To
The Editor-in-Chief,
The Siberian Electronic Mathematical Reports Date: March 29, 2024

Subject: Submission of manuscript

Dear Professor,

I hereby submit a manuscript of my article with the following details.

Title: *A note on constructing triangle- free graphs with arbitrarily large graceful* $\lambda_{3,2,1}-$*chromatic number.*
Author: Ushnish Sarkar.

ORCID and afiliation details of the author:-

Ushnish Sarkar (ORCID: 0000-0002-3884-2911)
Department of Mathematics, School of Sciences
Netaji Subhas Open University,
DD 26, Sector I, Salt Lake City
Kolkata- 700064, West Bengal, India.
e-Mails: usarkar.sosci@wbnsou.ac.in, usn.prl@gmail.com

Mathematics Subject Classification: 05C12, 05C15, 05C78.

I appreciate your time and look forward to your response. Please intimate me of your decision at your earliest convenience.

Yours sincerely,
Ushnish Sarkar

A NOTE ON CONSTRUCTING TRIANGLE- FREE GRAPHS WITH ARBITRARILY LARGE GRACEFUL $\lambda_{3,2,1}-$CHROMATIC NUMBER

USHNISH SARKAR

ABSTRACT. Let G be a finite simple connected graph. Motivated from frequency assignment problem (FAP), an $L(3,2,1)-$coloring of G is a distance constrained vertex coloring using non-negative integers as colors. Here the adjacent vertices must receive colors of difference three; vertices at distance two and three are assigned colors of difference two and one respectively. The minimum span of an $L(3,2,1)-$coloring on G is referred as $\lambda_{3,2,1}-$chromatic number of G and such a coloring of minimum span is referred as $\lambda_{3,2,1}-$coloring of G. If a $\lambda_{3,2,1}-$coloring of $G = (V, E)$ gives a bijection from V to $\{0,\ 1,\ \ldots,\ |V| - 1\}$, then the $\lambda_{3,2,1}-$coloring is referred as graceful and its span is referred as graceful $\lambda_{3,2,1}-$chromatic number of G. This article provides a recursive construction of an infinite family of triangle- free graphs having arbitrarily large graceful $\lambda_{3,2,1}-$chromatic number. This work has an analogy to the effect of Mycielski construction.

1. INTRODUCTION

Frequency assignment problem (FAP) is the task of assigning frequencies to radio and TV transmitters in a communication network in a cost effective manner. The increasing proximity of transmitters demands greater mutual differences among the frequencies assigned otherwise the transmission will be adversely affected due to interference. Therefore efficiency in minimizing the span of frequencies used in the assignment, while satisfying the interfering constraints simultaneously, is extremely important. Hale [7] gave this problem a graph theoretic interpretation. In the graph theoretic representation of the problem, the transmitters are viewed as vertices and nearby stations are connected by an edge.

Interestingly, the radio signals attenuate over distance. This allows assignment of same frequency to different transmitters without inviting the risk of interference if they are at a fairly large distance depending on the interference constraints. Such critical distance is called reuse distance [3] of the assignment.

Let $G = (V, E)$ be a simple connected graph. An $L(p_1, p_2, \ldots, p_m)-$coloring of a graph is a function $f : V \longrightarrow \mathbb{N} \bigcup \{0\}$ such that $|f(u) - f(v)| \geq p_i$ when the distance of the vertices $u,\ v$ is $d(u, v) = i$, for $i = 1,\ 2,\ \ldots,\ m$. Given $p_1 \geq p_2 \geq \ldots \geq p_m$, this distance constrained coloring problem is the graph theoretic interpretation of the FAP we have discussed. Note that here $m + 1$ is the reuse distance. For an $L(p_1, p_2, \ldots, p_m)-$coloring f of G, the *span* of f is defined by $(\max_{v \in V} f(v) - \min_{v \in V} f(v))$ and the $\lambda_{p_1,p_2,\ldots,p_m}-$chromatic number of the graph G, denoted by $\lambda_{p_1,p_2,\ldots,p_m}(G)$, is defined as $\min_f\{span(f) :\ f$ is an $L(p_1, p_2, \ldots, p_m)-$coloring of $G\}$. An optimal $L(p_1, p_2, \ldots, p_m)-$ coloring or a $\lambda_{p_1,p_2,\ldots,p_m}-$coloring of G is an $L(p_1, p_2, \ldots, p_m)-$coloring of G with span $\lambda_{p_1,p_2,\ldots,p_m}(G)$. For $2 \leq k \leq$ diameter of G, the $L(k, k-1, k-2, \ldots, 2, 1)-$coloring problem is also referred as radio $k-$coloring problem [2, 6, 9, 14, 15, 22]. For $k =$ diameter of G and $k =$ diameter of $G - 1$, this is known as radio coloring

2020 *Mathematics Subject Classification.* Primary: 05C12, 05C15, 05C78.

Key words and phrases. $L(3,2,1)-$coloring , Graceful $\lambda_{3,2,1}-$ chromatic number, Radio graceful graph, triangle-free graph.

problem [1, 4, 5, 21] and antipodal coloring problem [10, 11, 18] respectively. Interestingly no two vertices can have same color in a radio coloring of a graph. Thus there is no scope for reusing the same frequency in radio coloring of a graph.

From the perspective of the reuse distance, the $L(3,2,1)-$coloring of graphs [3, 13, 23] is a relevant area of interest where the reuse distance is four.

This may be noted that understanding of the radio $k-$coloring problem, especially the $L(3,2,1)-$coloring problem for even some basic families of graphs remain elusive and have been observed to be quite challenging.

An interesting question of an optimal $L(p_1, p_2, \ldots, p_m)-$coloring of a graph is its gracefulness. An optimal $L(p_1, p_2, \ldots, p_m)-$coloring f of a graph $G = (V,\ E)$ is *graceful* if f is a bijection between V and the set $\{0,\ 1,\ \ldots,\ |V|-1\}$. The span of f is then called graceful $\lambda_{p_1,p_2,\ldots,p_m}-$chromatic number of G.

A graph G is said to be radio graceful if it admits a radio graceful coloring [17]. Few progress have been made in determining if some graph is radio graceful. In [20], Saha and Basunia have given a necessary and sufficient for radio graceful graphs. Niedzialomski extensively used Cartesian product to investigate the problem. In [17], Niedzialomski has shown that Hamming graph $H = K_{n_1}\square K_{n_2}\square\ldots\square K_{n_s}$ is radio graceful where $n_1,\ n_2,\ \ldots,\ n_s$ relatively prime and K_n is the complete graph of order n. The author has also shown that K_n^t is radio graceful for $1 \le t \le n$, where K_n^t is the $t-$th Cartesian power of K_n. This provides example of radio graceful graphs of arbitrary diameter. Advancing the study further, in [16] Locke and Niedzialomski has shown Cartesian product of K_n and Peterson graph is radio graceful. In [25], Wyels and Tomova provided upper bound of graceful radio number of $G\square G$ if the diameter of the graph G is 2. They have also proved that Cartesian product of Peterson garph with itself is radio graceful.

2. Our contribution

In this note, we attack this problem from a different angle. So far in the literature, all the radio graceful graphs with arbitrarily large radio numbers have been observed to have large clique size, up to the best of our knowledge. Also, the graceful $\lambda_{3,2,1}-$coloring is one the least explored area. In this article, we prove existence of triangle- free graphs admitting graceful $\lambda_{3,2,1}-$coloring. In fact, we have given an infinite family of three- diameter graphs having arbitrarily large graceful $\lambda_{3,2,1}-$chromatic number. In other words, this gives an infinite family of triangle- free radio graceful graphs of diameter three. This resembles the *effect* of Mycielski's construction [24] ensuring existence of an infinite family of triangle- free graphs with arbitrarily large chromatic number.

3. The construction and its properties

Here we recursively construct a family of simple graphs $\mathbb{G}_n$, for $n \ge 6$, with $n+1$ number of vertices.

For $n = 6$, we construct $\mathbb{G}_6$ with the vertex set $V(\mathbb{G}_6) = \{v_0,\ v_1,\ v_2,\ v_3,\ v_4,\ v_5,\ v_6\}$ and the edge set $E(\mathbb{G}_6) = \{(v_0, v_3),\ (v_0, v_5),\ (v_1, v_4),\ (v_1, v_6),\ (v_2, v_5),\ (v_3, v_6)\}$.

Moreover, for $n \ge 6$, we recursively construct $\mathbb{G}_{n+1}$ with the vertex set $V(\mathbb{G}_{n+1}) = \{v_0,\ v_1,\ \ldots,\ v_{n+1}\}$, i.e., $V(\mathbb{G}_{n+1}) = V(\mathbb{G}_n)\bigcup\{v_{n+1}\}$ and the edge set $E(\mathbb{G}_{n+1}) =\ E(\mathbb{G}_n)\bigcup$ $\{(v_{n-2j-2}, v_{n+1}):\ j \in \mathbb{N}\cup\{0\}$ and $n \ge 2j+2\}$.

It is obvious that $\mathbb{G}_n$, for $n \ge 6$, is a family of connected simple graphs.

Theorem 3.1. *For $n \ge 6$, $\mathbb{G}_n$ is bipartite and hence triangle- free graph.*

Proof : It can be seen from the construction of $\mathbb{G}_n$, for $n \geq 6$, that two adjacent vertices must be of opposite parity. Therefore $\mathbb{G}_n$, for $n \geq 6$, is bipartite and hence triangle- free. □

The following results will be helpful in proving that the diameter of $\mathbb{G}_n$ is three, for $n \geq 9$. These results will play important role in finding our desired graceful $\lambda_{3,2,1}$−coloring.

Lemma 3.2. *For $n \geq 7$, $d(v_j,\ v_{j+1}) = 3$, for j with $0 \leq j \leq n-1$, in $\mathbb{G}_n$.*

Proof : The vertices $v_j,\ v_{j+3},\ v_{j+6},\ v_{j+1}$, for $0 \leq j \leq n-6$; the vertices $v_j,\ v_{j+3},\ v_{j-2},\ v_{j+1}$, for $2 \leq j \leq n-3$ and the vertices $v_j,\ v_{j-5},\ v_{j-2},\ v_{j+1}$, for $5 \leq j \leq n-1$ induce paths of length three in $\mathbb{G}_n$.

Suppose $d(v_j,\ v_{j+1}) = 2$, for j with $0 \leq j \leq n-1$. Then there exists some suitable p, $0 \leq p \leq n$ and $j \neq p \neq j+1$, such that v_p is adjacent to both v_j and v_{j+1}.

Case I: Let $p > j+1$. Now $p = j+3$ or $j+3+2t$, $t \geq 1$, since v_p is adjacent to v_j. Also, $p = j+4$ or $j+4+2s$, $s \geq 1$, since v_p is adjacent to v_{j+1}.

Since v_p is adjacent to both v_j and v_{j+1}, therefore $j+3 \neq p \neq j+4$. Hence $j+3+2t = j+4+2s$, for some natural numbers $t,\ s \geq 1$, implying $2(t-s) = 1$, a contradiction.

Case II: Let $p < j$. Then $j = p+3$ or $p+3+2t$, $t \geq 1$, since v_p is adjacent to v_j. Again, $j = p+2$ or $p+2+2s$, $s \geq 1$ since v_p is adjacent to v_{j+1}. Clearly $p+3 \neq j \neq p+2$, arguing as in Case I. Therefore we have $2(s-t) = 1$, a contradiction.

Hence $d(v_j,\ v_{j+1}) \neq 2$ and so $d(v_j,\ v_{j+1}) = 3$, in $\mathbb{G}_n$ for $n \geq 7$. □

Lemma 3.3. *For $n \geq 7$, $d(v_j,\ v_{j+2}) = 2$, for j with $0 \leq j \leq n-2$, in $\mathbb{G}_n$.*

Proof : Clearly, v_j and v_{j+2} are not adjacent in $\mathbb{G}_n$, by construction. Now the vertices $v_{j+2},\ v_{j+5},\ v_j$, for $0 \leq j \leq n-5$, and the vertices $v_j,\ v_{j-3},\ v_{j+2}$, for $3 \leq j \leq n-2$, induce paths of length two. This completes the proof. □

Considering the paths of length two induced by the vertices $v_{j+4},\ v_{j+7},\ v_j$, for $0 \leq j \leq n-7$, and the vertices $v_j,\ v_{j-3},\ v_{j+4}$, for $3 \leq j \leq n-4$, we have the following lemma.

Lemma 3.4. *For $n \geq 9$, $d(v_j,\ v_{j+4}) = 2$, for j with $0 \leq j \leq n-4$, in $\mathbb{G}_n$.*

Remark. In $\mathbb{G}_6$, $d(v_0,\ v_4) = d(v_1,\ v_5) = d(v_2,\ v_6) = 4$. In $\mathbb{G}_7$, $d(v_1,\ v_5) = 4 = d(v_2,\ v_6)$ and in $\mathbb{G}_8$, $d(v_2,\ v_6) = 4$.

Moreover, considering the path of length two induced by the vertices $v_j,\ v_{j+3+2(t-1)},\ v_{j+4+2t}$, for $t \geq 1$, we have the following result.

Lemma 3.5. *For $n \geq 6$, $d(v_j,\ v_{j+4+2t}) = 2$, for $t \geq 1$ and j with $0 \leq j \leq n-4-2t$, in $\mathbb{G}_n$.*

Now we are in a position to determine the diameter of $\mathbb{G}_n$.

Theorem 3.6. *For $n \geq 9$, the diameter of $\mathbb{G}_n$ is three.*

Proof : Together with the lemmas 3.2, 3.3, 3.4, 3.5 and the adjacency of the vertices v_j, v_{j+5+2t} for $t \geq 0$, the proof is done. □

Remark. For $n = 6,\ 7,\ 8$, the diameters of $\mathbb{G}_n$ are six, four and four respectively.

4. The graceful $\lambda_{3,2,1}-$coloring

Now we are in a position to define our desired graceful $\lambda_{3,2,1}-$coloring on $\mathbb{G}_n$.

Theorem 4.1. *For $n \geq 9$, the vertex coloring f on $\mathbb{G}_n$ defined by $v_i \mapsto i,\ i = 0,\ 1,\ 2, \ldots,\ n$, gives a graceful $\lambda_{3,2,1}-$coloring of $\mathbb{G}_n$.*

Proof : Let $j \in \{0,\ 1,\ \ldots,\ n\}$.

Using Lemma 3.2, we have

$$|f(v_j) - f(v_{j+1})| + d(v_j, v_{j+1}) > 3,\ 0 \leq j \leq n-1.$$

Using Lemma 3.3, we have

$$|f(v_j) - f(v_{j+2})| + d(v_j, v_{j+2}) > 3,\ 0 \leq j \leq n-2.$$

Also,

$$|f(v_j) - f(v_{j+3})| + d(v_j, v_{j+3}) > 3,\ 0 \leq j \leq n-3.$$

and

$$|f(v_j) - f(v_{j+t})| + d(v_j, v_{j+t}) > 4,\ 0 \leq j \leq n-t,\ \forall t \geq 4.$$

Therefore f is an $L(3, 2, 1)-$coloring of span n.

Since diameter of $\mathbb{G}_n$ is three by Theorem 3.6, therefore all the vertices of $\mathbb{G}_n$ must receive distinct colors under any $L(3, 2, 1)-$coloring. Hence, f is a graceful $\lambda_{3,2,1}-$coloring of $\mathbb{G}_n$. Thus the graceful $\lambda_{3,2,1}-$chromatic number of $\mathbb{G}_n$ is n, for any $n \geq 9$. □

Remark. Note that $\lambda_{3,2,1}-$chromatic numbers of $\mathbb{G}_6$, $\mathbb{G}_7$ and $\mathbb{G}_8$ are $6,\ 7,\ 8$ respectively. In fact, the vertex coloring on $\mathbb{G}_n$ given by $v_i \mapsto i,\ i = 0,\ 1,\ 2, \ldots,\ n$, gives a graceful $\lambda_{3,2,1}-$coloring, for $n = 6,\ 7,\ 8$.

Hence we have the final theorem.

Theorem 4.2. *For $n \geq 6$, there exists an $n-$vertex triangle- free graph with a graceful $\lambda_{3,2,1}-$coloring.*

5. Conclusion and open directions

In this article, we have given a recursive construction of an infinite family of triangle- free graphs having arbitrarily large graceful $\lambda_{3,2,1}-$chromatic number. From a bigger perspective, obtaining families of graphs with arbitrarily large graceful $\lambda_{p_1,p_2,\ldots,p_m}-$chromatic number, while keeping the clique size in check, may be an interesting as well as important question of future study.

Acknowledgement. The research is supported by Netaji Subhas Open University, West Bengal, India (Reference No. Reg/ 1285 dated 22/11/2021, Office Order No. AC/140/2021-22 dated 01/11/2021)

References

[1] D. Bantva, D.D.-F. Liu, Optimal radio labellings of block graphs and line graphs of trees, *Theoretical Computer Science*, 891 (2021) 90-104.
[2] C. Bloomfield, D.D.-F. Liu, J. Ramirez, Radio$-k-$labeling of cycles for large k, *Discrete Applied Mathematics*, 316 (2022) 60-70.
[3] T. Calamoneri, Optimal $L(\delta_1, \delta_2, 1)-$ labeling of eight-regular grids, *Inform. Process. Lett.*, 113 (2013) 361-364.

[4] G. Chartrand, D. Erwin, F. Harrary, P. Zhang, Radio labeling of graphs. *Bull. Inst. Combin. Appl.*, 33 (2001) 77-85.
[5] G. Chartrand, D. Erwin, P. Zhang, A graph labeling problem suggested by FM channel restrictions, *Bull. Inst. Combin. Appl.*, 43 (2005) 43-57.
[6] S. Das, S. C. Ghosh, S. Nandi, S. Sen, A lower bound technique for radio k-coloring, *Discrete Math.*, 340 (2017) 855-861.
[7] W. K. Hale, Frequency assignment: theory and applications. *Proc. IEEE.*, 68 (1980) 1497-1514.
[8] J. S. Juan, D. D-F. Liu, Antipodal labelings of cycles, preprint (2006).
[9] M. Kchikech, R. Khennoufa, O. Togni, Radio k-labelings for cartesian products of graphs. *Discuss. Math. Graph Theory.*, 28/1 (2008), 165-178.
[10] R. Khennoufa, O. Togni, A note on radio antipodal colorings of paths, *Math. Bohem.*, 130(1) (2005) 277-282.
[11] R. Khennoufa, O. Togni, The radio antipodal and radio numbers of the hypercube. *Ars Combin.*, 102 (2011) 447-461.
[12] S.R. Kola, P. Panigrahi, On radio $(n-4)$-chromatic number the path P_n,. *AKCE Int. J. Graphs Combin.*, 6(1) (2009) 209-217.
[13] D. King, C.J. Ras, S. Zhou, The $L(h,1,1)-$labelling problem for trees, *European J. Combin.*, 31 (2010) 1295–1306.
[14] D. Korže, Z. Shao, A. Vesel, New results on radio $k-$labelings of distance graphs, *Discrete Appl. Math.*, 319 (2022) 472-479.
[15] A. Chavez, D.D.-F. Liu, M. Shurman, Optimal radio-k-labelings of trees, *European Journal of Combinatorics*, 91 (2021) 103203.
[16] Sarah Locke, Amanda Niedzialomski, $K_n \square P$ is radio graceful, *Le Matematiche* LXXIII (2018) - Fasc. I, 127–137
[17] Amanda Niedzialomski, Radio Graceful Hamming Graphs, *Discuss. Math. Graph Theory* 36 (2016), 1007-1020
[18] L. Saha, P. Panigrahi, Antipodal number of some powers of cycles, *Discrete Math.*, 312 (2012) 1550-1557.
[19] D. B. West, Introduction to Graph Theory, *Prentice Hall*, 2001.
[20] L. Saha, A. R. Basunia, Radio Graceful Labelling of Graphs, *Theory and Applications of Graphs*, 7(1) (2012) p. 7.
[21] D. D-F. Liu, L. Saha, S. Das, Improved lower bounds for the radio number of trees, *Theoretical Computer Science*, 851 (2021) 1-13.
[22] U. Sarkar, A. Adhikari, On characterizing radio k-coloring problem by path covering problem, *Discrete Math.*, 338 (2015) 615-620.
[23] U. Sarkar, A. Adhikari, On relationship between Hamiltonian path and holes in $L(3,2,1)-$coloring of minimum span, *Discrete Applied Math.*, 222 (2017) 227-234.
[24] D. B. West, Introduction to Graph Theory, *Prentice Hall*, 2001.
[25] Cindy Wyels, Maggy Tomova, Radio Labeling Cartesian Graph Products *9thCologne-Twente Workshop on Graphs and Combinatorial Optimization, Cologne,Germany, May 25-27, 2010. Extended Abstracts*, (2010) 163-167

Department of Mathematics, School of Sciences,
Netaji Subhas Open University,
DD 26, Sector I, Salt Lake City,
Kolkata- 700064, West Bengal, India.
e-Mails (Ushnish Sarkar): usarkar.sosci@wbnsou.ac.in, usn.prl@gmail.com