

VIRTUAL BRAIDS AND CLUSTER ALGEBRAS

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ABSTRACT. In 2015 Hikami and Inoue constructed the R -operator from cluster mutations which are related to the subdivision of knot complements into ideal hyperbolic tetrahedra. Also they established the representation of the braid group using this R -operator. In this note we construct a representation of the virtual braid group based on the Hikami – Inoue representation.

1. INTRODUCTION

Let us start with recalling braid groups and related groups. For $n \geq 2$, the *braid group* B_n is defined as a group with generators $\sigma_1, \dots, \sigma_{n-1}$ and the following defining relations [Ar47]:

$$(1) \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad i = 1, 2, \dots, n-2,$$

$$(2) \quad \sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i - j| \geq 2.$$

A geometric interpretation of B_n is well known, it is isomorphic to a group of geometric braids on n strings, and a mapping class group of an n -punctured disc [KaTu]. By adding the relations

$$(3) \quad \sigma_i^2 = 1, \quad i = 1, 2, \dots, n-1.$$

we get the *flat braid group* FB_n on n strings.

The *virtual braid group* VB_n on n sitings is the group with two families of generators, classical and virtual, denoted by $\sigma_1, \dots, \sigma_{n-1}$ and $\rho_1, \dots, \rho_{n-1}$, with the following defining relations: (1) and (2) for classical generators; (4), (5) and (6) for virtual generators,

$$(4) \quad \rho_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \rho_{i+1}, \quad i = 1, 2, \dots, n-2,$$

$$(5) \quad \rho_i \rho_j = \rho_j \rho_i, \quad |i - j| \geq 2,$$

$$(6) \quad \rho_i^2 = 1, \quad i = 1, 2, \dots, n-1,$$

and mixed relations (7) and (8) for classical and virtual generators both.

$$(7) \quad \sigma_i \rho_j = \rho_j \sigma_i, \quad |i - j| \geq 2,$$

$$(8) \quad \rho_i \rho_{i+1} \sigma_i = \sigma_{i+1} \rho_i \rho_{i+1}, \quad i = 1, 2, \dots, n-2.$$

It was observed in [GPV00] that relations (9) и (10)

$$(9) \quad \rho_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \rho_{i+1},$$

$$(10) \quad \rho_{i+1} \sigma_i \sigma_{i+1} = \sigma_i \sigma_{i+1} \rho_i$$

do not hold in VB_n , so these relations are called *forbidden* relations. By adding relation (3) to VB_n we get the *flat virtual braid group* FVB_n on n strings.

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Described above relation between braid groups and virtual braid groups admits to construct representations of VB_n by extending known representations of B_n by corresponding to ρ_i suitable involutions. In particular, Bardakov, Vesnin and Wiest [BVW12] constructed a representation of VB_n by extending Dynnikov representation [D02], and demonstrated that the representation from [BVW12] is faithful for $n = 2$ and distinguish virtual braids on three strings good enough. Gotin [Go17] constructed a representation of VB_n by extending a representation of B_n through rook algebras given by Bigelow, Ramos and Yi [BRY11].

In the present note we construct a representation of VB_n by extending a representation of B_n given by Hikami and Inoue in [HI15] in terms of a cluster algebra. It was demonstrated in [CYZ18] the the representation from [HI15] allows to compute volume of hyperbolic knot which is the closer of a braid.

2. CLUSTER MUTATIONS

Let V be a complex vector space. An automorphism R of the tensor product $V \otimes V$ is said to be an R -operator if it satisfies the following *Yang - Baxter equation*

$$(R \otimes \text{Id})(\text{Id} \otimes R)(R \otimes \text{Id}) = (\text{Id} \otimes R)(R \otimes \text{Id})(\text{Id} \otimes R),$$

where Id is the identity operator $\text{Id} : V \rightarrow V$.

Let us recall the constuction of R -operator from [HI15]. Denote by \mathbb{F}_N the field of rational functions over \mathbb{C} of N algebraically independent variables $\mathbf{x} = (x_1, \dots, x_N)$. A *cluster seed* is a pair (\mathbf{x}, \mathbf{B}) , where

- $\mathbf{x} = (x_1, \dots, x_N)$ is an ordered set of N algebraically independent variables,
- $\mathbf{B} = (b_{ij})$ is an antisymmetric $N \times N$ - matrix of integers.

For any $k = 1, \dots, N$ define a *mutation* μ_k of a seed (\mathbf{x}, \mathbf{B}) in direction k as follows

$$\mu_k(\mathbf{x}, \mathbf{B}) = (\tilde{\mathbf{x}}, \tilde{\mathbf{B}}),$$

where $\tilde{\mathbf{x}} = (\tilde{x}_1, \dots, \tilde{x}_N)$ is defined by the rule

$$(11) \quad \tilde{x}_i = \begin{cases} x_i, & \text{if } i \neq k, \\ \frac{1}{x_k} \left(\prod_{j: b_{jk} > 0} x_j^{b_{jk}} + \prod_{j: b_{jk} < 0} x_j^{-b_{jk}} \right), & \text{if } i = k, \end{cases}$$

and matrix $\tilde{\mathbf{B}} = (\tilde{b}_{ij})$ is calculating by the formula:

$$(12) \quad \tilde{b}_{ij} = \begin{cases} -b_{ij}, & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2}, & \text{otherwise.} \end{cases}$$

A pair $(\tilde{\mathbf{x}}, \tilde{\mathbf{B}})$ is a cluster seed again.

Using cluster variables \mathbf{x} we define cluster variables $\mathbf{y} = (y_1, \dots, y_N)$ by setting

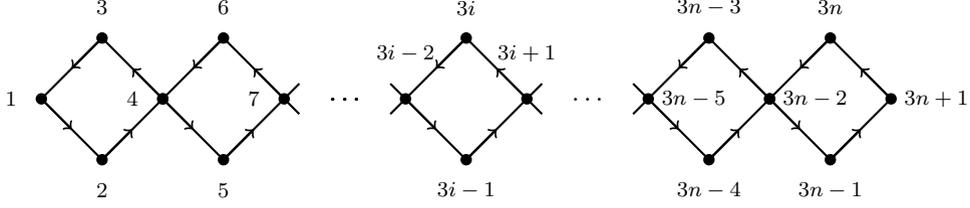
$$(13) \quad y_j = \prod_{k=1}^N x_k^{b_{kj}}.$$

Mutation μ_k induces a mutation of a pair (\mathbf{y}, \mathbf{B}) , $\mu_k(\mathbf{y}, \mathbf{B}) = (\tilde{\mathbf{y}}, \tilde{\mathbf{B}})$, where $\tilde{\mathbf{B}}$ is given by formula (12) and $\tilde{\mathbf{y}} = (\tilde{y}_1, \dots, \tilde{y}_N)$ is given by the following formulas:

$$(14) \quad \tilde{y}_i = \begin{cases} y_k^{-1}, & \text{if } i = k, \\ y_i(1 + y_k^{-1})^{-b_{ki}}, & \text{if } i \neq k \text{ and } b_{ki} \geq 0, \\ y_i(1 + y_k)^{-b_{ki}}, & \text{if } i \neq k \text{ and } b_{ki} \leq 0. \end{cases}$$

In [HI15] a matrix \mathbf{B} was taken equal to the adjacency matrix of a quiver (oriented graph) Γ presented in figure 1. Graph Γ has $N = 3n + 1$ vertices. Namely \mathbf{B} is $(3n + 1) \times (3n + 1)$ -matrix with enters determined by the quiver Γ :

$$b_{ij} = \begin{cases} 1, & \text{if there is an edge going from vertex } i \text{ to vertex } j, \\ -1, & \text{if there is an edge from vertex } j \text{ to vertex } i, \\ 0, & \text{if vertices } i \text{ and } j \text{ are not adjacent.} \end{cases}$$


 FIGURE 1. Quiver Γ with $3n + 1$ vertices

In particular, if $n = 2$ then matrix \mathbf{B} is of the form

$$(15) \quad \mathbf{B} = \begin{pmatrix} 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \end{pmatrix}.$$

Let us denote by $\Phi : \mathbb{F}_{3n+1} \rightarrow \mathbb{F}_{3n+1}$, $n \geq 2$, the operator defined in [HH15, Formula 2-13] as a composition of mutations. If $n = 2$ then we get $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ and Φ is of the form

$$\Phi(\mathbf{x}) = \begin{pmatrix} \Phi_1(\mathbf{x}) \\ \Phi_2(\mathbf{x}) \\ \Phi_3(\mathbf{x}) \\ \Phi_4(\mathbf{x}) \\ \Phi_5(\mathbf{x}) \\ \Phi_6(\mathbf{x}) \\ \Phi_7(\mathbf{x}) \end{pmatrix}^T = \begin{pmatrix} x_1 \\ x_5 \\ \frac{x_1 x_3 x_5 + x_3 x_4 x_5 + x_1 x_2 x_6}{x_2 x_4} \\ \frac{x_1 x_3 x_4 x_5 + x_3 x_4^2 x_5 + x_1 x_3 x_5 x_7 + x_3 x_4 x_5 x_7 + x_1 x_2 x_6 x_7}{x_2 x_4 x_6} \\ \frac{x_1 x_3 x_5 + x_3 x_4 x_5 + x_1 x_2 x_6}{x_4 x_6} \\ x_3 \\ x_7 \end{pmatrix}^T.$$

We denote by $\Psi : \mathbb{F}_{3n+1} \rightarrow \mathbb{F}_{3n+1}$, $n \geq 2$, the operator inverse to Φ . If $n = 2$ then

$$\Psi(\mathbf{x}) = \begin{pmatrix} \Psi_1(\mathbf{x}) \\ \Psi_2(\mathbf{x}) \\ \Psi_3(\mathbf{x}) \\ \Psi_4(\mathbf{x}) \\ \Psi_5(\mathbf{x}) \\ \Psi_6(\mathbf{x}) \\ \Psi_7(\mathbf{x}) \end{pmatrix}^T = \begin{pmatrix} x_1 \\ \frac{x_1 x_3 x_5 + x_1 x_2 x_6 + x_2 x_4 x_6}{x_3 x_4} \\ x_6 \\ \frac{x_1 x_2 x_4 x_6 + x_2 x_4^2 x_6 + x_1 x_3 x_5 x_7 + x_1 x_2 x_6 x_7 + x_2 x_4 x_6 x_7}{x_3 x_4 x_5} \\ x_2 \\ \frac{x_2 x_4 x_6 + x_3 x_5 x_7 + x_2 x_6 x_7}{x_4 x_5} \\ x_7 \end{pmatrix}^T.$$

Following [HH15, Formula 2-13] we go from \mathbf{x} -variables to \mathbf{y} -variables. If $n = 2$ then $\mathbf{y} = (y_1, y_2, y_3, y_4, y_5, y_6, y_7)$ and R-operator Φ will take a form φ , where

$$(16) \quad \varphi(\mathbf{y}) = \begin{pmatrix} \varphi_1(\mathbf{y}) \\ \varphi_2(\mathbf{y}) \\ \varphi_3(\mathbf{y}) \\ \varphi_4(\mathbf{y}) \\ \varphi_5(\mathbf{y}) \\ \varphi_6(\mathbf{y}) \\ \varphi_7(\mathbf{y}) \end{pmatrix}^T = \begin{pmatrix} \frac{y_1(1+y_2+y_2y_4)}{y_2y_4y_5y_6} \\ \frac{1+y_2+y_6+y_2y_6+y_2y_4y_6}{1+y_2+y_4+y_2y_6+y_2y_4y_6} \\ \frac{y_2y_4}{y_4} \\ \frac{(1+y_2+y_2y_4)(1+y_6+y_4y_6)}{1+y_2+y_6+y_2y_6+y_2y_4y_6} \\ \frac{y_4y_6}{y_2y_3y_4y_6} \\ \frac{1+y_2+y_6+y_2y_6+y_2y_4y_6}{(1+y_6+y_4y_6)y_7} \end{pmatrix}^T,$$

as well as Ψ will takes a form ψ , where

$$(17) \quad \psi(\mathbf{y}) = \begin{pmatrix} \psi_1(\mathbf{y}) \\ \psi_2(\mathbf{y}) \\ \psi_3(\mathbf{y}) \\ \psi_4(\mathbf{y}) \\ \psi_5(\mathbf{y}) \\ \psi_6(\mathbf{y}) \\ \psi_7(\mathbf{y}) \end{pmatrix}^T = \begin{pmatrix} \frac{y_1y_3y_4}{1+y_4+y_3y_4} \\ \frac{y_5}{1+y_4+y_3y_4+y_4y_5+y_3y_4y_5} \\ \frac{(1+y_4+y_3y_4+y_4y_5+y_3y_4y_5)y_6}{(1+y_4+y_3y_4)(1+y_4+y_4y_5)} \\ \frac{y_3y_4y_5}{y_2(1+y_4+y_3y_4+y_4y_5+y_3y_4y_5)} \\ \frac{y_3}{1+y_4+y_3y_4+y_4y_5+y_3y_4y_5} \\ \frac{y_4y_5y_7}{1+y_4+y_4y_5} \end{pmatrix}^T.$$

The following property easily follows from the above formulae.

Lemma 2.1. *By setting $y_1 = y_4 = y_7 = -1$ in formulae (16) and (17) we get*

$$\varphi_1 = \varphi_4 = \varphi_7 = -1 \quad \text{and} \quad \psi_1 = \psi_4 = \psi_7 = -1.$$

3. VIRTUAL BRAID GROUPS

For a vector $\mathbf{z} = (z_1, z_2, z_3, z_4) = (y_2, y_3, y_5, y_6)$ of length four we define two operators

$$(18) \quad S \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}^T = \begin{pmatrix} -\frac{z_1z_3z_4}{1+z_1+z_4} \\ -\frac{z_1}{1+z_1+z_4} \\ -\frac{1+z_1+z_4}{z_4} \\ -\frac{z_1z_2z_4}{1+z_1+z_4} \end{pmatrix}^T, \quad S^{-1} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}^T = \begin{pmatrix} -\frac{z_3}{z_2+z_3+z_2z_3} \\ -(z_2+z_3+z_2z_3)z_4 \\ -z_1(z_2+z_3+z_2z_3) \\ -\frac{z_2}{z_2+z_3+z_2z_3} \end{pmatrix}^T$$

and an involution

$$(19) \quad T(z_1, z_2, z_3, z_4) = (z_3, z_4, z_1, z_2).$$

Now for $n \geq 2$ we define operators $S_i^{\pm 1}$ and T_i , $i = 1, \dots, n-1$, which act on vector $\mathbf{z} = (z_1, z_2, \dots, z_{2n})$ of length $2n$ by the following rule. Operators $S_i^{\pm 1}$ and T_i act on 4-tuple $(z_{2i-1}, z_{2i}, z_{2i+1}, z_{2i+2})$ in the same way as operators $S^{\pm 1}$ and T act on 4-tuple (z_1, z_2, z_3, z_4) , and do not change other components of \mathbf{z} :

$$S_i^{\pm 1} = I^{2i-2} \otimes S^{\pm 1} \otimes I^{2n-2i-2}, \quad T_i = I^{2i-2} \otimes T \otimes I^{2n-2i-2}.$$

For $n \geq 2$ we denote by Θ_n the group generated by S_i, T_i , $i = 1, \dots, n-1$, with composition as a group operation. Define a map $F: VB_n \rightarrow \Theta_n$ by setting

$$(20) \quad F(\sigma_i) = S_i, \quad F(\rho_i) = T_i, \quad i = 1, \dots, n-1.$$

Lemma 3.1. *Let w be a word in VB_n . Then for a vector of algebraically independent variables $\mathbf{z} = (z_1, z_2, \dots, z_{2n})$ in the image of $F(w)(\mathbf{z})$ no coordinate turns into zero or infinity.*

Proof. Consider $2n$ -tuple $\mathbf{z}' = (-1, -1, \dots, -1)$. It is easy to see from (18) and (19) that $S_i^{\pm 1}(\mathbf{z}') = \mathbf{z}'$ and $T_i(\mathbf{z}') = \mathbf{z}'$ for each i . Hence $F(w)(\mathbf{z}') = \mathbf{z}' = (-1, -1, \dots, -1)$. Hence, in the image of $F(w)(\mathbf{z})$ no coordinate can turn into zero or infinity, because for $z_i = -1$, $i = 1, \dots, 2n$, all coordinates of the image will be equal to -1 . \square

Theorem 3.1. *Map $F : VB_n \rightarrow \Theta_n$, $n \geq 2$, defined by (20) is a homomorphism.*

Proof. Let us check that operators S_i and T_i , $i = 1, \dots, n-1$, act on \mathbf{z} in such a way that the following identities hold.

- (1) $S_i S_{i+1} S_i = S_{i+1} S_i S_{i+1}$, where $i = 1, 2, \dots, n-2$.
- (2) $S_i S_j = S_j S_i$, where $|i - j| \geq 2$.
- (3) $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$, where $i = 1, 2, \dots, n-2$.
- (4) $T_i T_j = T_j T_i$, where $|i - j| \geq 2$.
- (5) $T_i^2 = 1$, where $i = 1, 2, \dots, n-1$.
- (6) $T_i T_{i+1} S_i = S_{i+1} T_i T_{i+1}$, where $i = 1, 2, \dots, n-2$.

Obviously, it is enough to consider the case $i = 1$. Identities (1) and (2) are particular cases of [HI15, Theorem 2.3]. Nevertheless, we present a straightforward proof of (1) for a reader convenience. Let $\mathbf{z} = (z_1, z_2, z_3, z_4, z_5, z_6)$. Consider the left-side part of (1)

$$\begin{aligned} S_1 S_2 S_1(\mathbf{z}) &= S_1 S_2 S_1(z_1, z_2, z_3, z_4, z_5, z_6) \\ &= S_1 S_2 \left(-\frac{z_1 z_3 z_4}{1 + z_1 + z_4}, -\frac{1 + z_1 + z_4}{z_1}, -\frac{1 + z_1 + z_4}{z_4}, -\frac{z_1 z_2 z_4}{1 + z_1 + z_4}, z_5, z_6 \right) \\ &= S_1 \left(-\frac{z_1 z_3 z_4}{1 + z_1 + z_4}, -\frac{1 + z_1 + z_4}{z_1}, -\frac{(1 + z_1 + z_4) z_5 z_6}{1 + z_1 - z_4 z_6}, -\frac{1 + z_1 - z_4 z_6}{1 + z_1 + z_4}, \frac{1 + z_1 - z_4 z_6}{z_4 z_6}, \frac{z_1 z_2 z_4 z_6}{1 + z_1 - z_4 z_6} \right) \\ &= \left(\frac{z_1 z_3 z_5 z_6}{1 - z_1 z_3 + z_6}, \frac{1 - z_1 z_3 + z_6}{z_1 z_3}, \frac{z_4(1 - z_1 z_3 + z_6)}{1 + z_1 - z_4 z_6}, \frac{z_3(1 + z_1 - z_4 z_6)}{1 - z_1 z_3 + z_6}, \frac{1 + z_1 - z_4 z_6}{z_4 z_6}, \frac{z_1 z_2 z_4 z_6}{1 + z_1 - z_4 z_6} \right). \end{aligned}$$

The right-side part of (1) is equal

$$\begin{aligned} S_2 S_1 S_2(\mathbf{z}) &= S_2 S_1 S_2(z_1, z_2, z_3, z_4, z_5, z_6) \\ &= S_2 S_1 \left(z_1, z_2, -\frac{z_3 z_5 z_6}{1 + z_3 + z_6}, -\frac{1 + z_3 + z_6}{z_3}, -\frac{1 + z_3 + z_6}{z_6}, -\frac{z_3 z_4 z_6}{1 + z_3 + z_6} \right) \\ &= S_2 \left(\frac{z_1 z_2 z_5 z_6}{1 - z_1 z_2 + z_6}, \frac{1 - z_1 z_2 + z_6}{z_1 z_2}, -\frac{1 - z_1 z_2 + z_6}{1 + z_3 + z_6}, \frac{z_1 z_2(1 + z_3 + z_6)}{-1 + z_1 z_3 - z_6}, -\frac{1 + z_3 + z_6}{z_6}, -\frac{z_3 z_4 z_6}{1 + z_3 + z_6} \right) \\ &= \left(\frac{z_1 z_3 z_5 z_6}{1 - z_1 z_3 + z_6}, \frac{1 - z_1 z_3 + z_6}{z_1 z_3}, \frac{z_4(1 - z_1 z_3 + z_6)}{1 + z_1 - z_4 z_6}, \frac{z_3(1 + z_1 - z_4 z_6)}{1 - z_1 z_3 + z_6}, \frac{1 + z_1 - z_4 z_6}{z_4 z_6}, \frac{z_1 z_2 z_4 z_6}{1 + z_1 - z_4 z_6} \right). \end{aligned}$$

Thus, the identity (1) holds.

Let us demonstrate that the identity (6) holds, Indeed from the one hand,

$$\begin{aligned} T_1 T_2 S_1(\mathbf{z}) &= T_1 T_2 S_1(z_1, z_2, z_3, z_4, z_5, z_6) = T_1 T_2(S(z_1), S(z_2), S(z_3), S(z_4), z_5, z_6) \\ &= T_1(S(z_1), S(z_2), z_5, z_7, S(z_3), S(z_4)) = (z_5, z_6, S(z_1), S(z_2), S(z_3), S(z_4)), \end{aligned}$$

and from the other hand,

$$\begin{aligned} S_2 T_1 T_2(\mathbf{z}) &= S_2 T_1 T_2(z_1, z_2, z_3, z_4, z_5, z_6) = S_2 T_1(z_1, z_2, z_5, z_6, z_3, z_4) \\ &= S_2(z_5, z_6, z_1, z_2, z_3, z_4) = (z_5, z_6, S(z_1), S(z_2), S(z_3), S(z_4)). \end{aligned}$$

Remaining identities (2), (3), (4) and (5) hold obviously. \square

Theorem 3.1 allows to distinguish elements of the virtual braid group VB_n by computing their images which are vectors of lengths $2n$.

Example 3.1. Let $w_1 = \sigma_1 \rho_1 \sigma_1 \in VB_2$. By formulae (18) and (19) the operator $F(w_1)$ acts on $(1, 2, 2, 1)$ in the following way:

$$F(w_1)(1, 2, 2, 1) = \left(-\frac{6}{5}, -\frac{5}{3}, -\frac{5}{3}, -\frac{6}{5} \right) \neq (1, 2, 2, 1).$$

Therefore, the homomorphism F distinguishes w_1 from a trivial braid.

Example 3.2. It is known [Ma05] that a generalized Burau representation does not distinguish a braid $w_2 = (\sigma_1^2 \rho_1 \sigma_1^{-1} \rho_1 \sigma_1^{-1} \rho_1)^2 \in VB_2$ from a trivial braid. By acting $F(w_2)$ on the vector $(1, 2, 2, 1)$ we get

$$F(w_2)(1, 2, 2, 1) = \left(-\frac{44}{19}, -\frac{19}{22}, -\frac{19}{22}, -\frac{44}{19} \right) \neq (1, 2, 2, 1).$$

Therefore, the homomorphism F distinguishes w_2 from a trivial braid.

Example 3.3. Consider

$$w_3 = \sigma_1 \rho_2 \sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2 \sigma_1^{-1} \rho_1 \sigma_2 \rho_1 \sigma_1 \rho_2 \sigma_1^{-1} \rho_2 \sigma_2^{-1} \sigma_1^{-1} \sigma_2 \sigma_1^{-1} \rho_2 \sigma_1^{-1} \in VB_3.$$

It is known that a representation from [BVW12] does not distinguish w_3 from a trivial braid. By acting $F(w_3)$ on $(1, 2, 2, 1, 1, 2)$ we get

$$F(w_3)(1, 2, 2, 1, 1, 2) = \left(\frac{2488285076682521504}{1290542656863845663}, \frac{1290542656863845663}{1244142538341260752}, \frac{1290542656863845663}{1290542656863845663}, \frac{1127136134852291178}{1268603408}, \frac{574648281}{574648281}, \frac{2537206816}{574648281} \right) \neq (1, 2, 2, 1, 1, 2).$$

Therefore, the homomorphism F distinguishes w_3 from a trivial braid.

4. FORBIDDEN RELATIONS

In this section we demonstrate that the forbidden relations do not hold in the group Θ_n .

Lemma 4.1. Let $\mathbf{z} = (z_1, z_2, \dots, z_{2n-1}, z_{2n})$ and $S_i, S_{i+1}, T_i, T_{i+1} \in \Theta_n$.

(a) *The forbidden relation*

$$(21) \quad T_i S_{i+1} S_i(\mathbf{z}) = S_{i+1} S_i T_{i+1}(\mathbf{z})$$

does not hold if and only if the vector \mathbf{z} is such that $z_j \neq -1$ for $j = 2i - 1, 2i + 2$ u $2i + 4$.

(b) *The forbidden relation*

$$(22) \quad T_{i+1} S_i S_{i+1}(\mathbf{z}) = S_i S_{i+1} T_{i+1}(\mathbf{z})$$

does not hold if and only if the vector \mathbf{z} is such that $z_j \neq -1$ for $j = 2i - 1, 2i + 1$ u $2i + 4$.

Proof. (a) Without loss of generality, we can assume $i = 1$. The left-hand side of (21) is

$$\begin{aligned} & T_1 S_2 S_1(\mathbf{z}) = T_1 S_2 S_1(z_1, z_2, z_3, z_4, z_5, z_6) \\ &= T_1 \left(-\frac{z_1 z_3 z_4}{1 + z_1 + z_4}, -\frac{1 + z_1 + z_4}{z_1}, -\frac{(1 + z_1 + z_4) z_5 z_6}{1 + z_1 - z_4 z_6}, -\frac{1 + z_1 - z_4 z_6}{1 + z_1 + z_4}, \frac{1 + z_1 - z_4 z_6}{z_4 z_6}, \frac{z_1 z_2 z_3 z_6}{1 + z_1 - z_4 z_6} \right) \\ &= \left(-\frac{(1 + z_1 + z_4) z_5 z_6}{1 + z_1 - z_4 z_6}, -\frac{1 + z_1 - z_4 z_6}{1 + z_1 + z_4}, -\frac{z_1 z_3 z_4}{1 + z_1 + z_4}, -\frac{1 + z_1 + z_4}{z_1}, \frac{1 + z_1 - z_4 z_6}{z_4 z_6}, \frac{z_1 z_2 z_4 z_6}{1 + z_1 - z_4 z_6} \right) \end{aligned}$$

and the right-hand side is equal

$$\begin{aligned} & S_2 S_1 T_2(\mathbf{z}) = S_2 S_1 T_2(z_1, z_2, z_3, z_4, z_5, z_6) = S_2 S_1(z_1, z_2, z_5, z_6, z_3, z_4) \\ &= \left(-\frac{z_1 z_5 z_6}{1 + z_1 + z_6}, -\frac{1 + z_1 + z_6}{z_1}, -\frac{(1 + z_1 + z_6) z_3 z_4}{1 + z_1 - z_4 z_6}, -\frac{1 + z_1 - z_6 z_4}{1 + z_1 + z_6}, \frac{1 + z_1 - z_6 z_4}{z_6 z_4}, \frac{z_1 z_2 z_6 z_4}{1 + z_1 - z_6 z_4} \right). \end{aligned}$$

Here we used formulae for $S_2S_1(\mathbf{z})$ from theorem 3.1. The fifth and sixth coordinates are equal. Comparison of the third and fourth coordinates leads to the equation

$$(23) \quad z_1(1 + z_1 - z_4z_6) = (1 + z_1 + z_4)(1 + z_1 + z_6),$$

which is equivalent to

$$(24) \quad (z_1 + 1)(z_4 + 1)(z_6 + 1) = 0.$$

Therefore, to obtain the relation (a) the necessary condition is that at least one of number z_1 , z_4 or z_6 is equal to -1 . But if at least one of numbers z_1 , z_4 or z_6 is equal to -1 , then the left and right parts of (a) coincide. Indeed, if $z_1 = -1$ then

$$T_1S_2S_1(-1, z_2, z_3, z_4, z_5, z_6) = S_2S_1T_2(-1, z_2, z_3, z_4, z_5, z_6) = (z_5, z_6, z_3, z_4, -1, z_2).$$

Analogously, if $z_4 = -1$ then

$$\begin{aligned} & T_1S_2S_1(z_1, z_2, z_3, -1, z_5, z_6) = S_2S_1T_2(z_1, z_2, z_3, -1, z_5, z_6) \\ & = \left(-\frac{z_1z_5z_6}{1 + z_1 + z_6}, -\frac{1 + z_1 + z_6}{z_1}, z_3, -1, -\frac{1 + z_2 + z_6}{z_6}, -\frac{z_1z_2z_6}{1 + z_1 + z_6} \right), \end{aligned}$$

and if $z_6 = -1$ then

$$\begin{aligned} & T_1S_2S_1(z_1, z_2, z_3, z_4, z_5, -1) = S_2S_1T_1(z_1, z_2, z_3, z_4, z_5, -1) \\ & = \left(z_5, -1, -\frac{z_1z_3z_4}{1 + z_1 + z_4}, -\frac{1 + z_2 + z_4}{z_1}, -\frac{1 + z_1 + z_4}{z_4}, -\frac{z_1z_2z_4}{1 + z_1 + z_4} \right). \end{aligned}$$

Therefore, the above necessary condition is also sufficient.

(b) The left-hand part of the relation (22) is equal to

$$\begin{aligned} & T_2S_1S_2(\mathbf{z}) = T_2S_1S_2(z_1, z_2, z_3, z_4, z_5, z_6) \\ & = T_2 \left(\frac{z_1z_3z_5z_6}{1 - z_1z_3 + z_6}, \frac{1 - z_1z_3 + z_6}{z_1z_3}, -\frac{1 - z_1z_3 + z_6}{1 + z_3 + z_6}, -\frac{z_1z_2(1 + z_3 + z_6)}{1 - z_1z_3 + z_6}, -\frac{1 + z_3 + z_6}{z_6}, -\frac{z_3z_4z_6}{1 + z_3 + z_6} \right) \\ & = \left(\frac{z_1z_3z_5z_6}{1 - z_1z_3 + z_6}, \frac{1 - z_1z_3 + z_6}{z_1z_3}, -\frac{1 + z_3 + z_6}{z_6}, -\frac{z_3z_4z_6}{1 + z_3 + z_6}, -\frac{1 - z_1z_3 + z_6}{1 + z_3 + z_6}, -\frac{z_1z_2(1 + z_3 + z_6)}{1 - z_1z_3 + z_6} \right) \end{aligned}$$

and the right-hand part is equal to

$$\begin{aligned} & S_1S_2T_1(\mathbf{z}) = S_1S_2T_1(z_1, z_2, z_3, z_4, z_5, z_6) = S_2S_1(z_3, z_4, z_1, z_2, z_5, z_6) \\ & = \left(\frac{z_3z_1z_5z_6}{1 - z_3z_1 + z_6}, \frac{1 - z_3z_1 + z_6}{z_3z_1}, -\frac{1 - z_1z_3 + z_6}{1 + z_1 + z_6}, -\frac{z_3z_4(1 + z_1 + z_6)}{1 - z_3z_1 + z_6}, -\frac{1 + z_1 + z_6}{z_6}, -\frac{z_1z_2z_6}{1 + z_1 + z_6} \right). \end{aligned}$$

Here we used formulae for $S_1S_2(\mathbf{z})$ from theorem 3.1. By comparing the third and fourth coordinates we get the equation

$$(25) \quad (1 + z_3 + z_6)(1 + z_1 + z_6) = z_6(1 - z_1z_3 + z_6),$$

which is equivalent to

$$(26) \quad (z_1 + 1)(z_3 + 1)(z_6 + 1) = 0.$$

Therefore, to obtain (b) the necessary condition is that at least one of z_1 , z_3 or z_6 is equal to -1 . But if at least one of this numbers is equal to -1 then left and right parts of (22) coincide. Indeed,

if $z_1 = -1$ then

$$\begin{aligned} T_2 S_1 S_2(-1, z_2, z_3, z_4, z_5, z_6) &= S_2 S_1 T_2(-1, z_2, z_3, z_4, z_5, z_6) \\ &= \left(-\frac{z_3 z_5 z_6}{1 + z_3 + z_6}, -\frac{1 + z_3 + z_6}{z_3}, -\frac{1 + z_3 + z_6}{z_6}, -\frac{z_3 z_4 z_6}{1 + z_3 + z_6}, -1, z_2 \right). \end{aligned}$$

Analogously, if $z_3 = -1$ then

$$\begin{aligned} T_2 S_1 S_2(z_1, z_2, -1, z_4, z_5, z_6) &= S_2 S_1 T_2(z_1, z_2, -1, z_4, z_5, z_6) \\ &= \left(-\frac{z_1 z_5 z_6}{1 + z_3 + z_6}, -\frac{1 + z_1 + z_6}{z_1}, -1, z_4, -\frac{1 + z_1 + z_6}{z_6}, -\frac{z_1 z_2 z_6}{1 + z_1 + z_6} \right) \end{aligned}$$

and if $z_6 = -1$ then

$$T_2 S_1 S_2(z_1, z_2, z_3, z_4, z_5, -1) = S_2 S_1 T_2(z_1, z_2, z_3, z_4, z_5, -1) = (z_5, -1, z_3, z_4, z_1, z_2).$$

Therefore, the above necessary condition is also sufficient. \square

The obvious consequence of this lemma is the following theorem, which concludes the section.

Theorem 4.1. *Let $S_i, S_{i+1}, T_i, T_{i+1} \in \Theta_n$.*

- (a) *Operators $T_i S_{i+1} S_i$ and $S_{i+1} S_i T_{i+1}$ are different.*
- (b) *Operators $T_{i+1} S_i S_{i+1}$ and $S_i S_{i+1} T_{i+1}$ are different.*

So the forbidden relations do not hold in Θ_n .

5. FLAT BRAID GROUPS

Let us consider vector \mathbf{z} of the form $(z_1, \frac{1}{z_1}, z_3, \frac{1}{z_3})$. Notice that

$$S(\mathbf{z}) = (\zeta_1, \frac{1}{\zeta_1}, \zeta_3, \frac{1}{\zeta_3}),$$

where

$$\zeta_1 = -\frac{z_1 z_3}{1 + z_3 + z_1 z_3}, \quad \zeta_3 = -(1 + z_3 + z_1 z_3).$$

Also notice that $S^2(\mathbf{z}) = \mathbf{z}$. These observations inspire to obtain the representation for flat braids.

Consider a vector of algebraically independent variables $\mathbf{t} = (t_1, t_2, \dots, t_n)$. Let's define the operators R_i , $i = 1, \dots, n-1$, according to the rule:

$$R_i : \begin{cases} t_i \rightarrow -\frac{t_i t_{i+1}}{1 + t_{i+1} + t_i t_{i+1}}, \\ t_{i+1} \rightarrow -(1 + t_{i+1} + t_i t_{i+1}). \end{cases}$$

Let F_{FB} be a map that match operators R_i with generators σ_i , $i = 1, \dots, n-1$, of the flat braid group T_n :

$$F_{FB}(\sigma_i) = R_i.$$

For $n \geq 2$, denote by Ω_n the group generated by operators R_i , $i = 1, \dots, n-1$, with composition as a group operation.

Lemma 5.1. *Let w be a word in FB_n . Then for a vector of algebraically independent variables $\mathbf{t} = (t_1, t_2, \dots, t_n)$ in the image of $F_{FB}(w)(\mathbf{t})$ no coordinate turns into zero or infinity.*

Proof. Consider n -tuple $\mathbf{t}' = (-1, -1, \dots, -1)$. It is easy to see that $R_i^{\pm 1}(\mathbf{t}') = \mathbf{t}'$ for each i . Hence $F_{FB}(w)(\mathbf{t}') = \mathbf{t}'$. Hence, in the image of $F_{FB}(w)(\mathbf{t})$ no coordinate can turn into zero or infinity, because for $t_i = -1$, $i = 1, \dots, n$, all coordinates of the image will be equal to -1 . \square

Theorem 5.1. *Correspondence $F_{FB} : FB_n \rightarrow \Omega_n$ is a homomorphism for any $n \geq 2$.*

Proof. Let us check that for the operators R_i , $i = 1, \dots, n-1$, act on \mathbf{t} in such a way that the following identities hold.

- (1) $R_i^2 = 1$, where $i = 1, 2, \dots, n-2$.
- (2) $R_i R_{i+1} R_i = R_{i+1} R_i R_{i+1}$, where $i = 1, 2, \dots, n-2$.
- (3) $R_i R_j = R_j R_i$, where $|i-j| \geq 2$.

We present a proof for the case of $i = 1$, which also works for an arbitrary $i = 1, \dots, n-1$. Consider $\mathbf{t} = (t_1, t_2, t_3)$. The relation (1) is easily verified. Indeed:

$$R_1^2(\mathbf{t}) = R_1^2(t_1, t_2, t_3) = R_1\left(-\frac{t_1 t_2}{1 + t_1 + t_1 t_2}, -(1 + t_2 + t_1 t_2), t_3\right) = ((t_1, t_2, t_3))$$

Let us now prove identity (2). Its left-hand side is

$$\begin{aligned} R_1 R_2 R_1(\mathbf{t}) &= R_1 R_2 R_1(t_1, t_2, t_3) = R_1 R_2 \left(-\frac{t_1 t_2}{1 + t_1 + t_1 t_2}, -(1 + t_2 + t_1 t_2), t_3 \right) \\ &= R_1 \left(-\frac{t_1 t_2}{1 + t_1 + t_1 t_2}, -\frac{1 + t_2 t_3 + t_1 t_2 t_3}{-1 + t_2 t_3 + t_1 t_2 t_3}, -1 + t_2 t_3 + t_1 t_2 t_3 \right) \\ &= \left(\frac{t_1 t_2 t_3}{1 + t_3 - t_1 t_2 t_3}, \frac{1 + t_3 - t_1 t_2 t_3}{-1 + t_2 t_3 + t_1 t_2 t_3}, -1 + t_2 t_3 + t_1 t_2 t_3 \right). \end{aligned}$$

The right-hand side is

$$\begin{aligned} R_2 R_1 R_2(\mathbf{t}) &= R_2 R_1 R_2(t_1, t_2, t_3) = R_2 R_1 \left(-\frac{t_1, t_2 t_3}{1 + t_2 + t_2 t_3}, -(1 + t_3 + t_2 t_3) \right) \\ &= R_2 \left(\frac{t_1 t_2 t_3}{1 + t_3 - t_1 t_2 t_3}, -\frac{1 + t_3 - t_1 t_2 t_3}{1 + t_3 + t_2 t_3}, -(1 + t_3 + t_2 t_3) \right) \\ &= \left(\frac{t_1 t_2 t_3}{1 + t_3 - t_1 t_2 t_3}, \frac{1 + t_3 - t_1 t_2 t_3}{-1 + t_2 t_3 + t_1 t_2 t_3}, -1 + t_2 t_3 + t_1 t_2 t_3 \right). \end{aligned}$$

Thus, identity (2) holds. The fulfillment of identity (3) is obvious. \square

Example 5.1. Consider $w_4 = (\sigma_1 \sigma_2)^2 \in FB_3$. The operator $F_{FB}(w_4)$ acts on $(1, 2, 2)$ in the following way;

$$F_{FB}(w_4)(1, 2, 2) = \left(-\frac{2}{5}, -\frac{10}{7}, 7 \right) \neq (1, 2, 2).$$

Therefore, the homomorphism F_{FB} distinguishes w_4 from a trivial braid.

6. FLAT VIRTUAL BRAID GROUPS

Consider a vector of algebraically independent variables $\mathbf{t} = (t_1, t_2, \dots, t_n)$. In addition to the operators R_i , $i = 1, \dots, n-1$ introduced in the previous section, we define the operators V_i , $i = 1, \dots, n-1$, according to the rule:

$$V_i : \begin{cases} t_i \rightarrow t_{i+1}, \\ t_{i+1} \rightarrow t_i. \end{cases}$$

Let F_{FVB} be a map that match operators R_i and V_i with generators σ_i and ρ_i , $i = 1, \dots, n-1$, of the virtual flat braid group T_n :

$$F_{FVB}(\sigma_i) = R_i, \quad F_{FVB}(\rho_i) = V_i.$$

For $n \geq 2$, denote by Δ_n the group generated by operators R_i, V_i , $i = 1, \dots, n-1$, with composition as a group operation.

Lemma 6.1. *Let w be a word in FVB_n . Then for a vector of algebraically independent variables $\mathbf{t} = (t_1, t_2, \dots, t_n)$ in the image of $F_{FVB}(w)(\mathbf{t})$ no coordinate turns into zero or infinity.*

Proof. Consider n -tuple $\mathbf{t}' = (-1, -1, \dots, -1)$. It is easy to see that $R_i^{\pm 1}(\mathbf{t}') = \mathbf{t}'$ and $V_i^{\pm 1}(\mathbf{t}') = \mathbf{t}'$ for each i . Hence $F_{FVB}(w)(\mathbf{t}') = \mathbf{t}'$. Hence, in the image of $F_{FVB}(w)(\mathbf{t})$ no coordinate can turn into zero or infinity, because for $t_i = -1$, $i = 1, \dots, n$, all coordinates of the image will be equal to -1 . \square

Theorem 6.1. *Correspondence $F_{FVB} : FVB_n \rightarrow \Delta_n$ is a homomorphism for any $n \geq 2$.*

Proof. Let us check that the operators R_i and V_i , $i = 1, \dots, n-1$, act on \mathbf{t} in such a way that the following identities hold.

- (1) $R_i^2 = 1$, where $i = 1, 2, \dots, n-2$.
- (2) $R_i R_{i+1} R_i = R_{i+1} R_i R_{i+1}$, where $i = 1, 2, \dots, n-2$.
- (3) $R_i R_j = R_j R_i$, where $|i-j| \geq 2$.
- (4) $V_i V_{i+1} V_i = V_{i+1} V_i V_{i+1}$, where $i = 1, 2, \dots, n-2$.
- (5) $V_i V_j = V_j V_i$, where $|i-j| \geq 2$.
- (6) $V_i^2 = 1$, where $i = 1, 2, \dots, n-1$.
- (7) $V_i V_{i+1} R_i = R_{i+1} V_i V_{i+1}$, where $i = 1, 2, \dots, n-2$.

Identities (1), (2) and (3) are proved in theorem 5.1. The fulfillment of identities (5) and (6) is obvious. It remains to prove the relations (4) and (7). We present a proof for the case of $i = 1$, which also works for an arbitrary $i = 1, \dots, n-1$. Consider $\mathbf{t} = (t_1, t_2, t_3)$. Let us now prove the identity (4). Its left-hand side is

$$V_1 V_2 V_1(\mathbf{t}) = V_1 V_2 V_1(t_1, t_2, t_3) = V_1 V_2(t_2, t_1, t_3) = V_1(t_2, t_3, t_1) = (t_3, t_2, t_1).$$

The right-hand side is

$$V_2 V_1 V_2(\mathbf{t}) = V_2 V_1 V_2(t_1, t_2, t_3) = V_2 V_1(t_1, t_3, t_2) = V_2(t_3, t_1, t_2) = (t_3, t_2, t_1).$$

So the identity (4) holds. Let us now prove the identity (7). Its left-hand side is

$$\begin{aligned} V_1 V_2 R_1(\mathbf{t}) &= V_1 V_2 R_1(t_1, t_2, t_3) = V_1 V_2 \left(-\frac{t_1 t_2}{1 + t_1 + t_1 t_2}, -(1 + t_2 + t_1 t_2), t_3 \right) \\ &= V_1 \left(-\frac{t_1 t_2}{1 + t_1 + t_1 t_2}, t_3, -(1 + t_2 + t_1 t_2) \right) = \left(t_3, -\frac{t_1 t_2}{1 + t_1 + t_1 t_2}, -(1 + t_2 + t_1 t_2) \right). \end{aligned}$$

The right-hand side is

$$\begin{aligned} R_2 V_1 V_2(\mathbf{t}) &= R_2 V_1 V_2(t_1, t_2, t_3) = R_2 R_1(t_1, t_3, t_2) = R_2(t_3, t_1, t_2) \\ &= \left(t_3, -\frac{t_1 t_2}{1 + t_1 + t_1 t_2}, -(1 + t_2 + t_1 t_2) \right). \end{aligned}$$

So the identity (8) holds. \square

Example 6.1. Consider $w_5 = \sigma_2 \rho_1 \sigma_1 \rho_2 \in FVB_3$. The operator $F_{FVB}(w_5)$ acts on $(1, 2, 2)$ in the following way:

$$F_{FVB}(w_5)(1, 2, 2) = \left(-5, \frac{4}{11}, -\frac{11}{5} \right) \neq (1, 2, 2).$$

Therefore, the homomorphism F_{FVB} distinguishes w_5 from a trivial braid.

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