Hermite Expansions of $C-$regularized cosine Functions

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Abstract

The aim of this paper is to approximate the exponentially bounded $C-$regularized cosine function by the Hermite series, recalling the notions and the results used.

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1 Introduction and preliminaries

The series expansion of Hermite orthogonal polynomials has been an important tool in quantum mechanics and statistical studies, both theoretical and applied. The study of sufficient conditions for the convergence of Hermite series has been the subject of numerous works; for more details see [8]. In 2015, L. Abadias and P. Miana studied in their article [4], the Hermite expansion of C_0-groups and cosine functions.
The generation of this paper is motivated by the question asked in [10], to interested readers my try to reconsider the results from [4] for some other classes of (non-) degenerate resolvent operator families. Additionally, [10] has already investigated the Laguerre expansions of degenerate (a, k)-regulazed $C-$resolvent families.
in this works we will be interested in Hermite expansion of C-regularized cosine function, starting with a reminder of the notations, concepts and results used. For more details on cosine functions see [7], for regularized operator families see [14] and [16] and for C-regularized cosine functions see [15].
Throughout this paper E denotes a non-trivial complex Banach space, $\mathfrak{F}(E,F)$ denotes the set of all applications from E to another Banach space F, $B(E)$

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denotes the space of bounded linear operators from E into itself, and $L^1_{loc}(E)$ the set of all $f \in \mathfrak{L}(\mathbb{R}, E)$ locally integrable. For a closed linear operator A on E, $\mathcal{D}(A)$, $R(A)$ and $\rho(A)$ denote its domain, range and resolvent set, respectively. $\mathcal{D}(A)$ equipped with the graph norm $\| x \|_{\mathcal{D}(A)} = \| x \|_E + \| Ax \|_E$ become Banach space. Throughout this paper, $C \in B(E)$ will be an injective operator. The C-resolvent set of A, denoted by $\rho_C(A)$, is defined by $\rho_C(A) := \{\lambda \in \mathbb{C} \mid R(C) \subseteq R(\lambda I - A) \ \ and \ \ \lambda I - A$ is injective in $B(E)\}$ and by $R_C(\lambda, A) = (\lambda I - A)^{-1}C \ \ (\lambda \in \rho_C(A))$ the C-resolvent.

Hermite functions and Hermite Expansions on Banach spaces

For all $n \in \mathbb{N}$, the classical Hermite polynomial is defined by Rodrigues formula:

$$(\forall x \in \mathbb{R}) \ \ H_n(x) = e^{x^2}(-1)^n \frac{d^n}{dx^n} e^{-x^2}.$$

H_n is a polynomial with of degree n, the same parity as n, whose highest monomial degree is $2^n X^n$ and has real coefficients. Furthermore, they verify the following orthogonality condition:

$$\int_{\mathbb{R}} H_n(x) H_m(x) e^{-x^2} dx = \delta_{n,m} n! 2^n \sqrt{\pi},$$

where $\delta_{n,m}$ is the Kronecker delta. We also have recurrence relations, differential equations and the Muckenhoupt estimates:

$$(\exists c > 0) \ (\forall (n_0, t) \in \mathbb{N} \times \mathbb{R} : \ \ t^2 \leq 2(2n_0+1)) \ (\forall n \geq n_0) \mid H_n(t) \mid \leq c\left(\frac{e^{\frac{t^2}{2}} \sqrt{2^n n!}}{n^{\frac{1}{12}}}\right). \tag{1}$$

For more details on the classical theory of orthogonal polynomials see [21], [13], [3], [4] and [2].
The Hermite functions on $\mathbb{R}$ are defined by:

$$\phi_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n(x) e^{\frac{-x^2}{2}} = (-1)^n \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-x^2}. \tag{2}$$

$(\phi_n)_{n \in \mathbb{N}}$ is an orthonormal basis in the Hilbert space $L^2(\mathbb{R})$ and satisfied some recurrence relations, equality and inequality. For more details see [21], [13], [3], [4] and [2].
For $n \in \mathbb{N}$, we denote by φ_n, the function on $\mathbb{R}$ is defined by:

$$(\forall t \in \mathbb{R}) \ \ \varphi_n(t) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} e^{\frac{-t^2}{2}} \phi_n(t) = \frac{1}{2^n n! \sqrt{\pi}} e^{-t^2} H_n(t) = \frac{(-1)^n}{2^n n! \sqrt{\pi}} \frac{d^n}{dt^n} e^{-t^2}. \tag{3}$$

φ_n has the same parity as n, satisfies recurrence relations and differential equations, for example:

$$\left(\forall \, (n, m) \in \mathbb{N}^2\right) \ \ \varphi_n^{(m)} = (-1)^m 2^m (n+1)...(n+m) \varphi_{n+m}. \tag{4}$$

And the following useful inequality given by Cauchy-Schwartz inequality:

$$\| \varphi_n \|_1 \leq \frac{1}{\sqrt{2^n n!}}. \tag{5}$$

For more details see [4].
One of the most important properties of $(\varphi_n)_{n\in\mathbb{N}}$ family is that if $f : \mathbb{R} \to E$ is a differentiable function such that $\int_{-\infty}^{+\infty} e^{-t^2} \| f(t) \|^2 \, dt < +\infty$, then the series $\sum_{n\in\mathbb{N}} c_n(f)H_n(t)$ converges pointwise to $f(t)$ for each $t \in \mathbb{R}$, where

$$c_n(f) = \int_{-\infty}^{+\infty} \varphi_n(t)f(t)dt.$$

For more details see [13], [19], [4] and [8].

$C-$regularized semigroups and $C-$regularized cosine functions

C-regularized semigroups

A map $T : [0, +\infty[\to B(E)$ is called C-regularized semigroup or C-semigroup, if

1. $T(t + s)C = T(t)T(s)$ for all $t, s \in \mathbb{R}^+$.
2. $T(0) = C$.
3. $t \mapsto T(t)x$ is continuous on $\mathbb{R}^+$ for every $x \in E$.

Its generator W is defined by

$$\mathcal{D}(W) = \left\{ x \in E : \lim_{s\to 0^+} \frac{T(s)x - Cx}{s} \text{ exists in R(C)} \right\}$$

and

$$(\forall x \in \mathcal{D}(W)) \quad Wx = C^{-1} \lim_{s\to 0} \frac{T(s)x - Cx}{s}.$$

We say that $(T(t))_{t\geq 0}$ is exponentially bounded $C-$regularized semigroup if $(\exists M \geq 0) \quad (\exists \omega \geq 0) : \| T(t) \| \leq Me^{\omega t}, \ for \ all \ t \geq 0.$
$(T(t))_{t\geq 0}$ is no degenerate if $T(t)x = 0$ for all $t \geq 0$, then $x = 0$, see [6], [18] and [17].
Since we have for all (t, x) fixed in $\mathbb{R}^+ \times \mathcal{D}(W)$:
$\lim_{s\to 0^+} \frac{T(s)T(t)x - CT(t)x}{s} = T(t) \lim_{s\to 0^+} \frac{T(s)x - Cx}{s} = T(t)CWx = CT(t)Wx \in R(C)$, this means that $T(t)x \in \mathcal{D}(W)$ and $WT(t)x = T(t)Wx$.
The following results can be found in [6] and [5].
Let $(T(t))_{t\in\mathbb{R}^+}$ be a strongly continuous family such that $\| T(t) \| \leq Me^{\omega t}$ for all $t \geq 0$ and W is closed linear operator, to conclude that $(T(t))_{t\in\mathbb{R}^+}$ is C-regularized semigroup generated by W, it is sufficient that :

$$W = C^{-1}WC, \ (\omega, +\infty) \subset \rho_C(W), \ and \ R_C(\lambda, W)x = \int_0^{+\infty} e^{-\lambda t}T(t)xdt, \tag{6}$$

$for\ \lambda > \omega\ and\ x \in E$.
We refer to [25], [5] and [18] for more details.
Let $0 < \alpha \leq \frac{\pi}{2}$, $\Sigma_\alpha = \{\lambda \in \mathbb{C}/\lambda \neq 0 \ \ and \ \ \mid arg(\lambda) \mid < \alpha\}$ and let $(T(t))_{t\geq 0}$ be a C-regularized semigroup. Then we say that $(T(t))_{t\geq 0}$ is an analytic C-regularized semigroup of angle α, if there exists an analytic function $\mathbf{T} : \Sigma_\alpha \to B(E)$ which satisfies:

1. $\mathbf{T}(t) = T(t) \ \ for \ \ all \ \ t > 0.$
2. $\lim\limits_{z\to 0, \ \ z\in\Sigma_\gamma} \mathbf{T}(z)x = 0$ for all $\gamma \in]0, \alpha[$ and $x \in E$.

For more details see [25].

$C-$regularized cosine functions

A map $T : \mathbb{R} \to B(E)$ is called C-regularized cosine function or $C-$cosine function, if

1. $T(t+s)C + T(t-s)C = 2T(t)T(s)$ for all $t, s \in \mathbb{R}$.
2. $T(0) = C$.
3. $t \to T(t)x$ is continuous on $\mathbb{R}$ for every $x \in E$.

We say that $(T(t))_{t\in\mathbb{R}}$ is exponentially bounded $C-$regularized cosine if

$$(\exists M > 0) \quad (\exists \omega \geq 0) : \parallel T(t) \parallel \leq Me^{\omega|t|}, \quad for \ \ all \ \ t \in \mathbb{R}.$$

The associated sine operator function $S(.)$ is defined by $S(t) := \int_0^t T(s)ds \ \ for \ \ all \ \ t \in \mathbb{R}$. The operator W defined by

$$\mathcal{D}(W) = \left\{ x \in E : \lim_{s\to 0} \frac{2(T(s)x - Cx)}{s^2} \text{ exists in R(C)} \right\}$$

and

$$Wx = C^{-1} \lim_{s\to 0} \frac{2(T(s)x - Cx)}{s^2} \quad for \ \ all \ \ x \in \mathcal{D}(W).$$

is called the generator of $(T(t))_{t\in\mathbb{R}}$, see [12] and [17].
We notice that W is closed linear operator in E, T is odd, $T(t)x \in \mathcal{D}(W)$, $S(t)x \in \mathcal{D}(W)$, $\int_0^t S(s)xds \in \mathcal{D}(W)$ and $W\int_0^t S(s)xds = T(t)x - Cx$ for all $x \in E$ and $t \in \mathbb{R}$, $C^{-1}WC = W$ and if $(T(t))_{t\in\mathbb{R}}$ is exponentially bounded $C-$regularized cosine then the function sine $S := S(t))_{t\in\mathbb{R}}$ is also exponentially bounded, so for all $k \in \mathbb{N}$ and $x \in \mathcal{D}(W)$,

$$\lim_{t\to\pm\infty} t^k e^{-t^2} T(t)x = \lim_{t\to\pm\infty} t^k e^{-t^2} S(t)x = \lim_{t\to\pm\infty} t^k e^{-t^2} WS(t)x = 0, \quad (7)$$

because, if $\parallel T(t) \parallel \leq Me^{\omega t}$ for all $t \in \mathbb{R}$, then for all $x \in \mathcal{D}(W)$:
$\parallel t^k e^{-t^2} T(t)x \parallel \leq \mid t \mid^k e^{-|t|^2} \parallel T(t)x \parallel \leq \mid t \mid^k e^{\frac{\omega^2}{4}} e^{-(|t|-\frac{\omega}{2})^2} \parallel x \parallel$, see [12,

proposition 1.1 and 1.2] for more properties and more details.
Consider in E the well-posed Cauchy problem

$$((ACP(W, u_0, u_1, 0)_2) \begin{cases} u''(t) = Wu(t), \quad t \in \mathbb{R} \\ u(0) = u_0 \\ u'(0) = u_1. \end{cases}$$

Where W generates a C-cosine operator function $(T(t))_{t\in\mathbb{R}}$ then

$$u(t) = C^{-1}T(t)u_0 + C^{-1}S(t)u_1$$

is the unique solution of the above Cauchy problem for every pair (u_0, u_1) of initial values in $C(\mathcal{D}(A))$. For more details see [12].
A strongly continuous family $(T(t))_{t\in\mathbb{R}^+}$ such that $\| T(t) \| \leq Me^{\omega t}, \ for \ all \ t \geq 0$ is $C-$regularized cosine with generator the closed linear operator W if and only if the following condition holds:

$$C^{-1}WC = W, \lambda^2 \in \rho_C(W) \text{ and } \lambda R_C(\lambda^2, W) = \int_0^{+\infty} e^{-\lambda t}T(t)dt \text{ on } E \text{ for all } \lambda > \omega.$$

For more details see [11], [12] and [23] .
If W is the generator of exponentially bounded and $C-$regularized cosine $(T(t))_{t\in\mathbb{R}}$, then W is the generator of analytic $C-$regularized semigroup of angle $\frac{\pi}{2}$ defined by

$$(\forall z \in \Sigma_{\frac{\pi}{2}}), \ T_1(z)x = \int_0^{+\infty} \frac{e^{\frac{-s^2}{4z}}}{\sqrt{\pi z}} T(s)xds \ \ (for \ all \ z \in \Sigma_{\frac{\pi}{2}} \ , Re(z) > 0) \quad (8)$$

whose proof is similar for C_0-cosine operator-valued function, see [9] and [1].

2 Main results

Lemma 2.1. *Let $(T(t))_{t\in\mathbb{R}}$ be an exponentially bounded C-cosine function on a Banach space E with generator $(W, \mathcal{D}(W))$. For all $n \in \mathbb{N}$ and $x \in \mathcal{D}(W)$, we have :*

1. $\int_{-\infty}^{+\infty} \varphi_{2n+1}(t)T(t)xdt = 0$
2. $\int_{-\infty}^{+\infty} \varphi_{2n}(t)T(t)xdt = \frac{1}{2^{2n}(2n)!}W^n T_1(\frac{1}{4})x.$
3. *In the case where* $\sup_{t\in\mathbb{R}} \| T(t) \| < +\infty$, *we have :*
   - *(i)* $\| W^n T_1(\frac{1}{4})x \| \leq 2^n \sqrt{(2n)!} \sup_{t\in\mathbb{R}} \| T(t) \| \| x \|.$
   - *(ii)* $\| \int_{-\infty}^{+\infty} \varphi_{2n}(t)T(t)xdt \| \leq \frac{c_1\sqrt{(2(n-1))!}}{2^{n+1} \ (2n)!} \| Wx \|$ *with c_1 a positive constant.*

Proof

1. Let $n \in \mathbb{N}$ and $x \in E$. The function $t \mapsto \varphi_{2n+1}(t)T(t)x$ is continuous and odd, integrable in the senss of Bochner approach, then $\int_{-\infty}^{+\infty} \varphi_{2n+1}(t)T(t)xdt = 0$.

2. Let $x \in \mathcal{D}(W)$. For all $n \in \mathbb{N}$, let $I_n(x) = \int_{-\infty}^{+\infty} \frac{d^{2n}}{dt^{2n}}(e^{-t^2})T(t)xdt$, then:

$$\begin{aligned} I_n(x) &= [\frac{d^{2n-1}}{dt^{2n-1}}(e^{-t^2})T(t)x]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \frac{d^{2n-1}}{dt^{2n-1}}(e^{-t^2})\frac{d}{dt}T(t)xdt \\ &= 0 - [\frac{d^{2n-2}}{dt^{2n-2}}(e^{-t^2})WS(t)x]_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} \frac{d^{2n-2}}{dt^{2n-2}}(e^{-t^2})\frac{d^2}{dt^2}T(t)xdt \\ &= 0 - 0 + \int_{-\infty}^{+\infty} \frac{d^{2n-2}}{dt^{2n-2}}(e^{-t^2})WT(t)xdt \\ &= W \int_{-\infty}^{+\infty} \frac{d^{2n-2}}{dt^{2n-2}}(e^{-t^2})T(t)xdt \\ &= W\ I_{n-1}(x). \end{aligned}$$

A simple recurrence on n gives

$$I_n(x) = W^n I_0(x) = W^n \int_{-\infty}^{+\infty} e^{-t^2}T(t)xdt$$

By definition of φ_n; from where we obtain:

$$\begin{aligned} \int_{-\infty}^{+\infty} \varphi_{2n}(t)\,T(t)\,xdt &= \int_{-\infty}^{+\infty} \frac{(-1)^{2n}}{2^{2n}(2n)!\sqrt{\pi}} \frac{d^{2n}}{dt^{2n}} e^{-t^2}T(t)xdt \\ &= \frac{1}{2^{2n}(2n)!\sqrt{\pi}} I_n(x) \\ &= \frac{W^n}{2^{2n}(2n)!\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-t^2}T(t)xdt \\ &= \frac{W^n}{2^{2n}(2n)!\sqrt{\pi}} 2\int_{0}^{+\infty} e^{-t^2}T(t)xdt \\ &= \frac{W^n}{2^{2n}(2n)!} T_1(\frac{1}{4})x. \end{aligned}$$

3. In the case where $\sup_{t\in\mathbb{R}} \| T(t) \| < +\infty$, we obtain:

- (i) For all $x \in \mathcal{D}(W)$:

$$\begin{aligned}
\| W^n T_1(\frac{1}{4})x \| &\leq 2^{2n}(2n)! \int_{-\infty}^{+\infty} \| \varphi_{2n}(t)T(t)x \| \, dt \\
&\leq 2^{2n}(2n)! \left(\sup_{t\in\mathbb{R}} \| T(t) \| \right) (\| x \|)(\| \varphi_{2n}(t) \|_1) \\
&\leq 2^{2n}(2n)! \sup_{t\in\mathbb{R}} \| T(t) \| \times \| x \| \times \frac{1}{\sqrt{2^{2n}(2n)!}} \quad (by \;\; 5) \\
&\leq 2^n \sqrt{(2n)!} \, \| x \| \sup_{t\in\mathbb{R}} \| T(t) \| \, .
\end{aligned}$$

- (ii) Let $x \in \mathcal{D}(W)$ and $n \geq 1$, if we posed $B = \| \int_{-\infty}^{+\infty} \varphi_{2n}(t)T(t)xdt \|$; then using (i) and integration by parts we get

$$\begin{aligned}
B &= \| \frac{1}{4(2n-1)(2n)} \int_{-\infty}^{+\infty} \varphi''_{2n-2}(t)T(t)xdt \| \quad (by \;\; 4) \\
&= \| -\frac{1}{4(2n-1)(2n)} \int_{-\infty}^{+\infty} \varphi'_{2n-2}(t)T'(t)xdt \| \\
&= \| \frac{1}{4(2n-1)(2n)} \int_{-\infty}^{+\infty} \varphi_{2n-2}(t)T''(t)xdt \| \\
&= \| \frac{1}{4(2n-1)(2n)} \int_{-\infty}^{+\infty} \varphi_{2n-2}(t)WT(t)xdt \| \quad (like \; 1) \\
&\leq \frac{1}{4(2n-1)(2n)} \left(\int_{-\infty}^{+\infty} | \varphi_{2n-2}(t) | \, dt \right) \left(\sup_{t\in\mathbb{R}} \| T(t) \| \right) (\| Wx \|) \\
&\leq \left(\sup_{t\in\mathbb{R}} \| T(t) \| \right) \frac{1}{4(2n-1)(2n)} \| \varphi_{2n-2} \|_1 \;\; \| Wx \| \\
&\leq \frac{c_1}{2^2(2n-1)(2n)} \frac{1}{2^{n-1}\sqrt{2(n-1)!}} \| Wx \| \quad (by \;\; 5) \;\; (c_1 = \sup_{t\in\mathbb{R}} \| T(t) \|) \\
&\leq \frac{c_1\sqrt{2(n-1)!}}{(2n-1)(2n)} \frac{1}{2^{n+1}(2(n-1))!} \| Wx \| \\
&\leq \frac{c_1\sqrt{2(n-1)!}}{2^{n+1}(2n)!} \| Wx \| \, .
\end{aligned}$$

Theorem 2.1. *Let $(T(t))_{t\in\mathbb{R}}$ be an exponentially bounded $C-$cosine function on a Banach space E ($T(t) \leq Me^{\omega|t|}, M > 0$ and $\omega \geq 0$) with a generator $(W, \mathcal{D}(W))$.*

1. *For any $x \in \mathcal{D}(W)$,*

$$T(t)x = \sum_{n=0}^{+\infty} \frac{1}{2^{2n}(2n)!} W^n T_1(\frac{1}{4})xH_{2n}(t), \;\; for \; all \; t \in \mathbb{R}.$$

2. *In the case where $\sup_{t\in\mathbb{R}} \| T(t) \| < +\infty$, we have for any $x \in \mathcal{D}(W)$:*

- *(i)*

$$T_1(z)x = \sum_{n=0}^{+\infty} \frac{1}{2^{2n} n!} W^n T_1(\frac{1}{4}) x (4z-1)^n, \ for\ all\ z \in \mathbb{C},\ | z - \frac{1}{4} | < \frac{1}{4}.$$

- *(ii) For all $t \in \mathbb{R}$, there is $m_t \in \mathbb{N}$ and $c_t > 0$ such that:*

$$(\forall m \geq m_t) \ \ \| T(t)x - \sum_{n=0}^{m} \frac{1}{2^{2n}(2n)!} W^n T_1(\frac{1}{4}) x H_{2n}(t) \| \leq \frac{C_t}{m^{\frac{1}{12}}} \| Wx \| .$$

Moreover, the convergence is uniform on any compact of $\mathbb{R}$.

Proof

1. Let $x \in \mathcal{D}(W)$, which implies that $Cx \in \mathcal{D}(W)$ because $WCx = CWx$. Let's remember that $T(.)x : \mathbb{R} \to E$ is in $C^2(\mathbb{R}, E)$ and like

$$\begin{aligned}
\int_{-\infty}^{+\infty} e^{-t^2} \| T(t)x \|^2 \, dt &\leq \int_{-\infty}^{+\infty} e^{-t^2} \| T(t) \|^2 \| x \|^2 \, dt \\
&\leq M^2 \int_{-\infty}^{+\infty} e^{-t^2} e^{2\omega|t|} \| x \|^2 \, dt \\
&\leq M^2 \| x \|^2 \int_{-\infty}^{+\infty} e^{-|t|^2 + 2\omega|t|} dt \\
&\leq M^2 \| x \|^2 e^{\omega^2} \int_{-\infty}^{+\infty} e^{-(|t|-\omega)^2} dt \\
&\leq M^2 \| x \|^2 e^{\omega^2} \int_{-\infty}^{+\infty} e^{-u^2} du \\
&< +\infty,
\end{aligned}$$

then, the series $\sum_{n \in \mathbb{N}} c_n(T(.)x) H_n$ with

$$c_n(T(.)x) = \textstyle\int_{-\infty}^{+\infty} \varphi_n(t) T(t) x dt = \frac{1}{2^n n!} W^n T_1(\frac{1}{4}) x$$

converges pointwise to $T(t)x$ for $t \in \mathbb{R}$, it is $T(t)x = \sum_{n=0}^{+\infty} \frac{1}{2^n n!} W^n T_1(\frac{1}{4}) x H_n(t)$.

2. - (i)
   According to 3.(i) of Lemma 2.1, for all $x \in \mathcal{D}(W)$ and $z \in \mathbb{C}$, we have

$$\begin{aligned}
\sum_{n=0}^{+\infty} \frac{\| W^n T_1(\frac{1}{4}) x \|}{2^{2n} n!} | 4z - 1 |^n &\leq \sum_{n=0}^{+\infty} \frac{2^n \sqrt{(2n)!}}{2^{2n} n!} | 4z - 1 |^n \| x \| \left(\sup_{t \in \mathbb{R}} \| T(t) \| \right) \\
&\leq c_1 \sum_{n=0}^{+\infty} \frac{\sqrt{(2n)!}}{2^n n!} | 4z - 1 |^n,
\end{aligned}$$

where $c_1 = \left(\sup_{t\in\mathbb{R}} \| T(t) \|\right) \| x \|$.

Stirling's formula $\frac{\sqrt{(2n)!}}{n!} \sim_{+\infty} 2^n n^{\frac{-1}{4}} (2\pi)^{\frac{-1}{4}} 2^{\frac{1}{4}}$ implies that:

$$c_1 \sum_{n=0}^{+\infty} \frac{\sqrt{(2n)!}}{2^n n!} \mid 4z-1 \mid^n \leq c \sum_{n=0}^{+\infty} \frac{\mid 4z-1 \mid^n}{n^{\frac{1}{4}}} < \infty, \text{ for } \mid z - \tfrac{1}{4} \mid < \tfrac{1}{4}.$$

Finally let a analytic family of operators :

$V : D(\frac{1}{4}, \frac{1}{4}) = \{z \in \mathbb{C} / \mid z - \frac{1}{4} \mid < \frac{1}{4}\} \to B(\mathcal{D}(W)), z \mapsto V(z) :$ $\mathcal{D}(W) \to \mathcal{D}(W),\ x \mapsto V(z)x = \sum_{n=0}^{+\infty} \frac{W^n T_1(\frac{1}{4})x}{2^{2n} n!} (4z-1)^n.$

So for $x \in D(W)$:

$$\begin{aligned} V'(z)x &= \sum_{n=1}^{+\infty} \frac{W^n T_1(\frac{1}{4})x}{2^{2n} n!} 4n(4z-1)^{n-1} \\ &= \sum_{n=0}^{+\infty} \frac{W^n T_1(\frac{1}{4})Wx}{2^{2n} n!} (4z-1)^n \\ &= V(z)Wx \\ &= WV(z)x. \end{aligned}$$

Like W a generator of analytic $C-$semigroups $(T_1(t))_{t\geq 0}$ of angle $\frac{\pi}{2}$ and $V(\frac{1}{4})x = T_1(\frac{1}{4})x \in \mathcal{D}(W)$ then $V(t)x = T_1(t)x$ for $t \in]0, \frac{1}{2}[$ because the first-order abstract Cauchy problem

$$\begin{cases} u'(t) = Wu(t), & t \in]0, \frac{1}{2}[\\ u(\frac{1}{4}) = T_1(\frac{1}{4})x \end{cases}$$

has a unique solution defined in $]0, \frac{1}{2}[$. On the other hand the two functions $z \mapsto T_1(z)x$ and $z \mapsto V(z)x$ are holomorphic on the connected open set $D(\frac{1}{4}, \frac{1}{4})$, likewise $]0, \frac{1}{2}[\subset \{u \in D(\frac{1}{4}, \frac{1}{4}),\ V(u)x = T_1(u)x\}$, by the principle of analytical continuation we get: $V(z)x = T_1(z)x$ for $\mid z - \frac{1}{4} \mid < \frac{1}{4}$.

- (ii) Let $t \in \mathbb{R}$ be fixed. We are looking to increase the quantity

$$\| \sum_{n=0}^{+\infty} \frac{1}{2^{2n}(2n)!} W^n T_1(\frac{1}{4}) x H_{2n}(t) - \sum_{n=0}^{m} \frac{1}{2^{2n}(2n)!} W^n T_1(\frac{1}{4}) x H_{2n}(t) \| . \tag{9}$$

There exists $m_t \in \mathbb{N}^*$ such that $0 \leq t^2 \leq 2(2m_t + 1)$, according to inequality (1)

$(\forall n \geq m_t) \ \mid H_{2n}(t) \mid \leq c \left(\frac{e^{\frac{t^2}{2}} \sqrt{2^{2n}(2n)!}}{(2n)^{\frac{1}{12}}} \right)$ (the c is independent of t and it is the inequality constant constant (1)),

let $x \in \mathcal{D}(W)$, according to the inequality $3 - (ii)$ of lemma 2.1, there is a constant $c_1 > 0$ such as for all $m \geq m_t$ from where:

$$
\begin{aligned}
(9) \quad & \leq \sum_{n=m+1}^{+\infty} \| \frac{1}{2^{2n}(2n)!} W^n T_1(\frac{1}{4}) x H_{2n}(t) \| \\
& \leq \sum_{n=m+1}^{+\infty} \| \int_{-\infty}^{+\infty} \varphi_{2n}(s) T(s) x ds \| \; | H_{2n}(t) | \\
& \leq \sum_{n=m+1}^{+\infty} \frac{c_1 \sqrt{(2n-2)!}}{2^{n+1}(2n)!} \| Wx \| c \frac{e^{\frac{t^2}{2}} \sqrt{2^{2n}(2n)!}}{(2n)^{\frac{1}{12}}} \quad (3.ii \text{ of lemma } 2.1) \\
& \leq \sum_{n=m+1}^{+\infty} \frac{c c_1 e^{2m_t+1} \sqrt{(2n-2)!(2n)!2^n}}{2^{n+1}(2n)! 2^{\frac{1}{12}} n^{\frac{1}{12}}} \| Wx \| \\
& \leq \sum_{n=m+1}^{+\infty} c_t' \sqrt{\frac{(2n-2)!}{(2n)!}} \frac{1}{n^{\frac{1}{12}}} \| Wx \| \quad (c_t' = c c_1 e^{2m_t+1}) \\
& \leq \sum_{n=m+1}^{+\infty} \frac{c_t'}{\sqrt{2n(2n-1)} n^{\frac{1}{12}}} \| Wx \| \\
& \leq \sum_{n=m+1}^{+\infty} \frac{c_t'}{n^{\frac{13}{12}}} \| Wx \| \quad (\text{because } \frac{1}{\sqrt{2n(2n-1)}} \leq \frac{1}{n}).
\end{aligned}
$$

The Riemann serie $\sum_{n \geq 1} \frac{1}{n^{\frac{13}{12}}}$ is convergent, so

$$
\sum_{n=m+1}^{+\infty} \frac{1}{n^{\frac{13}{12}}} \sim_{m \to +\infty} \frac{1}{\frac{13}{12} - 1} \frac{1}{m^{\frac{13}{12}-1}} = \frac{1}{\frac{1}{12}} \frac{1}{m^{\frac{1}{12}}} \text{ from where}
$$

$$
\begin{aligned}
\| T(t)x - \sum_{n=0}^{m} \frac{1}{2^{2n}(2n)!} W^n T_1(\frac{1}{4}) x H_{2n}(t) \| & \leq c_t' c_2 \frac{12}{m^{\frac{1}{12}}} \| Wx \| \\
& \leq \frac{c_t}{m^{\frac{1}{12}}} \| Wx \| .
\end{aligned}
$$

- If $K \subset \mathbb{R}$ is a compact set then there is $m_0 \in \mathbb{N}$ for all $t \in K$, $0 \leq t^2 \leq 2(2m_0 + 1)$, so the constant c_t' (from where c_t) is independent of t therefore, the reminder tends uniformly on K towards 0.

Example 2.2. *Let $m : \mathbb{R} \to \mathbb{R}^-$ be an even measurable function. In the Banach space $L^1(\mathbb{R})$, we consider the family $T := (T(t))_{t \in \mathbb{R}} \subset \mathfrak{F}(L^1(\mathbb{R}))$ defined by $T : \mathbb{R} \to B\left(L^1(\mathbb{R})\right), t \mapsto T(t) : L^1(\mathbb{R}) \to L^1(\mathbb{R}), f \mapsto T(t)(f) : \mathbb{R} \to \mathbb{R}, s \mapsto T(t)(f)(s) = \cos\left(t\sqrt{-m(s)}\right) f(-s)$.*
Clearly, $(T(t))_{t \in \mathbb{R}} \subset B(L^1(\mathbb{R}))$. If we put $T(0) = C$, then for all $s, t \in \mathbb{R}$ $T(t)(f)(s) = \cos\left(t\sqrt{-m(s)}\right) C(f)(s)$. We also have:

- *(i) C is an injective element of $B\left(L^1(\mathbb{R})\right)$.*

- *(ii) For all $t, s, z \in \mathbb{R}$:*

$$\begin{aligned}
D(z) &= \left(T(t+s)+T(t-s)\right)\left(C(f)\right)(z) \\
&= T(t+s)\left(C(f)\right)(z)+T(t-s)\left(C(f)\right)(z) \\
&= \cos\left((t+s)\sqrt{-m(z)}\right)C(f)(-z)+\cos\left((t-s)\sqrt{-m(z)}\right)C(f)(-z) \\
&= \cos\left((t+s)\sqrt{-m(z)}\right)f(z)+\cos\left((t-s)\sqrt{-m(z)}\right)f(z) \\
&= \left(\cos\left(t\sqrt{-m(z)}+s\sqrt{-m(z)}\right)+\cos\left(t\sqrt{-m(z)}-s\sqrt{-m(z)}\right)\right)f(z) \\
&= 2\cos\left(t\sqrt{-m(z)}\right)\cos\left(s\sqrt{-m(z)}\right)f(z) \\
&= 2\cos\left(t\sqrt{-m(z)}\right)\cos\left(s\sqrt{-m(z)}\right)\left(C(f)\right)(-z) \\
&= 2\cos\left(t\sqrt{-m(z)}\right)T(s)\left(C(f)\right)(z) \\
&= 2\cos\left(t\sqrt{-m(z)}\right)C\left(T(s)(f)\right)(z) \quad (m\ is\ an\ even\ function) \\
&= 2T(t)\left(T(s)(f)\right)(z) \\
&= 2T(t)T(s)(f)(z)
\end{aligned}$$

- *(iii) For each f fixed in $L^1(\mathbb{R})$. The function $\mathbb{R}\to L^1(\mathbb{R}), t\mapsto T(t)(f)$ is continuous on $\mathbb{R}$.*

- *(iv) For all $f, g \in L^1(\mathbb{R})$ such that $C^{-1}\lim_{t\to 0}\dfrac{2(T(t)f-C(f))}{t^2}=g$, we have :*

$$\begin{aligned}
0 &= \lim_{t\to 0}\parallel \frac{2\left(T(t)(f)-C(f)\right)}{t^2}-C(g)\parallel_{L^1(\mathbb{R})} \\
&= \lim_{t\to 0}\int_{\mathbb{R}}\mid \frac{2\left(\cos\left(t\sqrt{-m(s)}\right)C(f)(s)-C(f)(s)\right)}{t^2}-C(g)(s)\mid ds \\
&= \lim_{t\to 0}\int_{\mathbb{R}}\mid \frac{2\left(\cos\left(t\sqrt{-m(s)}\right)-1\right)}{t^2}C(f)(s)-C(g)(s)\mid ds \\
&= \int_{\mathbb{R}}\lim_{t\to 0}\mid \frac{2\left(\cos\left(t\sqrt{-m(s)}\right)-1\right)}{t^2}C(f)(s)-C(g)(s)\mid ds\ (Fatou's\ lemma) \\
&= \int_{\mathbb{R}}\mid m(s).C(f)(s)-C(g)(s)\mid ds \\
&= \int_{\mathbb{R}}\mid C\left(C(m).f-g\right)(s)\mid ds\ (\ because\ \forall\ h\in\ \mathfrak{F}(\mathbb{R},\mathbb{R})\quad C\circ C(h)=h\ ) \\
&= \parallel C\left(m.f-g\right)\parallel_{L^1(\mathbb{R})},\quad (C(m)=m\quad because\quad m\quad is\quad an\quad even\quad function)
\end{aligned}$$

- *(v) Let $t\in\mathbb{R}$ fixed, we have for all $f\in L^1(\mathbb{R})$:*
  $\parallel T(t)(f)\parallel_{L^1(\mathbb{R})}=\int_{\mathbb{R}}\mid\cos\left(t\sqrt{-m(s)}\right)f(-s)\mid ds\leq\parallel f\parallel_{L^1(\mathbb{R})}$.

Ultimately, $(T(t))_{t\in\mathbb{R}}$ is uniformly bounded $C-$regularized cosine function with generator $(W,\mathcal{D}(W))$ defined by W:

$\mathcal{D}(W) = \{f \in L^1(\mathbb{R})/\ m.f \in L^1(\mathbb{R})\} \to L^1(\mathbb{R}),\ f \mapsto W(f) = m.f$. *On the other hand, for all* $s \in \mathbb{R}$ *and* $f \in L^1(\mathbb{R})$,

$$\begin{aligned}
\frac{2}{\sqrt{\pi}}\int_0^{+\infty} e^{-t^2} T(t)(f)(s)dt &= \frac{2}{\sqrt{\pi}}\int_0^{+\infty} e^{-t^2}\cos\left(t\sqrt{-m(s)}\right) f(-s)dt \\
&= \frac{2}{\sqrt{\pi}}\int_0^{+\infty} e^{-t^2} Re\left(e^{i\left(t\sqrt{-m(s)}\right)}\right) f(-s)dt \\
&= Re\left(\tfrac{2}{\sqrt{\pi}}\int_0^{+\infty} e^{-t^2+it\sqrt{-m(s)}} f(-s)dt\right) \\
&= Re\left(\tfrac{2}{\sqrt{\pi}}\int_0^{+\infty} e^{\frac{m(s)}{4}} e^{-\left(t-i\frac{\sqrt{-m(s)}}{2}\right)^2} f(-s)dt\right) \\
&= e^{\frac{m(s)}{4}}\left(\tfrac{2}{\sqrt{\pi}}\int_0^{+\infty} e^{-u^2}du\right).f(-s) \\
&= e^{\frac{m(s)}{4}} f(-s),
\end{aligned}$$

then for all $s \in \mathbb{R}$ *and* $f \in L^1(\mathbb{R})$

$$T_1(\frac{1}{4})(f)(s) = \left(\tfrac{2}{\sqrt{\pi}}\int_0^{+\infty} e^{-t^2}\cos(t\sqrt{-m(s)})dt\right).f(-s) = e^{\frac{m(s)}{4}} f(-s),$$

Theorem 2.1 gives for $f \in \mathcal{D}(W)$ *and* $s, t \in \mathbb{R}$:

$$\begin{aligned}
T(t)(f)(s) &= \sum_{n=0}^{+\infty} \frac{1}{2^{2n}(2n)!} W^n T_1(\frac{1}{4})(f)(s) H_{2n}(t) \\
&= \sum_{n=0}^{+\infty} \frac{(m(s))^n}{2^{2n}(2n)!} e^{\frac{m(s)}{4}} f(-s) H_{2n}(t).
\end{aligned}$$

So $T(t)(f) = \displaystyle\sum_{n=0}^{+\infty} \frac{(m(.))^n}{2^{2n}(2n)!} e^{\frac{m(.)}{4}} H_{2n}(t) C(f)$.

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