On Numerical Radius Inequalities for Hilbert Space Operators

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Abstract

In this article , the aim of this study is to find special cases of some inequalities for numerical radii and spectral radii of a bounded linear operator on a Hilbert space, at the end of this paper we find several inequalities for numerical radii by using the spectral norm.

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1. Introduction

The concepts of numerical range and numerical radii play an important role in various fields of con-temporary mathematics, including operator theory, operator trigonometry, numerical analysis and others. Since 1997 the research on these objects has grown greatly. However, the size of the areas of applications for numerical ranges and radius is very difficult to estimate. The numerical range of an operator C on a complex Hilbert space H. For the subset of the complex plane $\mathbb{C}$ *given by*

$$W(\mathrm{C}) = \left\{ \langle Cx, x \rangle \ : x \in H, \|x\| = 1 \right\}.$$

The numerical radii ω(C) of an operator C on *H is given by :*

$$\omega(C) = sup\left\{ |\lambda|, \ \lambda \in W(C) \right\}$$

$$= sup\left\{\left|\left\langle Cx, x\right\rangle\right|, \|x\| = 1\right\}. \qquad (1.1.1)$$

Obviously, by (1.1.1), for any x $\in H$ *one has*

$$\left|\left\langle Cx, x\right\rangle\right| \le \omega(\mathrm{C})\|x\|^2.$$

This monograph focuses on numerical radius inequalities for restricted linear operators on complex Hilbert spaces for the case of one and two operators. Let H be a complex Hilbert space with an inner product $\langle .,. \rangle$ *and let B(H) denote the algebra of all bounded linear operators on H. The algebra of all complex matrices, will be denoted by* $M_n(\mathbb{C})$ *To achieve our goal; we need the following Theorems.*

Theorem 1.1[7]

(1) *If CD = DC, then*

$$\omega\left(CD\right) \le 2\omega(\mathrm{C})\omega(\mathrm{D}).$$

(2) *If C or D is normal such that CD = DC, then*

$$\omega\left(CD\right) \le \omega(\mathrm{C})\omega(\mathrm{D}).$$

(3) *If* $C^2 = I$ *and* $CD = DC$, *then*

Lemma 1.1 [1]

If E, C and D are self-adjoint operators in B(H),then

$$\omega^r(\mathrm{E}) \le \frac{1}{2}\left\|\left|CC^*\right|^{\alpha r} + \left|DD^*\right|^{(1-\alpha)r}\right\|.$$

Theorem 1.2 [1]

Let C and D be self-adjoint operators in B(H), and r ≥ 1. Then

$$\omega^r\left(C + D\right) \le 2^{r-1}\left\|\left|C\right|^r + \left|D\right|^r\right\|.$$

Theorem 1.3[6]

If $\mathrm{C}, \mathrm{D} \in B(H),$ *then*

$$\omega^r(\mathrm{C}) \leq \frac{1}{2}\left\| |C|^{2r\alpha} + |C^*|^{2r(1-\alpha)} \right\|,$$

$$\omega^r(\mathrm{C}+D) \leq 2^{r-2}\left\| |C|^{2r\alpha} + |C^*|^{2r(1-\alpha)} + |D|^{2r\alpha} + |D^*|^{2r(1-\alpha)} \right\|$$

for 0 < α < 1 and r ≥ 1.

٢. New Results for Numerical Radius and Spectral Radius Inequalities

we present some new inequalities for numerical radius and spectral radius.

Theorem 2.1

Let E, C and D operators in B(H) such that A and B are positive and

$\|Ex\| \leq \|Cx\|, \|E^*x\| \leq \|Dx\|$. *Then*

$$r(E) \leq \frac{1}{\sqrt[3]{2}} \left\| |CC^*| + |DD^*|^2 \right\|^{1/3}.$$

Proof:

We know that $r(E) \leq \omega(E)$.

Also by **Lemma 1.1**, *we have*

$$\omega^r(E) \leq \frac{1}{2}\left\| |CC^*|^{\alpha r} |DD^*|^{(1-\alpha)\mathrm{r}} \right\| \qquad (2.1)$$

Put r = 3 and $\alpha = \frac{1}{3}$. *Then*

$$r^3(E) \le \omega^3(E) \le \frac{1}{2}\left\| \left|CC^*\right| + \left|DD^*\right|^2 \right\|. \tag{2.2}$$

$$r^3(E) \le \frac{1}{2}\left\| \left|CC^*\right| + \left|DD^*\right|^2 \right\|. \tag{2.3}$$

By taking the cube root of both sides of (2.3) we obtain the result.

Theorem 2.2

Let C and D be self-adjoint operators in B(H). Then

$$r(C+D) \le \left\| |C| + |D| \right\| \tag{2.4}$$

Also, $$\omega(C+D) \le \sqrt[3]{2}\left\| |C|^{\frac{3}{2}} + |D|^{\frac{3}{2}} \right\|^{2/3}$$

Proof:

By Theorem 1.1, we have

$$\omega^r(C+D) \le 2^{r-1}\left\| |C|^r + |D|^r \right\| \quad \textit{for } r \ge 1$$

If r =1.5, we have

$$\omega^{\frac{3}{2}}(C+D) \le \sqrt{2}\left\| |C|^{\frac{3}{2}} + |D|^{\frac{3}{2}} \right\|$$

So, we get the result.

and by letting r = 1, we have $\omega(A+B)\le \||A|+|B|\|$*, and since .Hence, we get the result.*

Now we prove the famous inequality $\omega(C)\le\|C\|$

as following

Theorem 2.3

Let C be self-adjoint operator in B(H). Then

$$\omega(C)\le\|C\|$$

Proof:

Taking D = C and r = 1.5 in Theorem 1.2, we have

$$\omega^{\frac{3}{2}}(2C)\le\sqrt{2}\left\|2|C|^{\frac{3}{2}}\right\|$$

$$\omega(2C)\le\sqrt[3]{2}\sqrt[3]{4}\left\||C|\right\|.$$

$$\omega(2C)\le 2\left\||C|\right\|.$$

$$\omega(C)\le\left\||C|\right\|.$$

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Theorem 2.4

If C, D $\in$ *B(H), then*

$$\omega(\mathrm{C})\le\frac{1}{\sqrt[3]{4}}\left\||C|^{\frac{3}{2}}+|C^*|^{\frac{3}{2}}\right\|^{2/3}$$

Proof:

By Theorem 1.3, we have

$$\omega^r(\mathrm{C}) \le \frac{1}{2}\left\| |C|^{2r\alpha} + |C^*|^{2r(1-\alpha)} \right\|, \text{ for } 0 < \alpha < 1,\ r \ge 1$$

If $r = 1.5$, $\alpha = \frac{1}{2}$, *we have*

$$\omega^{\frac{3}{2}}(\mathrm{C}) \le \frac{1}{2}\left\| |C|^{\frac{3}{2}} + |C^*|^{\frac{3}{2}} \right\|$$

$$\omega(\mathrm{C}) \le \frac{1}{\sqrt[3]{4}}\left\| |C|^{\frac{3}{2}} + |C^*|^{\frac{3}{2}} \right\|^{2/3}$$

Theorem 2.5

If $C \in B(H)$, *then*

$$\omega(\mathrm{C}) \le \frac{1}{\sqrt[3]{4}}\left\| \left(|C|^{\frac{3}{2}} + |C^*|^{\frac{3}{2}} \right) \right\|^{\frac{2}{3}}$$

Proof:

By Theorem 1.3, we have

$$\omega^r\left(C + D\right) \le 2^{r-2}\left\| |C|^{2r\alpha} + |C^*|^{2r(1-\alpha)} + |D|^{2r\alpha} + |D^*|^{2r(1-\alpha)} \right\|,$$

for $0 < \alpha < 1,\ r \ge 1$

If D = C, r = 1.5, and $\alpha = \frac{1}{2}$*, we have*

$$\omega^{\frac{3}{2}}\left(C+C\right) \le \frac{1}{\sqrt{2}}\left\| |C|^{\frac{3}{2}} + |C^*|^{\frac{3}{2}} + |C|^{\frac{3}{2}} + |C^*|^{\frac{3}{2}} \right\|$$

$$\omega^{\frac{3}{2}}(2\mathrm{C}) \le \frac{1}{\sqrt{2}}\left\| 2\left(|C|^{\frac{3}{2}} + |C^*|^{\frac{3}{2}} \right) \right\|$$

$$\omega(\mathrm{C}) \le \frac{\sqrt[3]{4}}{2\sqrt[3]{2}}\left\| \left(|C|^{\frac{3}{2}} + |C^*|^{\frac{3}{2}} \right) \right\|^{\frac{2}{3}}$$

$$\omega(\mathrm{C}) \le \frac{1}{\sqrt[3]{4}}\left\| \left(|C|^{\frac{3}{2}} + |C^*|^{\frac{3}{2}} \right) \right\|^{\frac{2}{3}}$$

Theorem 4.6

If $C \in B(H)$*, then*

$$\omega\left(C^2\right) \le 2\|C\|$$

Proof:

By Theorem 1.1, we have

$$\omega\left(CD\right) \le 2\omega(\mathrm{C})\omega(\mathrm{D}).$$

put D = 2C, we have

$$\omega\left(2C^2\right) \le 2\omega(\mathrm{C})\omega(2\mathrm{C})$$

$$\omega\left(2C^2\right) \le 4\omega^2(\mathrm{C}) \le 4\|C\|^2$$

Hence, $\omega\left(2C^2\right) \le 4\|C\|^2$

$$\omega\left(C^2\right) \le 2\|C\|^2$$

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