CRITICAL POINT ORBIT STRUCTURE AND THERMODYNAMIC FORMALISM FOR ERGODIC CRITICAL CIRCLE MAPPINGS OF POINTS

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Abstract: Consider the space X_ρ^{cr} of analytic, critical circle homeomorphisms with rotation number $\rho = [k_1, k_2, k_1, k_2, ..., k_1, k_2, ...]$, $k_1, k_2 \in \mathbb{N}$. It is well known, that the renormalization transformation $\mathfrak{R} = \mathfrak{R}(k_1, k_2)$ in X_ρ^{cr} has a unique fixed point $T_{cr} = T(k_1, k_2)$. We study the behaviour of Poincare's map $\pi_n = (T_{cr}^{q_n}, T_{cr}^{q_{n+1}})$ with first return time q_n and the structure of infinite orbit $\mathbb{O}(x_c) = \{T_{cr}^i(x_c), i \geq 0\}$ of critical point $x_c = 0$. Denote by $Cr(T_{cr})$ the set of all C^1−conjugated to T_{cr} critical circle maps. We constrict unique pair of potentials (U_o, U_e) corresponding to all maps from $Cr(T_{cr})$. algebra.

Keywords: circle maps, critical point, rotation number, orbit, invariant measure.

1 Introduction

Ya. G. Sinai in his fundamental work [3] first applied the ideas of thermodynamic formalism (TF) for investigating Anosov diffeomorphisms. Later, the TF for dynamical systems devoloped by D. Ruelle [5].The thermodynamic formalism for the unimodal Feigenbaum map of the interval was built by E. Vul, Y. Sinai

and K. Khanin in [1]. A. Dzhalilov in [4] used thermodynamic formalism for investigating the singular invariant measures of critical circle maps.

The introduction of thermodynamic formalism within the mathematical field of dynamical systems occurred in the 1970s, and was primarily due to Y. Sinai, D. Ruelle and R. Bowen. In this paper we build the thermodynamic formalism for the circle maps with a critical point. The thermodynamic formalism for the unimodal Feigenbaum map was built by E. Vul, Y. Sinai and K. Khanin [1]. In fact, our work is an analogy of their work for the critical circle maps.

The main purpose of this work to study the orbit of critical point of circle map f with universal renormalization properties (see [1]-[9]) and on the basic this orbit build the thermodynamic formalism.

Let $k_1 \in \mathbb{N}$. Consider the space X pairs $(\xi(x), \eta(x))$ of real analytic, strictly increasing homeomorphisms of real line $\mathbb{R}^1$ satisfying the following conditions [4]

a) $0 < \xi(0) < 1$;

b) $\xi(0) = \eta(0) + 1$;

c) $\xi(\eta(0)) = \eta(\xi(0))$;

d) $\xi(\eta(0)) < 0, \ \xi^2(\eta(0)) < 0, \ ..., \ \xi^{k_1-1}(\eta(0)) < 0$;

e) $\xi^{k_1}(\eta(0)) > 0$;

g) $\xi'(0) = \eta'(0) = \xi''(0) = \eta''(0) = 0; \ \xi'''(0) \neq 0, \ \eta'''(0) \neq 0$;

h) $(\xi \circ \eta)'''(0) = (\eta \circ \xi)'''(0)$.

Here and later on f^i denotes $i-$ iteration of f. Let $(\xi, \eta) \in X$. Conditions $(a) - (h)$ allows to build homeomorphism on the circle $[\eta(0), \xi(0))$:

$$T_{\xi,\eta}(x) = \left\{ \begin{array}{ll} \xi(x), & \text{if } x \in [\eta(0), 0), \\ \eta(x), & \text{if } x \in [0, \xi(0)). \end{array} \right.$$
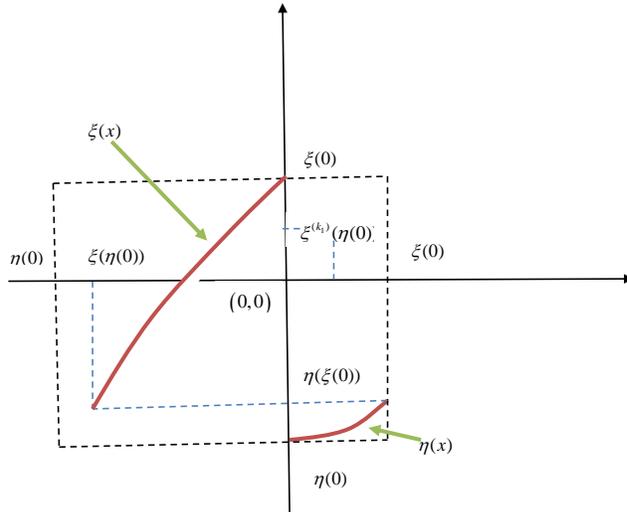

Figure 1

We denote by X_ρ^{cr} the subset of X, consisting all pairs (ξ, η), such that the rotation number

$$\rho(T_{\xi,\eta}(x)) = \rho = \frac{-k_2 + \sqrt{k_2^2 - \frac{4 \cdot k_2}{k_1}}}{2}, \ k_1, \ k_2 \in \mathbb{N}.$$

Note that the continued fraction expansion of irrational rotation number ρ has the form: $\rho = [k_1, k_2, , k_1, k_2, ...]$. We denote by $\frac{p_n}{q_n}$ the $n-$th convergent of ρ. The numbers q_n are called the of first return times and satisfy the difference equation

$$q_{2n+1} = k_1 \cdot q_{2n} + q_{2n-1},$$

$$q_{2n} = k_2 \cdot q_{2n-1} + q_{2n-2},$$

with initial conditions $q_0 = 1, \ q_1 = k_1, \ q_2 = k_1 \cdot k_2 + 1$.

Now we define the renormalization transformation R_{k_i} (see [4]):

$$R_{k_i}(\xi, \eta) = (\alpha_i \xi^{k_i-1}(\eta(\alpha_i^{-1}x)), \alpha_i \xi^{k_i-1}(\eta(\xi(\alpha_i^{-1}x)))),$$

where $\alpha_i = \alpha_i(\xi, \eta) = (\xi^{k_i-1}(\eta(0)) - \xi^{k_i}(\eta(0)))^{-1}$. The conditions (a) and (b) implies that $\alpha_i < -1, \ i = 1, 2$. We define the renormalization transformation $\mathfrak{R} := R_{k_2} \circ R_{k_1}$ on X_ρ^{cr}.

The renormalization group transformation $\mathfrak{R}$ has a single hyperbolic periodic point (ξ, η) (see [2], [4].)

Note, that $\xi(x)$ and $\eta(x)$ are analytic functions of x^3.

$$\eta(x) = \eta(0) + \frac{\eta^{(3)}(0)}{3!} \cdot x^3 + \frac{\eta^{(6)}(0)}{6!} \cdot x^6 + ... + \frac{\eta^{(3)}(0)}{(3n)!} \cdot x^{3n} + ...,$$

$$\xi(x) = \xi(0) + \frac{\xi^{(3)}(0)}{3!} \cdot x^3 + \frac{\xi^6(0)}{6!} \cdot x^6 + ... + \frac{\xi^{(3n)}(0)}{(3n)!} \cdot x^{3n} + ...$$

Let **T** be a circle homeomorphism generated by (ξ, η) and defined on the unit circle $[\eta(0), \xi(0))$.

Definition 1. *Two orientation-preserving circle Homeomorphisms T_1 and T_2 are said to be C^r-conjugate, $r \geq 1$, if there exists an orientation-preserving circle homeomorphism $T_\varphi \in C^r(\mathbb{S}^1)$ here if $r \geq 1$, then it is required that $T_\varphi^{-1} \in C^1(\mathbb{S}^1)$), such that*

$$T_\varphi \circ T_1 = T_2 \circ T_\varphi.$$

Note that critical circle maps possess the "rigidity"property. Namely, any two real-analytic, critical circle maps T_f and T_g with cubic critical point $x_0 = 0$ and with irrational rotation number $\rho = \rho(f) = \rho(g)$ of "bounded type"(it means that, the elements of continued fraction expansion of ρ are uniformly bounded) $C^{1+\alpha}-$ conjugate with some $0 < \alpha < 1$, depending on rotation number ρ. [2].

Denote by $E(T_{cr})$ the set of all circle homeomorphisms whose are C^1-

conjugated with **T** and defined on the standard circle $\mathbb{S}^1$. It is well known (see [1]-[8]) that any two topological conjugated homeomorphisms have the

same rotation number. Therefore, the rotation numbers of homeomorphisms of $E(T_{cr})$ are the same and equal to ρ.

The purpose of this paper is to build thermodynamic formalism for the set $E(T_{cr})$. To formulate our main result, we introduce further notations. Let

$$\frac{p_n}{q_n} = \underbrace{[k_1, k_2, k_1, k_2, ..., k_1, k_2]}_{n \ times}$$

be the sequence of rational convergents of the continued fraction

$\rho = [k_1, k_2, k_1, k_2, ..., k_1, k_2, ...]$. The coprime numbers p_n and q_n satisfy the recurrence relations $p_n = k_n p_{n-1} + p_{n-2}$ and $q_n = k_n q_{n-1} + q_{n-2}$ for $n \geq 1$, where, for convenience we set $p_0 = 0, q_0 = 1$ and $p_{-1} = 1, q_{-1} = 0$. Taking the critical point $x_0 = 0$, we define the nth *fundamental segment* $I_0^n := I_0^n$ as the circle arc $[x_0, T_{cr}^{q_n}(x_0)]$ if n is even and $[T_{cr}^{q_n}(x_0), x_0]$ if is odd. Certain number of images of fundamental segments I_0^{n-1} and I_0^n, under iterates of T_{cr}, cover whole circle without overlapping beyond the endpoints and form the nth *dynamical partition* of the circle

$$\mathbb{P}_n = \{I_j^n := T_{cr}^j(I_0^n), 0 \leq j < q_{n-1}\} \cup \{I_i^{n-1} := T_{cr}^i(I_0^{n-1}), 0 \leq i < q_n\}.$$

Obviously, the partition $\mathbb{P}_{n+1}$ is a refinement of the partition $\mathbb{P}_n$. Indeed, the "short" intervals I_j^n are members of $\mathbb{P}_{n+1}$ and each "long" interval $I_i^{n-1} \in \tau_n, 0 \leq i < q_n$, is partitioned into $k_{n+1} + 1$ intervals belonging to $\mathbb{P}_{n+1}$ such that

$$I_i^{n-1} = I_i^{n+1} \cup \bigcup_{s=0}^{k_{n+1}-1} I_{j+q_{n-1}+sq_n}^n.$$

Next, using the sequence of dynamical rartition $(\mathbb{P}_n)_n$ we introduce a certain symbolic representation for the dynamics of T_{cr} as follows. Let $\mathcal{A} = \{a, 0, 1, ..., k_1\}$ be an *alphabet*. Consider the set of *infinite words* $\mathcal{L} = \{\underline{a} := (a_1, a_2, ..., a_n, ...) : |a_i \in \mathcal{A}\}$ corresponding to $\mathbb{S}^1 \setminus O_{T_{cr}}(x_0)$, where $O_{T_{cr}}(x_0 = \{x_0, T_{cr}(x_0), ...\}$ and defined as follows. Take an arbitrary $x \in \mathbb{S}^1 \setminus O_{T_{cr}}(x_0)$ we associate the unique word $\underline{a} := (a_1, a_2, ..., a_n, ...)$ defined inductively as: $a_n = 0$ if $x \in I_j^n, 0 \leq j < q_{n-1}; a_n = k_1 - s$, if $x \in I_{i+q_{n-1}+sq_n}^n$, $0 \leq s \leq k_{n+1} - 1, 0 \leq i < q_n$; and $a_n = 0$, if $I_i^{n+1}, 0 \leq i < q_n$. Thus, we obtain a one-to-one correspondence

$$\mathbb{S}^1 \setminus O_{T_{cr}}(x_0) \leftrightarrow \mathcal{L}.$$

Notice that, the finite word $(a_1, a_2, ..., a_n)$ of the length n corresponds to an interval I^n of the dynamical partition $\mathbb{P}_n$. We set $I^n = I(a_1, a_2, ..., a_n)$. Denote by λ_0 a probability measure on the space of sequences $\mathcal{L}$ induced by Lebesgue measure on the circle, namely, $\lambda_0(a_1, a_2, ..., a_n) = |I(a_1, a_2, ..., a_n)|$. Consider another space of one-sided sequences,

$$\Omega_+ = \{\underline{\varepsilon} = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_n, ...) \mid \varepsilon_i \in \mathcal{A} \ and \ \varepsilon_{i+1} = 0 \ if \ \varepsilon_i = a, \ i \geq 1\}.$$

Now, consider another $\widetilde{T} \in E(T_{cr})$. Since $\widetilde{T}$ and T_{cr} are topological conjugated the sets $O_{T_{cr}}(x_0)$ and $O_{\widetilde{T}_{cr}}(y_0)$ have the same order, where $y_0 = h(x_0)$ and h

is the conjugation between $\widetilde{T}$ and T_{cr}. Therefore, the symbolic representations of the points $x \in \mathbb{S}^1 \backslash O_{T_{cr}}(x_0)$ and $h \in \mathbb{S}^1 \backslash O_{\widetilde{T}_{cr}}(y_0)$ are the same in $\mathcal{L}$ i.e., there exists a unique $\underline{a} \in \mathcal{L}$ such that $\underline{a} = \psi(x)$ and $\underline{a} = \psi(h(x))$. We set

$$\underline{\gamma}(\varepsilon) := \left\{ \begin{array}{l} (0, a, 0, a, ...), \ if \ \varepsilon = a, \\ (a, 0, a, 0, ...), \ if \ \varepsilon = a. \end{array} \right.$$

$\triangle_i^n := h(I_i^n)$ and $V_1 := \triangle_1^{(1)} \cup \triangle_1^{(2)}$.

Our main aim is to prove the following result.

Theorem 1. *For any $T \in E(T_{cr})$ there exists a pair (U_o, U_e), continuous (in the Tychonoff topology) mappings $U_o, U_e : \Omega_+ \to R^1$, such that the following properties hold.*

1. For any $\underline{\varepsilon} = (\varepsilon_1, ..., \varepsilon_p, \varepsilon_{p+1}, ..., \varepsilon_n, ...)$ and $\underline{b} = (\varepsilon_1, ..., \varepsilon_p, b_{p+1}, ..., b_n, ...)$ belonging to the space Ω_+ there exists a constant $C_1 = C_1(T) > 0$ such that

$$|U_o(\underline{\varepsilon}) - U_o(\underline{b})| \leq C_1 |\alpha|^{-p}, \ if \ p \ is \ odd \ number;$$

$$|U_e(\underline{\varepsilon}) - U_e(\underline{b})| \leq C_1 |\alpha|^{-p}, \ if \ p \ is \ even \ number,$$

where $\alpha = \alpha_1 \alpha_2 < 1$.

2. If $I(a_1, ..., a_r, a_{r+1}..., a_n) \subset I(a_1, ..., a_r) \subset V_1, 1 \leq r < n$, and r, n are odd numbers. Then

$$\frac{|I(a_1, ..., a_n)|}{|I(a_1, ..., a_r)|} = (1 + \psi_o(a_1, ..., a_n)) \cdot exp\{ \sum_{s=r+1}^{n} U_o(a_s, a_{s+1}, ..., a_r, ..., a_1, \underline{\gamma}(a_1))\}$$

If $I(a_1, ..., a_r, a_{r+1}..., a_n) \subset I(a_1, ..., a_r) \subset V_1, 1 \leq r < n$, and r, n are even numbers. Then

$$\frac{|I(a_1, ..., a_n)|}{|I(a_1, ..., a_r)|} = (1 + \psi_e(a_1, ..., a_n)) \cdot exp\{ \sum_{s=r+1}^{n} U_e(a_s, a_{s+1}, ..., a_r, ..., a_1, \underline{\gamma}(a_1))\},$$

where $|\psi(a_1, ..., a_n)| \leq Const \cdot |\alpha|^{-r}$.

Remark 1. *1) Theorem 1 generalizes the main theorem of [4].*

2) The second assertion of Theorem 1 implies that, the length of any interval $I^n \in \mathbb{P}_n$ can be expressed in the "Cibbs"form:

$$const \leq \frac{I^n}{exp\{ \sum_{s=r+1}^{n} U(a_s, a_{s+1}, ..., a_r, ..., a_1, \underline{\gamma}(a_1))\}} \leq Const$$

3) Following the terminology of statistical mechanics, we call U the potential corresponding to the map $\mathbf{T}$. Due to Teorem 1 the potential depends on the long-range variables exponentially weakly, i.e., on the statistical mechanical point of view, it is "good".

4) One can easily verify that the potential U is unique and invariant under a smooth change of variable.

2 The orbit of critical point

As we have mentioned above the homemorphism T_{cr} is analytic, has the cubic critical point $x_0 = 0$ and its rotation number is irrational. Due to Yoccoz's [9] theorem T_{cr} and the linear rotation $T_\rho(x) = x + \rho \ (mod1)$ are topological equivalent. Hence the topological properties of T_{cr} and T_ρ are the same. In this section we study the metric properties of the $T_{cr}-$ orbit of the critical point $x_0 = 0$. We note that the commuting ξ, η is a fixed point of renormalization operator $\mathcal{R}$ i.e.,

$$\begin{cases} \alpha_2 \cdot \alpha_1(\underbrace{\xi^{k_1-1}(\eta...\xi^{k_1-1}(\eta(\xi(\alpha_2^{-1}\alpha_1^{-1}x)))))...}_{q_2 \ times}) = \xi(x) & \text{если } x \in [\eta(0), 0), \\ \alpha_2 \cdot \alpha_1(\underbrace{\xi^{k_1-1}(\eta...\xi^{k_1-1}(\eta(\xi^{k_1}(\eta(\alpha_2^{-1}\alpha_1^{-1}x)))))...}_{q_2+q_1 \ times}) = \eta(x) & \text{если } x \in [0, \xi(0)). \end{cases}$$

We use this fact in the proof of the following theorem.

Theorem 2. *For all $n \geq 1$ the following relations hold*

$$T_{cr}^{q_{2n}}(\alpha^{-n}x) = \alpha^{-n}\xi(x), \ x \in [\eta(0), 0) \tag{1}$$

$$T_{cr}^{q_{2n}+q_{2n-1}}(\alpha^{-n}x) = \alpha^{-n}\eta(x), \ x \in [0, \xi(0)), \tag{2}$$

where $\alpha = \alpha_1 \cdot \alpha_2$.

Proof. We prove the theorem by induction. Let $x \in [\eta(0), 0)$. If $n = 1$, then $q_2 = k_1 \cdot k_2 + 1$ and $\alpha^{-1}x \in [0, \xi(0))$. Therefore

$$T_{cr}^{q_2}(\alpha^{-1}x) = T_{cr}^{k_1 \cdot k_2 + 1}(\alpha^{-1}x) = T_{cr}^{k_1 \cdot k_2}(T_{cr}(\alpha^{-1}x)) = T_{cr}^{k_1 \cdot k_2}(\xi(\alpha^{-1}x)) =$$

$$\underbrace{T_{cr}^{(k_1-1+1)+...+(k_1-1+1)}}_{k_2 \ time} (\xi(\alpha^{-1}x)) =$$

$$\underbrace{T_{cr}^{(k_1-1+1)+...+(k_1-1+1)}}_{k_2-1 \ time} (\xi^{k_1-1}(\eta(\xi(\alpha^{-1}x)))) =$$

$$\underbrace{\xi^{k_1-1}(\eta...\xi^{k_1-1}(\eta(\xi(\alpha^{-1}x)))))...}_{q_2 \ time}).$$

Relation (1) implies $\underbrace{\xi^{k_1-1}(\eta...\xi^{k_1-1}(\eta(\xi(\alpha^{-1}x)))))...}_{q_2 \ time}) = \alpha^{-1}\xi(x)$. Hence

Suppose, that equation (2) holds $l \leq n$. The relation (1) and $q_{2n+2} = k_2 q_{2n+1} + q_{2n}$ imply

$$T_{cr}^{q_{2l+2}}(\alpha^{-(l+1)}x) = T_{cr}^{k_2 q_{2l+1}+q_{2l}}(\alpha^{-(l+1)}x) = T_{cr}^{k_2 q_{2l+1}}(T_{cr}^{q_{2l}}(\alpha^{-(l+1)}x)) =$$

$$T_{cr}^{k_2 q_{2l+1}}((\alpha^{-l}\xi(\alpha^{-1}x))) = T_{cr}^{k_2(k_1 q_{2l}+q_{2l-1})}((\alpha^{-l}\xi(\alpha^{-1}x))) =$$

$$T_{cr}^{k_2 k_1 q_{2l}+k_2 q_{2l-1}}((\alpha^{-l}\xi(\alpha^{-1}x))) =$$

$$T_{cr}^{(k_2 k_1-1)q_{2l}+(k_2-1)q_{2l-1}}((\alpha^{-l}\eta(\xi(\alpha^{-1}x)))) = ...$$

$$= \alpha^{-l}\underbrace{\xi^{k_1-1}(\eta...\xi^{k_1-1}(\eta(\xi(\alpha^{-1}x)))))...}_{q_2 \ time}) = \alpha^{-l}\alpha^{-1}\eta(x) = \alpha^{-(l-1)}\eta(x)$$

This proves the first assertion of theorem. The proof of the second assertion is similar.

As a consequence of Theorem 2 we have.

Corollary 1. *Let $x_0 = 0$ and $x_i = T_{cr}^i(x_0)$, $i \geq 1$. For all $n \geq 1$ we have*

$$x_{q_{2n}} = \alpha^{-n}\xi(0), \quad x_{q_{2n}+q_{2n-1}} = \alpha^{-n}\eta(0),$$

where $\alpha = \alpha_1 \cdot \alpha_2$.

Let A_n be a subset of $O_{T_{cr}}(x_0)$ which is generation the nth dynamical partition i.e., $A_n = \{x_0, x_1, .., x_{q_n+q_{n-1}-1}\}$. We set

$A_n^{(1)} = A_n \cap [0, \xi(0))$, $A_{n,1}^{(1)} = A_n \cap [0, x_{q_2})$, $A_{n,2}^{(1)} = A_n \cap [x_{q_2}, \xi(0))$,

$A_n^{(k_1+1)} = A_n \cap [x_{k_1}, 0)$, $A_{n,1}^{(k_1+1)} = A_n \cap [x_{q_2+q_1}, 0)$,

$A_n^{(m+1)} = A_n \cap [x_m, x_{m+1}), 1 \leq m \leq k_1 - 1$,

$A_{n,2}^{(k_1+1)} = A_n \cap [x_{k_1}, x_{q_2+q_1})$,

where $1 \leq m \leq k_1$. It is clear, that $A_n = \cup_{m=1}^{k_1+1} A_n^{(m)}$ and $A_n^{(1)} = A_{n,1}^{(1)} \cup A_{n,2}^{(1)}$, $A_n^{(k_1+1)} = A_{n,1}^{(k_1+1)} \cup A_{n,2}^{(k_1+1)}$.
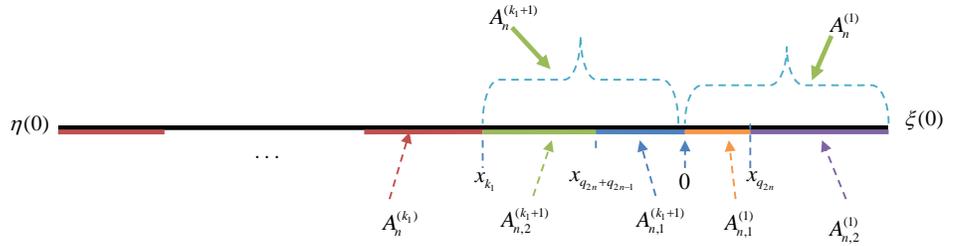

Figure 2

Using definition $A_n^{(i)}, i = \overline{1, (k_1 + 1)}$ and the structure of dynamical partition one can show that

$|A_n^{(i)}| = \frac{k_n q_{n-1} + k_{n-1} q_{n-2}}{k_1}, i = \overline{1, k_1}$, $|A_n^{(k_1+1)}| = q_{n-2} + q_{n-3}$,

where $|\cdot|-$ the cardinality of the finite set.

Lemma 1. *For the number of plurality $A_n^{(i)}, i \geq 1$ the following equations are just*

$$|A_n^{(1)}| = |A_n^{(2)}| = |A_n^{(3)}| = ... = |A_n^{(k_1)}| = \frac{k_n q_{n-1} + k_{n-1} q_{n-2}}{k_1} \qquad (3)$$

$$|A_n^{(k_1+1)}| = q_{n-2} + q_{n-3}, \ q_{-1} = q_{-2} = q_{-3} = 0. \qquad (4)$$

Proof. We verity the (3) and (4) correlation for $n = 1$. Dynamical partition $\mathbb{P}_1$ has following segments:

$$I_0^0, I_1^0, ..., I_{q_1-1}^0, I_0^1$$

Now we look through $\mathbb{P}_2$ partition. Here in this partition each segment of 0-rank is divided into $k_2 + 1$ segments, and the I_0^1 segments are not divided. Here in the result the news segments of division will have 1 and 2 rank, etc.

$$I_j^0 = I_j^2 \cup \bigcup_{s=0}^{k_2-1} I_{j+q_0+sq_1}^1, 1 \leq j \leq k_1 - 1$$

$|A_2^{(1)}| = |A_2^{(2)}| = |A_2^{(3)}| = ... = |A_2^{(k_1)}| = k_2+1 = \frac{k_2 k_1 + k_1}{k_1} = \frac{k_2 q_1 + k_1 q_0}{k_1}$, $|A_2^{(k_1+1)}| = q_0 = 1$.

Thus we have k_1 items of 2-rank segments and $k_1 \cdot (k_2+1) = k_1 \cdot (\frac{k_2 k_1 + k_1}{k_1}) = k_2 q_1 + q_1$ items of 1-rank segments number $k_2 q_1 + q_1 + q_0 = q_2 + q_1$.

So, the correctness of (3) and (4) are proved. By the method of mathematical induction we add for $n + 1$, supposing that the correlation is true for all n.

The number of segment of $\mathbb{P}_n$ partition is equal for $k_1 \cdot \frac{k_n \cdot q_{n-1} + k_{n-1} \cdot q_{n-2}}{k_1} + q_{n-2} + q_{n-3} = q_n + q_{n-1}$, and the number of segments $\mathbb{P}_{n+2}$ is equal to $q_{n+2} + q_{n+1}$

In dynamical partition $\mathbb{P}_{n+2}$, we have I^{n+1} segments of $n + 1$ in the quantity of q_{n+1} and I^{n+2} of $n + 2$ rank in the quantity of q_{n+1}, here we have the elements lying on $[0, \xi(0)) \cap A_{n+2}$ is equal to $q_{n+2} + q_{n+1}$, As the elements $A_{n+2}^i, \overline{i = 1, k_1}$, are made up by the $[0, \xi(0)) \cap A_{n+2}$ iteration, so then their element numbers are equal:

$$|A_{n+2}^{(1)}| = |A_{n+2}^{(2)}| = |A_{n+2}^{(3)}| = ... = |A_{n+2}^{(k_1)}| = \frac{k_{n+2} \cdot q_{n+1} + k_{n+1} \cdot q_n}{k_1}$$

The equality (3) is proved. So, the correctness of (3) is proved. We proved the correctness of (4) for $A_{n+2}^{(k_1+1)}$. Suppose, $A_n^{(k_1+1)} = q_{n-2} + q_{n-3}$. It is known, that I_1^1 segment contents the $n-1$ rank segments in q_{n-2} quantity and n rank segments in q_{n-3} quantity. While partition $\mathbb{P}_{n+2}$, each of n rank segments is divided into one $n - 1$ rank and k_{n-1} items $n - 1$ rank segments. We get $(k_{n-1}+1)q_{n-1} = k_{n-1}q_{n-1}+q_{n-1}$. Now we add preserved n rank segments q_n items. Here we get number of n rank segments as $k_{n-1}q_{n-1} + q_{n-1} + q_{n-3} = q_{n-1} + q_{n-2}$. Thus we proved that $A_{n+2}^{(k_1+1)} = q_n + q_{n-1}$. Lemma 1 is proved.

The next theorem describes the transition from A_n to A_{n+2}.

Theorem 3. *For all $n \geq 1$ the following relations hold.*

$A_{n+2,2}^{(k_1+1)} = \alpha^{-1}(A_n \setminus (A_n^{(1)})), \ A_{n+2,1}^{(1)} = \alpha^{-1}(A_n^{(1)}),$

$A_{n+2,1}^{(k_1+1)} = T_{cr}^{k_1}(A_{n+2,1}^{(1)}), \ A_{n+2}^{(k_1+1)} = (A_{n+2,1}^{(k_1+1)} \cup A_{n+2,2}^{(k_1+1)}),$

$A_{n+2,2}^{(1)} = \sum_{j=0}^{k_2-1} T_{cr}^{1+j \cdot k_1}(A_{n+2}^{(k_1+1)}), \ A_{n+2}^{(1)} = A_{n+2,1}^{(1)} \cup A_{n+2,2}^{(1)},$

$A_{n+2}^{(s)} = T_{cr}^{s-1}(A_{n+2}^{(1)}), \quad 1 < s \leq k_1.$

Proof. Firstly, we prove that the second statement of the theorem 3. By the corollary 1 we have $x_{q_2+q_1} = \alpha^{-1} \cdot x_1$ and $x_{q_2} = \alpha^{-1} \cdot \xi(0)$, for every point $x_i \in \mathbb{P}_n$ such that $\alpha^{-1}x_i \in \tau_{n+2}$ where $0 < \alpha < 1$ from that we take $\alpha^{-1}(A_n^{(1)}) \in A_{n+2,1}^{(1)}$. From the structure of dynamical partition we can see that

$$|A_{n+2,1}^{(k_1+1)}| = |A_{n+2,1}^{(1)}| = |A_n^{(1)}|$$

We need to prove $|A_{n+2,2}^{(k_1+1)}| = |A_n| - |A_n^{(1)}|$. It is easy to see that

$$|A_n| - |A_n^{(1)}| = q_n + q_{n-1} - \frac{k_n q_{n-1} + k_{n-1} q_{n-2}}{k_1}.$$

We evaluate the left side

$$|A_{n+2,2}^{(k_1+1)}| = |A_{n+2}^{k_1+1}| - |A_{n+1,1}^{(k_1+1)}|$$

$$= |A_{n+2}^{k_1+1}| - |A_n^{(1)}|$$

$$= q_n + q_{n-1} - \frac{k_n q_{n-1} + k_{n-1} q_{n-2}}{k_1}.$$

Other statements are not difficult. The theorem 3. is proved.

Lemma 2. *Let $I^n \in \mathbb{P}_n$ and $I^{n-k} \in \mathbb{P}_{n-k}$, $n > k$ such that $I^n \subset I^{n-k}$. There exists a constant $C = C(T_{cr}) > 0$ such that*

$$\frac{I^n}{I^{n-k}} \leq C\alpha^{-k}, \quad \alpha^{-3n} \leq C|I_0^n|.$$

Proof. The proof of lemma follows closely that of [1] for the Feigenbaum map.

To formulate our next result we define some subsets of $\mathbb{P}_n$ as follow

$$\mathbb{P}_n^1 = \{I^n \in \mathbb{P}_n : \ such \ that \ I^n \subset [\eta(0), 0]\},$$

$$\mathbb{P}_n^2 = \{I^n \in \mathbb{P}_n : \ such \ that \ I^n \subset [0, \xi(0)]\},$$

$$\mathbb{P}_{n,i} = \{I^n \in \mathbb{P}_n : \ such \ that \ I^n \subset I_0^i\},$$

where $i = (n - k - 1), (n - k)$. The following theorem plays an important role to build the thermodynamical formalism for T_{cr}.

Theorem 4. *The following relations hold, for any $0 < k < n$.*

$$\mathbb{P}_{n,n-k} = \alpha^{\frac{n-k}{2}} \mathbb{P}_k^1, \ \mathbb{P}_{n,n-k-1} = \alpha^{\frac{n-k}{2}} \mathbb{P}_k^2.$$

if $n - k$ is an odd number.

$$\mathbb{P}_{n,n-k-1} = \alpha^{\frac{n-k}{2}} \mathbb{P}_k^2, \ \mathbb{P}_{n,n-k} = \alpha^{\frac{n-k}{2}} \mathbb{P}_k^1.$$

if $n - k$ is an even number.

Proof. The proof of theorem follows Theorem 4.

Consider the interval $V_1^{n-k} = I_1^{n-k} \cup I_1^{n-k-1}$. Notice that this interval is a neighborhood of the point x_1. Further, taking two intervals I' and I such that $I' \subset I \subset V_1^{n-k}$ and $I' \in \mathbb{P}_{n+1}$, $I \in \mathbb{P}_n$ we provide an estimate for the ration $R_0 = I' \backslash I$ and its "iterations" $R_i = \frac{|T_{cr}^i(I')|}{|T_{cr}^i(I)|}$. Denote

$$s_{n-k} = \begin{cases} q_{n-k-1}, & \text{if } I \subset I_1^{n-k}, \\ q_{n-k}, & \text{if } I \subset I_1^{n-k-1} \end{cases}$$

The following lemma will be used below.

Lemma 3. [4] *There exists a constant $C = C(T_{cr}) > 0$ and a natural number $N_{cr} > N_{cr}(T_{cr})$ such that*

$$|\ln \frac{R_i}{R_0}| \leq C\alpha^{-k}, \, 0 < i < s_{n-k}.$$

for all $n > N_{cr}$.

3 Construction of a potential for ergodic critical mappings of a circle

In this section we prove Theorem 1 i.e., we build the potential U for T_{cr}. Note that the proof of Theorem 1 follows closely that of Theorem 1.1 [4]. To prove the theorem first we define the prelimit potentials $U_k^n(\varepsilon_1, \varepsilon_2, ..., \varepsilon_k)$ and $\widetilde{U}_k^n(\varepsilon_1, \varepsilon_2, ..., \varepsilon_k)$ for $1 \leq k \leq n$ and then we show that these potentials converge to the limit of $U_k(\varepsilon_1, \varepsilon_2, ..., \varepsilon_k)$ exponentially fast as $n \to \infty$. Finally we show that the limit of $U_k(\varepsilon_1, \varepsilon_2, ..., \varepsilon_k)$ exists and unique as $m \to \infty$. Take $m \geq 1$ and fix it. Consider the V_1^{n-m} neighborhood of x_1 for $n \geq m$. It is clear that $V_{n-k} \subset [\eta(0), x_{q_2+q_1}] \cup [x_{q_2}, \xi(0))$ for $n \geq k + 4$. For definiteness we assume $(n-k)-$ is even. The proof is similar for the case when $(n-k)-$ is odd. It is obvious

$$I_1^{n-k} \subset [x_1, x_{q_2+q_1}], \, I_1^{n-k-1} \subset [x_{q_2}, \xi(0)].$$

Consider two intervals $I' \in \mathbb{P}_n$ and $I \in \mathbb{P}_{n-2}$ such that $I' \subset I \subset I_1^{(n-k)}$. Let the words

$$I(\underbrace{a, 2, 0, a, ..., 0, a, \varepsilon_k, ..., \varepsilon_2, \varepsilon_1}_{n- \, times}) \, I(\underbrace{a, 2, 0, ..., a, \varepsilon_k, ..., \varepsilon_2}_{(n-2)- \, times}).$$

be the symbolic representations of I' and I respectively. Define the following functions

$$U_k^n(\varepsilon_1, \varepsilon_2, ..., \varepsilon_k) = \ln \frac{I(a, 2, 0, a, ..., 0, a, \varepsilon_k, ..., \varepsilon_2, \varepsilon_1)}{I(a, 2, 0, ..., a, \varepsilon_k, ..., \varepsilon_2)}.$$

and

$$\widetilde{U}_k^n(\varepsilon_1, \varepsilon_2, ..., \varepsilon_k) = \ln \frac{I(2, 0, a, ..., 0, a, 0, \varepsilon_k, ..., \varepsilon_2, \varepsilon_1)}{I(2, 0, a, ..., a, 0, \varepsilon_k, ..., \varepsilon_2)}.$$

It follows from the structure of dynamical partition and the definition of symbolical representation that

$$|I(2,0,a,0,...,a,\varepsilon_k,...,\varepsilon_2,a)| = |I(2,0,a,0,...,a,\varepsilon_k,...,\varepsilon_2)|.$$

It is easy to see that

$$U_k^n(a,0,\varepsilon_3,...,\varepsilon_k) = \widetilde{U}_k^n(a,0,\varepsilon_3,...,\varepsilon_k) = 0.$$

Using theorem 3 and theorem 4, we get:

$$I(a,2,0,a,...,0,a,\varepsilon_k,...,\varepsilon_2,\varepsilon_1) = \eta(I(0,a,0,a,...,0,a,\varepsilon_k,...,\varepsilon_2,\varepsilon_1)) =$$

$$= \eta(\alpha^{-\frac{n-k}{2}}(I(\varepsilon_k,...,\varepsilon_2,\varepsilon_1))), \tag{5}$$

where $I(\varepsilon_k,...,\varepsilon_2,\varepsilon_1) \in \mathbb{P}_k$. Similarly,

$$I(a,2,0,a,...,0,a,\varepsilon_k,...,\varepsilon_2) = \eta(I(0,a,0,a,...,0,a,\varepsilon_k,...,\varepsilon_2)) =$$

$$= \eta(\alpha^{-\frac{n-k}{2}}(I(\varepsilon_k,...,\varepsilon_2))) \tag{6}$$

where $I(\varepsilon_k,...,\varepsilon_2) \in \mathbb{P}_{k-2}$. Denote by (β_1,β_2) and (β_3,β_4) the intervals that correspond to $I(\varepsilon_k,...,\varepsilon_2,\varepsilon_1)$ and $I(\varepsilon_k,...,\varepsilon_2)$, respectively. It is obvious that $(\beta_1,\beta_2) \subset (\beta_3,\beta_4)$. Using the definition U_k^n and (5) and (6) we get:

$$U_k^n(\varepsilon_1,\varepsilon_2,...,\varepsilon_k) = \ln \frac{|\eta(\alpha^{-\frac{n-k}{2}}\beta_2) - \eta(\alpha^{-\frac{n-k}{2}}\beta_1)|}{|\eta(\alpha^{-\frac{n-k}{2}}\beta_4) - \eta(\alpha^{-\frac{n-k}{2}}\beta_3)|} =$$

$$= \ln \frac{\beta_2^3 - \beta_1^3}{\beta_4^3 - \beta_3^3} + O(\alpha^{-3\frac{n-k}{2}}) \equiv U_k(\varepsilon_1,\varepsilon_2,...,\varepsilon_k) + O(\alpha^{-3\frac{n-k}{2}}). \tag{7}$$

With the same manner one can show

$$\widetilde{U}_k^n(\varepsilon_1,\varepsilon_2,...,\varepsilon_k) = U_k(\varepsilon_1,\varepsilon_2,...,\varepsilon_k) + O(\alpha^{-3\frac{n-k}{2}}). \tag{8}$$

Next we prove the existence of the limit of $U_k(\varepsilon_1,\varepsilon_2,...,\varepsilon_k)$ when $k \to \infty$. For this we take a word $\underline{\varepsilon} = (\varepsilon_1,...,\varepsilon_n,...) \in \Omega_+$ and fix it. For a given k consider the expression $U_k := U_k(\varepsilon_1,\varepsilon_2,...,\varepsilon_k)$. Using (7) and (8), one can get

$$U_{k+m} = \ln \frac{|I(\underline{\gamma_{2n}}(\varepsilon_{k+m}),\varepsilon_{k+m},...,\varepsilon_1)|}{|I(\underline{\gamma_{2n}}(\varepsilon_{k+m}),\varepsilon_{k+m},...,\varepsilon_2)|} + O(\alpha^{-1,5n}), \tag{9}$$

Where $\underline{\gamma_{2n}}(\varepsilon_{k+m})$ be $2n-$ dimensional vector of the form

$$\underline{\gamma_{2n}}(\varepsilon_{k+m}) = \begin{cases} (2,0,a,0,...,a,0), \ if \ \varepsilon_{k+m} = a \\ (a,2,0,a,...,0,a), \ if \ \varepsilon_{k+m} = 0 \vee 1 \end{cases}$$

and the interval $\gamma_{2n}(\varepsilon_{k+m},\varepsilon_{k+m},...,\varepsilon_1)$ it is an element of dynamical partition $\mathbb{P}_{2n+k+m}$. It follows from the definition of symbolical dynamics, that the interval $\gamma_{2n}(\varepsilon_{k+m},\varepsilon_{k+m},...,\varepsilon_1)$ belongs to the trajectory of the interval $I(\underline{\gamma_{2n}}(\varepsilon_{k+m}),\varepsilon_{k+m},...,\varepsilon_1)$. More precisely

$$I(\underline{\gamma_{2n}}(\varepsilon_{k+m}),\varepsilon_{k+m},...,\varepsilon_{k+1},\varepsilon_k,...,\varepsilon_2,\varepsilon_1) = T_{cr}^i(I(\underline{\gamma_{2n+m}}(\varepsilon_k),\varepsilon_k,...,\varepsilon_2,\varepsilon_1)), \tag{10}$$

where $0 \le i < q_{2n+m}$, if $\varepsilon_k = a$ and $0 \le i < q_{2n+m+1}$ if $\varepsilon_k = 0 \vee 1$. Using (5)

$$|\ln\{\frac{|T_{cr}^i(I(\gamma_{n+m}(\varepsilon_k), \varepsilon_k, ..., \varepsilon_2, \varepsilon_1))|}{|T_{cr}^i(I(\underline{\gamma_{n+m}}(\varepsilon_k), \varepsilon_k, ..., \varepsilon_1))|} \cdot (\frac{|I(\gamma_{n+m}(\varepsilon_k), \varepsilon_k, ..., \varepsilon_2, \varepsilon_1)|}{|I(\underline{\gamma_{n+m}}(\varepsilon_k), \varepsilon_k, ..., , \varepsilon_1)|})^{-1}\}| \le Const \cdot \alpha^{-k}.$$

(11)

On the other hand, by (7) and (8) we have

$$\ln\frac{|I(\gamma_{n+m}(\varepsilon_k), \varepsilon_k, ..., \varepsilon_2, \varepsilon_1))|}{|I(\underline{\gamma_{n+m}}(\varepsilon_k), \varepsilon_k, ..., \varepsilon_1))|} = U_k(\varepsilon_1, \varepsilon_2, ..., \varepsilon_k) + O(\alpha^{-3\frac{n-k}{2}}).$$

(12)

Combining (9), (10) and (12) we get

$$|U_{k+m} - U_k| \le Const \cdot \alpha^{-k}, \tag{13}$$

where $Const$ depends only on T_{cr}. Here we see, that $\{U_{2k}(\varepsilon_1, \varepsilon_2, ..., \varepsilon_{2k})\}_{k=1}^{\infty}$ and $\{U_{2k-1}(\varepsilon_1, \varepsilon_2, ..., \varepsilon_{2k-1})\}_{k=1}^{\infty}-$ fundamental succession. So we have the existance this limits:

$$\lim_{k \to \infty} U_{2k-1}(\varepsilon_1, \varepsilon_2, ..., \varepsilon_{2k-1}) = U_{odd}(\underline{\varepsilon}),$$

$$\lim_{k \to \infty} U_{2k}(\varepsilon_1, \varepsilon_2, ..., \varepsilon_{2k}) = U_{even}(\underline{\varepsilon}).$$

where $\underline{\varepsilon} = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_{2k-1}, \varepsilon_{2k}, ...)$.

We designate the limit function by $U(\varepsilon_1, \varepsilon_2, ..., \varepsilon_n ...)$. Transiting into (13) to the limit of $m \to \infty$, we get:

$$|U(\varepsilon_1, \varepsilon_2, ..., \varepsilon_n ...) - U_k(\varepsilon_1, \varepsilon_2, ..., \varepsilon_k)| \le Const \cdot \alpha^{-k},$$

here

$$U = \begin{cases} U_{odd} \text{ if } k \text{ is odd number,} \\ U_{even} \text{ if } k \text{ is even number.} \end{cases}$$

Thus Theorem 1 is proved.

References

[1] Vul E. B., Sinai Ya. G., Khanin K. M. *Feigenbaum universality and the thermodynamic formalism.*, Russian Mathematical Surveys, 1984. Vol. 39. No. 3. P. 1–40.

[2] de Faria E., de Melo W. *Rigidity of critical circle mappings I, Journal of the European Mathematical Society*, 1999, vol. 1, issue 4, pp. 339–392.

[3] Sinai Ya. G. *Gibbs measures in ergodic theory*, Russian Mathematical Surveys,1972, vol. 27, no. 4,pp. 21–69.

[4] Dzhalilov A. A. *Thermodynamic formalism and singular invariant measures for critical circle maps*, Theoretical and Mathematical Physics,2003, vol. 134, no. 2, pp. 166–180.

[5] Ruelle D. *Thermodynamic formalism. The mathematical structures of classical equilibrium statistical mechanics*, Cambridge: Cambridge University Press, 2004.

[6] Dzhalilov A. A., Karimov J. J. *The thermodynamic formalism and exponents of singularity of invariant measure of circle maps with a single break*, Vestn. Udmurtsk. Univ. Mat. Mekh. Komp. Nauki, 2020, Volume 30, Issue 3, 343–366

[7] Dzhalilov A., Mayer D., Aliyev A. *The Thermodynamic Formalism and the Central Limit Theorem for Stochastic Perturbations of Circle Maps with a Break*,Rus. J. Nonlin. Dyn, 2022, Volume 18, Number 2, 253–287.

[8] Dzhalilov A.A., Abdukhakimov S. Kh. *A not on periodic points of Feigenbaum's type interval maps*, AIP Conference Proceedings, 2781, 020051 2023.

[9] Yoccoz J.-C. *Conjugaison differentiable des diffeomorphismes du cercle dont le nomber de rotation verifie une condition diophantienne*, Annales scientifiques de Ecole Normale Superieure. Serie 4. 1984. Vol. 17. No. 3. P. 333–359.

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