3\varphi^q$ is conjugate to v_3 in F_3. From this, according to the lemma about fixed elements, v_3 is conjugate with an element from $Fix\varphi$, that is, $v_3 = yuy^{-1}$, where $y \in F_3, u = u\varphi$.

Note that v_3 does not necessarily lie in $Fix\varphi$ (counterexample: for $v_3 = b^{-1}cb$ there are $x = b, u = c, v^d = ba^{kq}b^{-1}$, such that all conditions on v_3 are satisfied, the mapping is an automorphism, but $v_3 \notin Fix\varphi$).

Moreover, note that the sum of the powers of b in v_3 is equal to zero. This follows from the equality $v_3 = yuy^{-1}$, where $y \in \mathbb{F}_3, u \in Fix\varphi$.

We count the sums of powers a, b, c in the images of generators a, b, c in the commutator factor: $v \sim a^\alpha c^\gamma, v_2 \sim a^{\alpha_1} b^{\beta_1} c^{\gamma_1}, v_3 \sim a^{\alpha_2} c^{\gamma_2}$.

Note that $\gamma = 0$. Let it not be the case. Using χ we obtain a contradiction with condition (1).

Since the system $(v^s, t^l v_2, t^r v_3, t^q v^d)$ generates G_k, then there are words w_a, w_b, w_c, such that (we collect the powers of t from the left in view of ratios):

$$a = w_a(v^s, t^l v_2, t^r v_3, t^q v^d) = t^{l_1} w'_a(a, b, c),$$
$$b = w_b(v^s, t^l v_2, t^r v_3, t^q v^d) = t^{l_2} w'_b(a, b, c),$$
$$c = w_c(v^s, t^l v_2, t^r v_3, t^q v^d) = t^{l_3} w'_c(a, b, c).$$

Compare the sums of powers a, b, c on the right and left sides of the equality. Consider a vector of the form (the sum of powers a, the sum of powers b, the sum of powers c).

Then the vectors $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ are linear combinations of vectors $(\alpha, 0, 0), (\alpha_1, \beta_1, \gamma_1), (\alpha_2, 0, \gamma_2)$.

Hence,

$$\langle (\alpha, 0, 0), (\alpha_1, \beta_1, \gamma_1), (\alpha_2, 0, \gamma_2) \rangle_{\mathbb{Z}} \cong \mathbb{Z}^3.$$

Therefore, the matrix composed of these vectors must be invertible over $\mathbb{Z}$. It means that the determinant of such a matrix is equal to ± 1, which means that:

$$\begin{cases} \alpha = \pm 1, \\ \beta_1 = \pm 1, \\ \gamma_2 = \pm 1. \end{cases}$$

Note that in view of the obtained relations, the automorphisms χ and ψ do not affect the degree of t in the images a, t. Trace the sum of powers t in the images of generators b, c when taking the composition of automorphisms $\xi \circ \hat{x} \circ \chi^{\tilde{l}} \circ \psi^{\tilde{k}}$ and we find $\tilde{l}, \tilde{k}$ such that:

The sum of powers t in the image b under the action of the composition of automorphisms $\xi \circ \hat{x} \circ \chi^{\tilde{l}} \circ \psi^{\tilde{k}}$ is equal $l + \gamma_1 \tilde{l} + \beta_1 \tilde{k} = 0$,

the sum of t powers in the image c is equal to $r + (\pm 1)\tilde{l} = 0$.

Since $\beta_1 = \pm 1$, we obtain a system with respect to $\tilde{l}, \tilde{k}$ of the following form:

$$\begin{cases} l + \gamma_1 \tilde{l} \pm \tilde{k} = 0, \\ r + (\pm 1)\tilde{l} = 0. \end{cases}$$

The determinant of the matrix of this system is equal to ± 1, therefore, such a system is solvable and integer $\tilde{k}, \tilde{l}$ can be found.

The resulting composition of automorphisms acts as follows:

$$\xi \circ \hat{x} \circ \chi^{\tilde{l}} \circ \psi^{\tilde{k}} : \begin{cases} a \mapsto v^s, \\ b \mapsto v_2, \\ c \mapsto v_3, \\ t \mapsto t^q v^d, \end{cases}$$

where $v \in Fix\varphi, s, d \in \mathbb{Z}, v_2, v_3 \in F_3$.

For the presented composition to be an automorphism, it is necessary that $q = \pm\, 1$.

Appling if necessary $\omega(a \mapsto a^{-1}, t \mapsto t^{-1})$, we obtain the required.

□

Let us call a set of subwords of the form $a^m, ba^m b^{-1}, c^m$ **unchangeable blocks** (lie in $Fix\varphi$) and a set of subwords of the form b^{-1}, b **changeable blocks** (don't lie in $Fix\varphi$).

Note that any word is divided into these subwords.

Lemma 3. *Under the conditions of Lemma 2, two cases are possible:*

1. $v^s = a, d = 0, v_2 = uba^m, v_3 \in Fix\varphi$, *PiPɾPµu* $\in Fix\varphi$,
2. $v^s = a^{-1}, v^d = a^{-k}, v_2 = a^m b^{-1} u, v_3 = a^{m_1} b^{-1} v b a^{m_2}$,

where $u, v \in Fix\varphi, m, m_1, m_2 \in \mathbb{Z}$.

Proof. Apply the composition of automorphisms from Lemma 2 ($\xi \circ \hat{x} \circ \chi^{\tilde{l}} \circ \psi^{\tilde{k}}$) to the relations:

(1) $t^{-1}at = a \Rightarrow v^{-d} t^{-1} v^s t v^d = v^s$,
(2) $t^{-1}ct = c \Rightarrow v^{-d} t^{-1} v_3 t v^d = v_3$,
(3) $t^{-1}bt = ba^k \Rightarrow t^{-1} v_2 t = v^d v_2 v^{ks-d}$.

First, note that the sum of powers b in the word v_2 is equal to ± 1. This implies that the total number of changeable subwords in v_2 is odd.

For example, let v_2 contain three changeable subwords, that is

$$v_2 = w_1 h_1 w_2 h_2 w_3 h_3 w_4,$$

where $w_i \in Fix\varphi, h_i$ are changeable subwords. The word v_2 is reduced. Under the action of the automorphism φ we obtain a letter-by-letter equality.

$$(*) v_2\varphi = w_1 (h_1)\varphi w_2 (h_2)\varphi w_3 (h_3)\varphi w_4 \equiv v^d w_1 h_1 w_2 h_2 w_3 h_3 w_4 v^{ks-d} = v^d v_2 v^{ks-d}$$

Note that v^d is a power of a. Look at three possible options.

Let there be no contractions between v^d and v_2. We count the number of occurrences of $b^{\pm 1}$ in both sides of the equality (taking into account that the automorphism φ does not change the number of occurrences of $b^{\pm}$). Without loss of generality, we assume that $d > 0$, otherwise we consider $(v^{-1})^{|d|}$. We count the number of occurrences of the letters b ($|.|_b$) in both parts of $(*)$.

$$|v_2|_b = |v_2\varphi|_b = |v^d v_2 v^{ks-d}|_b = |v_2|_b + ks|v|_b$$

Since $k \neq 0, s \neq 0$, then $|v|_b = 0$.

Similar reasoning for the number of occurrences of the letters c leads to the fact that v is a power of a.

Let v^d contract from v_2, but v^d does not cancel completely. Then $v^d = xy, v^d v_2 = xyy^{-1}r = xr$ is a the given word. Note that in this case v^d is not cyclically reduced. It is a contradiction.

Let the reductions take place completely. Then, if v contains occurrences of the letters $b^{\pm 1}, c^{\pm 1}$, then the number of these letters in v_2 will decrease. Similarly to the previous cases, we count the number of occurrences of the letters $b^{\pm}, c^{\pm}$ in both sides of the equality $(*)$. We get a contradiction. It follows that v^d is a power of a, that is, $v = a^z, z \in \mathbb{Z}$.

Without loss of generality, it suffices for us to consider three variants of the v_2 structure.

Choose $h_1 = b, h_2 = b, h_3 = b^{-1}$ for the first example.

Let us use relation (3) ($v_2\varphi = t^{-1}v_2 t = v^d v_2 v^{ks-d} = a^{zd} v_2 a^{z(ks-d)}$):

$$w_1 b a^k w_2 b a^k w_3 a^{-k} b^{-1} w_4 = a^{dz} w_1 b w_2 b w_3 b^{-1} w_4 a^{z(ks-d)}.$$

Note that for cancellations to occur in this case, it is necessary that $w_3 = a^q, q \in \mathbb{Z}$. In this case, two of the three variable subwords merged into the block $bab^{-1} \in Fix\varphi$, the word v_2 has the form $w_1 b w_2$, where $p \in \mathbb{Z}, w_1, w_2\ in Fix\varphi$.

For the second example, let us take $h_1 = b^{-1}, h_2 = b, h_3 = b$.

Again, use relation (3):

$$w_1 a^{-k} b^{-1} w_2 b a^k w_3 b a^k w_4 = a^{dz} w_1 b^{-1} w_2 b w_3 b w_4 a^{z(ks-d)}.$$

Using similar reasoning to the previous example, we obtain a contradiction, since contractions in the center will not occur.

Take $h_1 = b, h_2 = b^{-1}, h_3 = b^{-1}$. We find that v_2 has the form $w_1 b^{-1} w_2, w_1, w_2 \in Fix\varphi, p \in \mathbb{Z}$

Note that for other cases, as well as for a larger odd number of subwords, similar reasoning leads to one of the presented options.

Examine, as an exception, two possible cases with one block.

Let $v_2 = w_1 b w_2 \Rightarrow w_1 b a^k w_2 = a^{zd} w_1 b w_2 a^{z(ks-d)}$.

We compare the subwords before and after the changeable subword in the second equality, we see that $d = 0, w_2 = a^n, zs = 1$, therefore $v^s = a^{zs} = a$ where $n\ in \mathbb{Z}$.

Let $v_2 = w_1 b^{-1} w_2 \Rightarrow w_1 a^{-k} b^{-1} w_2 = a^{dz} w_1 b^{-1} w_2 a^{z(ks-d)}$. We see that $w_1 = a^n, zs = -1$, hence, $v^s = a^{zs} = a^{-1}, zd = -k$, therefore, $v^d = a^{zd} = a^{-k}$.

We obtain two final options for the image of the generating set under the action of the composition of automorphisms:

Case 1: $\begin{cases} a \mapsto a \\ b \mapsto w_1 b a^n \\ c \mapsto v_3 \\ t \mapsto t \end{cases}$, where $w_1 \in Fix\varphi, p, n \in \mathbb{Z}, v_3 \in \mathbb{F}_3$,

Case 2: $\begin{cases} a \mapsto a^{-1} \\ b \mapsto a^n b^{-1} w_2 \\ c \mapsto v_3 \\ t \mapsto t a^{-k} \end{cases}$, where $w_2 \in Fix\varphi, p, n \in \mathbb{Z}, v_3 \in \mathbb{F}_3$.

In this case, the sum of powers b in v_3 is equal to 0. In each of the two cases, let us pay attention to the relation $((c^t)\xi \circ \hat{x} \circ \chi^{\tilde{l}} \circ \psi^{\tilde{k}} = (c)\xi \circ \hat{x} \circ \chi^{\tilde{l}} \circ \psi^{\tilde{k}} \Rightarrow a^{-zd} t^{-1} v_3 t a^{dz} = v_3)$.

Case 1: $\begin{cases} a \mapsto a, \\ b \mapsto w_1 b a^n, \\ c \mapsto v_3, \\ t \mapsto t. \end{cases}$

Applying the relation to this case, we get $v_3 \in Fix\varphi$.

Case 2: $\begin{cases} a \mapsto a^{-1}, \\ b \mapsto a^n b^{-1} w_2, \\ c \mapsto v_3, \\ t \mapsto t a^{-k}, \end{cases}$

where $w_2 \in Fix\varphi, n \in \mathbb{Z}, v_3 \in \mathbb{F}_3$, the sum of powers b in v_3 is equal to 0.

Since the sum of powers b in v_3 is equal to 0, then in addition to the subwords from $Fix\varphi$, v_3 can contain variable subwords, but always in pairs: if there is a certain number of subwords of the form b, then there is sure to be the same number of subwords of the form b^{-1}. We note that (using reasoning similar to that for v_2) given the relation, only one configuration is possible for v_3: $v_3 = a^{m_1} b^{-1} y b a^{m_2}, m_1, m_2\ in \mathbb{Z}, y \in Fix\varphi$, otherwise the necessary contractions will not occur in the center. □

Lemma 4. *Case 2 from Lemma 3 reduces to case 1 using automorphism* $\theta_3(a \mapsto a^{-1}, b \mapsto b^{-1}, c \mapsto b^{-1} c b, t \mapsto t a^{-k})$.

Proof. Consider the composition of the automorphism θ_3 and case number 2 from Lemma 3.

$$\begin{cases} a \xrightarrow{\theta_3} a^{-1} \xrightarrow{2} a, \\ b \xrightarrow{\theta_3} b^{-1} \xrightarrow{2} (w_2)^{-1} b a^{-n} = \tilde{w_2} b a^{-n}, \\ c \xrightarrow{\theta_3} b^{-1} c b \xrightarrow{2} (w_2)^{-1} b a^{-n} a^{m_1} b^{-1} y b a^{m_2} a^n b^{-1} w_2 = \tilde{w_2} b a^{m_1 - n} b^{-1} y b a^{m_2 + n} b^{-1} w_2, \\ t \xrightarrow{\theta_3} t a^{-k} \xrightarrow{2} t, \end{cases}$$

where $w_2, \tilde{w_2}, y \in Fix\varphi, m_1, m_2, n \in \mathbb{Z}$.

□

Formulate an auxiliary statement ([8] p. 20, 2.8):

Statement. Let $U = \{u_1, ..., u_m\}$ be the set of elements of the free group F with the basis $a_1, ..., a_m$. If the following conditions are met:

$(N1)$ $v_1 \neq 1$,
$(N2)$ $v_1 v_2 \neq 1 \Rightarrow |v_1 v_2| \geqslant |v_1|, |v_2|$,
$(N3)$ $v_1 v_2 \neq 1, v_2 v_3 \neq 1 \Rightarrow |v_1 v_2 v_3| > |v_1| - |v_2| + |v_3|$,
for all triplets $v_1, v_2, v_3 \in U^{\pm 1}$ Pë $\langle U \rangle = F$, then $U^{\pm 1} = \{a_1^{\pm 1}, ..., a_m^{\pm 1}\}$.
For convenience, we introduce the following automorphism:

$$\delta = [\kappa, \theta_2] = \kappa \circ \theta_2 \circ \kappa^{-1} \circ \theta_2^{-1} : \begin{cases} a \mapsto a, \\ b \mapsto ab, \\ c \mapsto c, \\ t \mapsto t. \end{cases}$$

Lemma 5. *The composition of automorphisms $\xi \circ \hat{x} \circ \chi^{\tilde{l}} \circ \psi^{\tilde{k}}$, acting on the generators as follows:*

$$\xi \circ \hat{x} \circ \chi^{\tilde{l}} \circ \psi^{\tilde{k}} : \begin{cases} a \mapsto a, \\ b \mapsto w_1 b a^n, \\ c \mapsto v_3, \\ t \mapsto t, \end{cases}$$

where $w_1, v_3 \in Fix\varphi, n \in \mathbb{Z}$, lies in the subgroup generated by automorphisms defined in Section 3.

Proof. Apply the automorphism β the required number of times, and since $w_1\beta = w_1, v_3\beta = v_3$ we obtain the following:

$$\eta = \xi \circ \hat{x} \circ \chi^{\tilde{l}} \circ \psi^{\tilde{k}} \circ \beta^{-n} : \begin{cases} a \mapsto a, \\ b \mapsto w_1 b, \\ c \mapsto v_3, \\ t \mapsto t, \end{cases}$$

where $w_1, v_3 \in Fix\varphi$, the sum of the powers of c in v_3 is equal to ± 1 (see Lemma 3).

Note that if $w_1 = 1$, then with the help of an auxiliary statement it is proved that in the word v_3 there is one occurrence of the letter $c^{\pm 1}$ and therefore η is expressed through $(\kappa^{-1})^{(\theta_1 \circ \theta_2)^{-1}}, \kappa^{(\theta_2^{-1})}, \mu, \theta_1$.

If $v_3 = c$, then similarly there are no subwords of the form bab^{-1} in the word w_1 and η is expressed through $\beta, \theta_2, \theta_1, \delta$.

Let $w_1 \neq 1, v_3 \neq c, |w_1| \geqslant |v_3|$. (The case $|v_3| \geqslant |w_1|$ is treated similarly). For convenience, we redesignate w_1 by u, v_3 by v, that is, in our case, the initial condition has the form $|u| \geqslant |v|$. Denote the length of the word u relative to generators $Fix\varphi$ by $|u|_\varphi$, which is a free subgroup of the free group $F_3 = \langle a, b, c \rangle$.

Let the statement of the lemma be false in this case. We choose a counterexample such that $|u|_\varphi + |v|_\varphi$ is minimal.

Note that if u or v starts with a, then we can apply one of the automorphisms δ or η to the required degree.

Let us further assume that these words do not begin with a. Let $u = ww_1, v = ww_2$, where w is the largest common prefix, $w_2^{-1}w_1$ is given.

We check the properties $(N1)-(N3)$ for the system (a, ww_1b, ww_2). If the properties are satisfied, then $(a, ww_1b, ww_2) = (a^{\pm1}, b^{\pm1}, c^{\pm1})$ is a contradiction. Failure to meet these properties implies the following conditions:

$$\begin{cases} |w_1| + 1 < |w|, \\ |w_2| < |w|. \end{cases}$$

Applying elementary Nielsen transformations, we pass to the system $(a, w_2^{-1}w_1b, ww_2)$, this is a system of reduced words, and checking the properties $(N1)-(N3)$ for it, we obtain the following system:

$$\begin{cases} |w_1| + 1 > |w_2|, \\ |w| > |w_2|, \end{cases}$$

which is fulfilled due to the conditions already obtained. Therefore, $(a, w_2^{-1}w_1b, ww_2) = (a^{\pm1}, b^{\pm1}, c^{\pm1})$ is a contradiction. It means that $w_2 = 1$, and initially the system had the form:

$$(a, ww_1b, w).$$

Using Nielsen transformations, we pass to the system (a, w_1b, w), for which the properties $(N1)-(N3)$ are satisfied if w_1 does not begin with a. Therefore, w_1 starts at $a^{\pm1}$ and the automorphism has the form:

$$\eta : \begin{cases} a \mapsto a, \\ b \mapsto wa^{\pm m}hb, \\ c \mapsto w, \\ t \mapsto t. \end{cases}$$

Consider the composition $\beta^{\pm m} \circ \theta_2 \circ \eta$ on generators:

$$\beta^{\pm m} \circ \theta_2 \circ \eta_2 : \begin{cases} a \xrightarrow{\beta^{\pm m}} a \xrightarrow{\theta_2} a \xrightarrow{\eta} a, \\ b \xrightarrow{\beta^{\pm m}} a^{\pm m}b \xrightarrow{\theta_2} a^{\pm m}c^{-1}b \xrightarrow{\eta} a^{\pm m}w^{-1}wa^{\mp m}hb = hb, \\ c \xrightarrow{\beta^{\pm m}} c \xrightarrow{\theta_2} c \xrightarrow{\eta} w, \\ t \xrightarrow{\beta^{\pm m}} t \xrightarrow{\theta_2} t \xrightarrow{\eta} t. \end{cases}$$

Note that $|u|_\varphi + |v|_\varphi > |h|_\varphi + |w|_\varphi$ is a contradiction with the stated counterexample. It means that the presented automorphism reduces to the identity automorphism. □

4 The Structure of $Out(G_k)$

Previously we proved that

$$Out(G_k) = \langle [\psi], [\chi], [\beta], [\kappa], [\mu], [\theta_2], [\omega], [\theta_1], [\theta_3] \rangle.$$

Next we will prove that this group decomposes into a semidirect product of subgroups

$$N = \langle [\psi], [\chi], [\beta], [\kappa], [\mu], [\theta_2] \rangle,$$
$$S = \langle [\omega], [\theta_1], [\theta_3] \rangle$$

and we will study their structure.

5 The Structure of S

Note that the subgroup $S \cong (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ due to the following relations between automorphisms:
$\theta_3^2 = \omega^2 = \theta_1^2 = id,$
$\theta_3 \circ \omega = \omega \circ \theta_3,$
$\theta_1 \circ \omega = \omega \circ \theta_1,$
$\theta_3 \circ \theta_1 = \theta_1 \circ \theta_3.$

6 The Structure of N

Study the structure of the subgroup

$$N = \langle [\psi], [\chi], [\beta], [\theta_2], [\mu], [\kappa] \rangle.$$

Let us prove that

$$N \cong \mathbb{Z}^2 \times \mathbb{Z}_k \times N_1,$$

where $N_1 = \langle [\mu], [\kappa], [\theta_2] \rangle$.

Lemma 6. *$N \cong (\mathbb{Z}^2 \times \mathbb{Z}_k) \times N_1$, where $N_1 = \langle [\mu], [\kappa], [\theta_2] \rangle$.*

Proof. The subgroup $\langle [\psi], [\chi], [\beta] \rangle$ is normal in the subgroup N, $\langle [\psi], [\chi], [\beta] \rangle \cong \mathbb{Z}^2 \times \mathbb{Z}_k$. Moreover,

$$N \cong (\mathbb{Z}^2 \times \mathbb{Z}_k) \times \langle [\mu], [\kappa], [\theta_2] \rangle.$$

Note that the subgroup $\langle [\beta], [\chi], [\psi] \rangle$ is contained in the center of the subgroup N. In addition, the classes $[\psi]$ and $[\chi]$ have an infinite order in the group $Out(G)$ and $\beta^k = \hat{t}$.

It remains to show that $\langle [\psi], [\chi], [\beta] \rangle \cap \langle [\mu], [\kappa], [\theta_2] \rangle = id$. Indeed, note that intersection is impossible due to the action of automorphisms on the generators b, c.

□

Lemma 7. *$Out(G) \cong N \rtimes S$*

Proof. It is enough to establish the following relations in the group $Aut(G)$:

$$\begin{array}{lllllllll}
\theta_1^{-1}\psi\theta_1 & = & \psi, & \omega^{-1}\psi\omega & = & \psi, & \theta_3^{-1}\psi\theta_3 & = & \psi^{-1}, \\
\theta_1^{-1}\chi\theta_1 & = & \chi^{-1}, & \omega^{-1}\chi\omega & = & \chi, & \theta_3^{-1}\chi\theta_3 & = & \chi, \\
\theta_1^{-1}\beta\theta_1 & = & \beta, & \omega^{-1}\beta\omega & = & \beta, & \theta_3^{-1}\beta\theta_3 & = & \beta\circ\hat{a}, \\
\theta_1^{-1}\theta_2\theta_1 & = & \theta_2, & \omega^{-1}\theta_2\omega & = & \theta_2, & \theta_3^{-1}\theta_2\theta_3 & = & \theta_2^{-1}, \\
\theta_1^{-1}\mu\theta_1 & = & \theta_2\mu^{-1}\theta_2^{-1}\beta, & \omega^{-1}\mu\omega & = & \mu^{-1}, & \theta_3^{-1}\mu\theta_3 & = & \theta_2\circ\kappa^{-1}\circ\theta_2^{-1}, \\
\theta_1^{-1}\kappa\theta_1 & = & \hat{a}^{-1}\beta\theta_2\kappa^{-1}\theta_2^{-1}, & \omega^{-1}\kappa\omega & = & \kappa^{-1}, & \theta_3^{-1}\kappa\theta_3 & = & \theta_2\circ\mu^{-1}\circ\theta_2^{-1}.
\end{array}$$

Check one of them; the rest are checked in the same way.

$$\theta_1^{-1}\kappa\theta_1 : \begin{cases}
a \xrightarrow{\theta_1^{-1}} a & \xrightarrow{\kappa} a & \xrightarrow{\theta_1} a \\
b \xrightarrow{\theta_1^{-1}} b & \xrightarrow{\kappa} ab & \xrightarrow{\theta_1} ab \\
c \xrightarrow{\theta_1^{-1}} c^{-1} & \xrightarrow{\kappa} c^{-1}a^{-1} & \xrightarrow{\theta_1} ca^{-1} \\
t \xrightarrow{\theta_1^{-1}} t & \xrightarrow{\kappa} t & \xrightarrow{\theta_1} t
\end{cases}$$

$$\hat{a}^{-1}\beta\theta_2\kappa^{-1}\theta_2^{-1} : \begin{cases}
a \xrightarrow{\hat{a}^{-1}} a & \xrightarrow{\beta} a & \xrightarrow{\theta_2} a & \xrightarrow{\kappa^{-1}} a & \xrightarrow{\theta_2^{-1}} a \\
b \xrightarrow{\hat{a}^{-1}} aba^{-1} & \xrightarrow{\beta} ab & \xrightarrow{\theta_2} ac^{-1}b & \xrightarrow{\kappa^{-1}} ac^{-1}b & \xrightarrow{\theta_2^{-1}} ab \\
c \xrightarrow{\hat{a}^{-1}} aca^{-1} & \xrightarrow{\beta} aca^{-1} & \xrightarrow{\theta_2} aca^{-1} & \xrightarrow{\kappa^{-1}} ca^{-1} & \xrightarrow{\theta_2^{-1}} ca^{-1} \\
t \xrightarrow{\hat{a}^{-1}} t & \xrightarrow{\beta} t & \xrightarrow{\theta_2} t & \xrightarrow{\kappa^{-1}} t & \xrightarrow{\theta_2^{-1}} t
\end{cases}$$

Note that $N \cap S = id$. This is true in view of the action of automorphisms on generators and in view of the finite order of automorphisms whose classes generate the subgroup S. □

Lemma 8. $N_1 \cong K \rtimes \mathbb{Z}$, *where* K *is a rectangular Artinian group.*

Proof. Denote $K = \langle\langle \mu, \kappa \rangle\rangle_{N_1}$. Then $N_1 \cong K \rtimes \mathbb{Z}$.

To prove this, replace the system of generators of the subgroup K as follows: $\kappa_i = \theta_2^i \circ \kappa \circ \theta_2^{-i}$, $\mu_i = \theta_2^i \circ \mu \circ \theta_2^{-i}$, where $i \in \mathbb{Z}$.

Let us show that K is a rectangular Artinian group, which has the following representation:

$$K \cong \langle\ \mu_i, \kappa_i, i \in \mathbb{Z} \mid [\kappa_i, \mu_i], [\mu_i, \mu_{i+1}], [\kappa_i, \kappa_{i+1}], i \in \mathbb{Z}\ \rangle.$$

Note that the system of generators $\mu_i, \kappa_i, i \in \mathbb{Z}$ is sufficient for the subgroup K, and the indicated commutation relations are obvious. Then consider $i \geq 1$. (Otherwise, the generator with a negative index can be corrected using the conjugation θ_2) Show that, in addition to the indicated commutation relations, there are no other relations in the representation K.

Denote: $\xi_k = \kappa_{k-1}^{-1} \circ \kappa_k$, then $\kappa_i = \kappa_0 \circ \xi_1 \circ ... \circ \xi_k$; $\delta_k = \mu_k \circ \mu_{k-1}^{-1}$, then $\mu_i = \delta_i \circ ... \circ \delta_1 \circ \mu_0$.

Such compositions of new generators of the subgroup K give the following actions on the generators of the group G_k (Generators a, t remain in place).

$$\xi_k : \begin{cases} b \mapsto (a^{-1})^{c^{k-1}} b \\ c \mapsto c \end{cases}$$

$$\delta_k : \begin{cases} b \mapsto (a^{-1})^{(c^{k-1}b)^{-1}} b \\ c \mapsto c^{(a^{-1})^{(c^{k-1}b)^{-1}}} \end{cases}$$

The subgroup $Aut(F_n(x_1...x_n))$ generated by partial conjugations $\alpha_{ij} = (x_i, x_j)$ was studied in [6].

Where:

$$(x_i, x_j) : \begin{cases} x_i \mapsto x_j^{-1} x_i x_j \\ x_k \mapsto x_k, k \neq i \end{cases}$$

The representation of this subgroup has the following relations:
(1) $\alpha_{ij}\alpha_{kj} = \alpha_{kj}\alpha_{ij}$,
(2) $\alpha_{ij}\alpha_{kl} = \alpha_{kl}\alpha_{ij}$,
(3) $\alpha_{ij}\alpha_{kj}\alpha_{ik} = \alpha_{ik}\alpha_{ij}\alpha_{kj}$.

We consider the set $(\xi_k, \delta_k, k \geq 1)$.

Note that the elements of this set generate partial conjugations on the subgroup $Fix\varphi.(\delta_k = \alpha_{23}^{k-1} \circ \alpha_{32} \circ \alpha_{23}^{-(k-1)}, \xi_k = \alpha_{23}^{k-1} \circ \alpha_{21} \circ \alpha_{23}^{-(k-1)})$

It follows that the set $(\xi_k, \delta_k, k \geq 1)$ is generated by partial conjugations $\alpha_{23}, \alpha_{21}, \alpha_{32}$.

Consider the factor group by normal closure:

$$\langle \alpha_{21}, \alpha_{13}, \alpha_{23}, \alpha_{21}, \alpha_{31}, \alpha_{32} \rangle / \langle \langle \alpha_{23}\alpha_{13}, \alpha_{21}\alpha_{31}, \alpha_{32}\alpha_{12} \rangle \rangle$$
$$\cong \langle \alpha_{23}, \alpha_{21}, \alpha_{32} \rangle \text{ is free.}$$

It means that there are no relations between the elements of the set $(\xi_k, \delta_k, k \geq 1)$.

Consider the word $\omega(\mu_0, \kappa_0, \xi_k, \delta_k, k \geqslant 1)$. Let the subgroup K contain a relation in addition to the indicated relations of commutation, i.e. $\omega = id$.

Using commutation, we can get rid of occurrences of μ_0.

We get $\omega_1(\xi_k, \delta_k, \kappa_0) = id$, which can be rewritten:

$$(**)\omega_2(\xi_k, \delta_k) = \kappa_0^{s_1} \circ v_1(\delta_k) \circ \kappa_0^{s_2} \circ v_2(\xi_k)...v_n(\xi_k)$$

The left side of the relation $(**)$ acts on the generators of the group G_k as partial conjugations. Consider the action of the right side of the relation $(**)$ on the generator c (without loss of generality it is sufficient to consider the action on only one generator) $c \mapsto a^{s_1} p_1^{-1} a^{s_2} p_2^{-1} ... a^{s_n} p_n^{-1} \mathbf{c} p_n ... p_1$, where $p_i, i = 1...n$ do not change with respect to each other under the influence of mappings. Therefore, $p_i = a^{t_i}, i = 1...n$. This is a contradiction. □

Thus, the following theorem follows from Lemmas 6-8.

Theorem 2.

$$Out(G_k) \cong ((\mathbb{Z}^2 \times \mathbb{Z}_k) \times N_1) \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2),$$

where $N_1 = \langle \mu, \kappa, \theta_2 \rangle$.

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