Vertex-disjoint cycles of different lengths in tournaments of cycles

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Abstract

We show in this paper that every tournament of k cycles $D = (V, A)$ with $k \geq 2$ and minimum out-degree 3, except the digraph D_8^3, contains two disjoint cycles of different lengths.

Key words: digraph, tournament of cycles, vertex-disjoint cycles, cycles of different lengths

AMS Mathematics Subject Classification (2020): Primary 05C20, Secondary 05C38.

1 Introduction

We consider here only a *finite simple digraph*, i.e., a digraph that has a finite number of vertices, no loop, and no multiple arc. Unless otherwise indicated, our graph-theoretic terminology will follow [2]. We will also adopt notation and basic definitions that are used in [16].

Let D be a digraph. Then the vertex set and the arc set of D are denoted by $V(D)$ and $A(D)$ (or by V and A for short), respectively. A vertex $v \in V$ is called an *out-neighbor* of a vertex $u \in V$ if $(u, v) \in A$. We denote the set of all out-neighbors of u by $N_D^+(u)$. The *out-degree* of $u \in V$, denoted by $d_D^+(u)$, is $|N_D^+(u)|$. The *minimum out-degree* of D is $\min\{d_D^+(u) \mid u \in V\}$. Similarly, a vertex $w \in V$ is called an *in-neighbor* of a vertex $u \in V$ if $(w, u) \in A$. We denote the set of all in-neighbors of u by $N_D^-(u)$. The *in-degree* of $u \in V$, denoted by $d_D^-(u)$, is $|N_D^-(u)|$. If $W \subseteq V$, then the subdigraph of D induced by W is denoted by $D[W]$.

Let $D = (V, A)$ be a digraph. Then we write uv for an arc $(u, v) \in A$ for short. By a *cycle* (resp., *path*) in D, we always mean a directed cycle (resp., directed path). By *disjoint cycles* in D, we always mean vertex disjoint cycles. A *chord* of a cycle C of D is an arc $uv \in A \setminus A(C)$ with $u, v \in V(C)$.

Thomassen in [18] has proved that every digraph with minimum out-degree at least 3 contains two vertex-disjoint cycles. Recently, in connection with 2-coloring of hypergraphs, Henning and Yeo have begun to study in

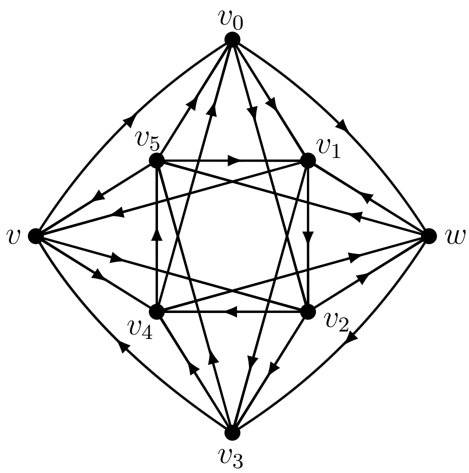

Figure 1: The digraph D_8^3

[7] the existence of vertex-disjoint cycles of different lengths in digraphs and they have posed there several conjectures. One of these conjectures has been solved by Lichiardopol in [10], which asserts that every digraph with minimum out-degree at least 4 contains two vertex-disjoint cycles of different lengths. By the results obtained by Thomassen [18] and Lichiardopol [10], the investigation of structure for digraphs without vertex-disjoint cycles of different lengths can be restricted to digraphs with minimum out-degree 3. Up to now there has been a lot of research on this topic, especially the results achieved recently (see [4, 5, 6, 8, 8, 11, 12, 13, 14, 15]). In this article, we continue to contribute a research result on this topic.

An *oriented graph* is a digraph with no cycle of length 2. A *tournament of k cycles* with $k \geq 2$ is an oriented graph $D = (V, A)$ with a partition $V = V_1 \cup V_2 \cup \ldots \cup V_k$ such that the subdigraphs $D[V_1], D[V_2], \ldots, D[V_k]$ are cycles and for every two vertices $u \in V_i$ and $v \in V_j$ with $i, j \in \{1, 2, \ldots, k\}$ and $i \neq j$ exactly one of the arcs uv and vu is in A. We study the existence of vertex-disjoint cycles of different lengths in tournaments of k cycles with $k \geq 2$ and minimum out-degree 3.

In [7], Henning and Yeo have given an example of 3-regular digraphs having no vertex-disjoint cycles of different lengths. We denote this digraph here by D_8^3. This digraph has the vertex set $V(D_8^3) = \{v_0, v_1, \ldots, v_5, v, w\}$ and the arc set $A(D_8^3) = A_1 \cup A_2 \cup A_3$, where $A_1 = \{v_i v_j \mid j - i = 1 \text{ or } 2 \pmod 6\}$, $A_2 = \{vv_i, v_i w \mid i = 0, 2, 4\}$ and $A_3 = \{wv_i, v_i v \mid i = 1, 3, 5\}$. The digraph D_8^3 is illustrated in Figure 1.

We note that the digraph D_8^3 is a tournament of two cycles with minimum out-degree 3. So, it is natural to ask whether there are other tournaments of

k cycles with $k \geq 2$ and minimum out-degree 3 without vertex-disjoint cycles of different lengths. In this paper, we will give the answer to this question by proving the following main result.

Theorem 1. *Every tournament of k cycles $D = (V, A)$ with $k \geq 2$ and minimum out-degree 3, except the digraph D_8^3, contains two disjoint cycles of different lengths.*

Further information can be found for tournaments in the recent surveys [3, 19].

2 Preliminary results

Let $D = (V, A)$ be a tournament of k cycles with $k \geq 2$ and minimum out-degree 3, having no disjoint cycles of different lengths. Further, let $V = V_0 \cup V_1 \cup \ldots \cup V_{k-1}$ be the partition of D and $D[V_j] = A^j$, where $j \in \{0, 1, \ldots, k-1\}$, be the cycle. $A^0, A^1, \ldots, A^{k-1}$ must have same length $t \geq 3$ because D has no disjoint cycles of different lengths. Let

$$A^j = (a_0^j, a_1^j, \ldots, a_{t-1}^j, a_0^j), \text{ where } j \in \{0, 1, \ldots, k-1\}.$$

A collection $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$ of cycles $C_1, C_2, \ldots, C_k$ in a digraph is called *bad* if there exist two vertex-disjoint cycles of different lengths in $\mathcal{C}$.

Let X and X' be two disjoint subsets of the vertex set V of a digraph $D = (V, A)$. We say that X *dominates* X' or X' *is dominated by* X if $X' \subseteq N_D^+(x)$ for each $x \in X$. We write $X \to X'$ if X dominates X'. If $X \to X'$ and $X = \{v\}$, then we simply say that v dominates X' and simply write $v \to X'$. Similarly, if $X \to X'$ and $X' = \{v'\}$, then we simply say that X dominates v' and simply write $X \to v'$. If $X = \{x\}$ and $X' = \{x'\}$, then we simply say that x dominates x', which means that $xx' \in A$, and simply write $x \to x'$. The set X dominates a subdigraph D' if X dominates $V(D')$. Similarly, a subdigraph D' dominates a subset X' if $V(D')$ dominates X'.

Then we have the following trivial observations.

Observation 2. *Each of A^0, A^1,... and A^{k-1} has no chord.*

Observation 3. *D has no bad collection of cycles.*

Observation 4. *By renaming the vertices of D, we may assume that $d_D^+(a_0^0) = 3$, $a_0^0 \to a_0^1$ and $a_1^1 \to a_0^0$.*

By these observations and assumtion about D, we have immediately the following Lemma 5 and Lemma 6

Lemma 5. $k = 2$.

Proof. For a contradiction, let $k \geq 3$. If $a_0^0 \to a_2^1$ (resp. $a_2^1 \to a_0^0$) then $(a_0^0, a_0^1, a_1^1, a_0^0)$, $(a_0^0, a_2^1 A^1 a_1^1, a_0^0)$ and A^2 (resp. $(a_0^0, a_0^1, a_1^1, a_0^0)$, $(a_0^0, a_0^1, a_1^1, a_2^1, a_0^0)$ and A^2) form a bad collection of cycles, a contradiction to Observation 3. Thus, we must have $k = 2$. □

Lemma 6. $t \geq 4$.

Proof. For a contradiction, let $t = 3$. Then

$$|A(D)| \leq \binom{6}{2} = 15 < 18 \leq \sum_{v \in V(D)} d^+(v),$$

a contradiction. Thus, we must have $t \geq 4$. □

From now on, we always assume that

(1) $d_D^+(a_0^0) = 3$, $N_D^+(a_0^0) = \{a_1^0, a_0^1, a_m^1\}$ with $1 < m < t$, $a_1^1 \to a_0^0$,

(2) $a_m^1 \to \{a_p^0, a_q^0\}$ with $0 < p < q < t$ such that if $0 < i < q, i \neq p$ then $a_i^0 \to a_m^1$.

Lemma 7. *a_p^0 and a_q^0 cannot both be out-neighbors of a_0^1.*

Proof. Suppose, on the contrary, that a_p^0 and a_q^0 are both out-neighbors of a_0^1. Since $d_D^+(a_p^0) \geq 3$, there exists a_n^1 in A^1 with $1 < n < t$ such that $a_p^0 \to a_n^1$. Then $a_n^1 \to a_0^0$ because $d_D^+(a_0^0) = 3$.

First, we assume that $1 < n < m$. If $m > n + 1$ then $(a_0^0, a_0^1, a_1^1, a_0^0)$ and $(a_p^0, a_n^1 A^1 a_m^1, a_p^0)$ form a bad collection of cycles, a contradiction. Thus, $m = n + 1$. If $a_1^1 \to a_p^0$ then $(a_p^0, a_n^1, a_m^1, a_p^0)$ and $(a_0^0, a_0^1, a_1^1, a_q^0 A^0 a_0^0)$ form a bad collection of cycles, a contradiction. Thus, $a_p^0 \to a_1^1$. It follows that $(a_p^0, a_n^1, a_m^1, a_p^0)$ and $(a_q^0, a_1^1, a_0^0, a_0^1, a_q^0)$ form a bad collection of cycles, a contradiction.

Now, we assume that $m < n < t$. Then $q = n$ (if $q \neq n$ then $(a_0^0, a_m^1, a_q^0 A^0 a_0^0)$ and $(a_0^1, a_p^0, a_n^1 A^1 a_0^1)$ form a bad collection of cycles, a contradiction) and $t - q = m - 1$ (if $t - q \neq m - 1$ then $(a_0^1, a_p^0, a_n^1 A^1 a_0^1)$, $(a_0^1, a_1^1, a_p^0, a_n^1 A^1 a_0^1)$ and $(a_0^0, a_m^1, a_q^0 A^0 a_0^0)$ form a bad collection of cycles, a contradiction). If $a_1^1 \to a_p^0$ then $(a_0^1, a_p^0, a_n^1 A^1 a_0^1)$, $(a_0^1, a_1^1, a_p^0, a_n^1 A^1 a_0^1)$ and $(a_0^0, a_m^1, a_q^0 A^0 a_0^0)$ form a bad collection of cycles, a contradiction. Thus, $a_p^0 \to a_1^1$. Since $d_D^+(a_1^1) \geq 3$, there exists a_h^0 in A^0 such that $a_1^1 \to a_h^0$. If $0 < h < p$ then $a_h^0 \to a_m^1$. So $(a_0^0, a_m^1, a_q^0 A^0 a_0^0)$, $(a_h^0, a_m^1, a_q^0 A^0 a_h^0)$ and $(a_p^0, a_n^1 A^1 a_0^1, a_p^0)$ form a bad collection of cycles, a contradiction. If $p < h < q$ then again $a_h^0 \to a_m^1$. So $(a_0^0, a_0^1, a_1^1, a_0^0)$, $(a_p^0 A^0 a_h^0, a_m^1, a_p^0)$ form a bad collection of cycles if $h > p+1$ and $(a_p^0, a_h^0, a_m^1, a_p^0)$,

$(a_p^0, a_1^1, a_h^0, a_m^1, a_p^0)$, $(a_0^0, a_0^1, a_q^0 A^0 a_0^0)$ form a bad collection of cycles if $h = p+1$, a contradiction. If $q < h < t$ then $(a_m^1, a_h^0 A^0 a_0^0, a_m^1)$, $(a_p^0, a_n^1 A^1 a_0^1, a_p^0)$ form a bad collection of cycles if $a_m^1 \to a_h^0$ and $(a_m^1, a_q^0 A^0 a_h^0, a_m^1)$, $(a_0^1, a_p^0, a_n^1 A^1 a_0^1)$ form a bad collection of cycles if $a_h^0 \to a_m^1$, a contradiction. □

Lemma 8. *a_p^0 and a_q^0 cannot both be in-neighbors of a_0^1.*

Proof. Suppose, on the contrary, that a_p^0 and a_q^0 are both in-neighbors of a_0^1. Since $d_D^+(a_0^1) \geq 3$, there exists a_h^0 and a_l^0 in A^0 with $0 < h < l < t$ such that $a_0^1 \to \{a_h^0, a_l^0\}$.

First, we assume that $0 < h < l < p$. It is clear that $\{a_h^0, a_l^0\} \to a_m^1$. Then $(a_0^1, a_h^0 A^0 a_p^0, a_0^1)$, $(a_0^1, a_l^0 A^0 a_p^0, a_0^1)$ and $(a_m^1, a_q^0 A^0 a_0^0, a_m^1)$ form a bad collection of cycles, a contradiction.

Next, we assume that $0 < h < p < l < q$. It is clear that $\{a_h^0, a_l^0\} \to a_m^1$. If $a_1^1 \to a_h^0$ then $(a_0^1, a_h^0 A^0 a_p^0, a_0^1)$ and $(a_0^1, a_1^1, a_h^0 A^0 a_p^0, a_0^1)$ form a bad collection of cycles, a contradiction. Thus, $a_h^0 \to a_1^1$. So $(a_0^1, a_1^1, a_0^0, a_0^1)$, $(a_0^1, a_h^0, a_1^1, a_0^0, a_0^1)$ and $(a_m^1, a_p^0 A^0 a_l^0, a_m^1)$ form a bad collection of cycles, a contradiction.

Next, we assume that $0 < h < p < q < l < t$. It is clear that $a_h^0 \to a_m^1$. If $a_l^0 \to a_m^1$ (resp., $a_m^1 \to a_l^0$), then $(a_l^0, a_m^1, a_p^0 A^0 a_l^0)$, $(a_l^0, a_m^1, a_q^0 A^0 a_l^0)$ and $(a_0^0, a_0^1, a_1^1, a_0^0)$ (resp., $(a_0^1, a_h^0 A^0 a_p^0, a_0^1)$, $(a_0^1, a_h^0 A^0 a_q^0, a_0^1)$ and $(a_m^1, a_l^0 A^0 a_0^0, a_m^1)$) form a bad collection of cycles, a contradiction.

Next, we assume that $p < h < l < q$. It is clear that $\{a_h^0, a_l^0\} \to a_m^1$. Then $(a_m^1, a_p^0 A^0 a_h^0, a_m^1)$, $(a_m^1, a_p^0 A^0 a_l^0, a_m^1)$ and $(a_0^0, a_0^1, a_1^1, a_0^0))$ form a bad collection of cycles, a contradiction.

Next, we assume that $p < h < p < l < t$. It is clear that $a_h^0 \to a_m^1$. If $a_l^0 \to a_1^1$ (resp., $a_1^1 \to a_l^0$), then $(a_0^0, a_0^1, a_1^1, a_0^0)$, $(a_0^0, a_0^1, a_l^0, a_1^1, a_0^0)$ and $(a_m^1, a_p^0 A^0 a_h^0, a_m^1)$ (resp., $(a_0^0, a_0^1, a_l^0 A^0 a_0^0)$, $(a_0^0, a_0^1, a_1^1, a_l^0 A^0 a_0^0)$ and $(a_m^1, a_p^0 A^0 a_h^0, a_m^1)$) form a bad collection of cycles, a contradiction.

Finally, we assume that $q < h < l < t$. If $a_p^0 \to a_1^1$ (resp., $a_q^0 \to a_1^1$), then $(a_0^0, a_0^1, a_h^0 A^0 a_0^0)$, $(a_0^0, a_0^1, a_l^0 A^0 a_0^0)$ and $(a_p^0, a_1^1 A^1 a_m^1, a_p^0)$ (resp., $(a_0^0, a_0^1, a_h^0 A^0 a_0^0)$, $(a_0^0, a_0^1, a_l^0 A^0 a_0^0)$ and $(a_q^0, a_1^1 A^1 a_m^1, a_q^0)$) form a bad collection of cycles, a contradiction. Thus, $a_1^1 \to \{a_p^0, a_q^0\}$. So, if $a_m^1 \to a_l^0$ (resp., $a_l^0 \to a_m^1$) then $(a_p^0, a_0^1, a_1^1, a_p^0)$, $(a_p^0 A^0 a_q^0, a_0^1, a_1^1, a_p^0)$ and $(a_0^0, a_m^1, a_l^0 A^0 a_0^0)$ (resp., $(a_l^0, a_m^1, a_p^0 A^0 a_l^0)$, $(a_l^0, a_m^1, a_q^0 A^0 a_l^0)$ and $(a_0^0, a_0^1, a_1^1, a_0^0)$) form a bad collection of cycles, a contradiction. □

3 Proof of Theorem 1

Let $D = (V, A)$ be a tournament of k cycles with $k \geq 2$ and minimum out-degree 3, having no disjoint cycles of different lengths. By Lemma 7 and Lemma 8, we have the following cases.

Case 1. $a_p^0 \to a_0^1$ and $a_0^1 \to a_q^0$.

Since $d_D^+(a_q^0) \geq 3$, there exists a_n^1 in A^1 with $1 < n < t$ such that $a_q^0 \to a_n^1$. It is clear that $a_n^1 \to a_0^0$. We again divide Case 1 into several subcases.

Subcase 1.1. $1 < n < m$.

If $m > n+1$ then $(a_0^0, a_0^1, a_1^1, a_0^0)$ and $(a_q^0, a_n^1 A^1 a_m^1, a_q^0)$ form a bad collection of cycles, a contradiction. So $m = n+1$. It follows that $(a_q^0, a_n^1, a_m^1, a_q^0)$ and $(a_0^0 A^0 a_p^0, a_0^1, a_1^1, a_0^0)$ form a bad collection of cycles, a contradiction.

Subcase 1.2. $m < n < t$.

Since $d_D^+(a_0^1) \geq 3$, there exists a_h^0 in A^0 with $h \notin \{0, p, q\}$ such that $a_0^1 \to a_h^0$.

First, we assume that $0 < h < p$. It is clear that $a_h^0 \to a_m^1$. If $a_n^1 \to a_h^0$ then $(a_0^0, a_0^1, a_1^1, a_0^0)$ and $(a_n^1, a_h^0 A^0 a_q^0, a_n^1)$ form a bad collection of cycles, a contradiction. So, $a_h^0 \to a_n^1$. If $a_1^1 \to a_p^0$ then $(a_h^0, a_n^1, a_0^0 A^0 a_h^0)$, $(a_h^0, a_m^1 A^1 a_n^1, a_0^0 A^0 a_h^0)$ and $(a_0^1, a_1^1, a_p^0, a_0^1)$ form a bad collection of cycles, a contradiction. So, $a_p^0 \to a_1^1$. Thus, $(a_p^0, a_1^1 A^1 a_m^1, a_p^0)$, $(a_p^0, a_0^1, a_1^1 A^1 a_m^1, a_p^0)$ and $(a_h^0, a_n^1, a_0^0 A^0 a_h^0)$ form a bad collection of cycles, a contradiction.

Next, we assume that $p < h < q$. It is clear that $a_h^0 \to a_m^1$. If $h > p+1$ then $(a_0^0, a_0^1, a_1^1, a_0^0)$ and $(a_m^1, a_p^0 A^0 a_h^0, a_m^1)$ form a bad collection of cycles, a contradiction. So $h = p+1$. It follows that $(a_m^1, a_p^0, a_h^0, a_m^1)$ and $(a_0^0, a_0^1, a_q^0, a_n^1, a_0^0)$ form a bad collection of cycles, a contradiction.

Finally, we assume that $q < h < t$. If $a_p^0 \to a_1^1$ then $(a_0^0, a_0^1, a_h^0 A^0 a_0^0)$, $(a_0^0, a_0^1, a_q^0 A^0 a_0^0)$ and $(a_p^0, a_1^1 A^1 a_m^1, a_p^0)$ form a bad collection of cycles, a contradiction. So, $a_1^1 \to a_p^0$. It follows that $(a_p^0, a_0^1, a_1^1, a_p^0)$ and $(a_0^0, a_m^1, a_q^0, a_n^1, a_0^0)$ form a bad collection of cycles, a contradiction.

Case 2. $a_0^1 \to a_p^0$ and $a_q^0 \to a_0^1$.

Since $d_D^+(a_p^0) \geq 3$, there exists a_n^1 in A^1 with $1 < n < t$ such that $a_p^0 \to a_n^1$. Then $a_n^1 \to a_0^0$ because $d_D^+(a_0^0) = 3$. We again divide Case 1 into several subcases.

Subcase 2.1. $1 < n < m$.

If $m > n+1$ then $(a_0^0, a_0^1, a_1^1, a_0^0)$ and $(a_p^0, a_n^1 A^1 a_m^1, a_p^0)$ form a bad collection of cycles, a contradiction. So $m = n+1$. If $a_1^1 \to a_q^0$ then $(a_p^0, a_n^1, a_m^1, a_p^0)$ and $(a_0^0, a_0^1, a_1^1, a_q^0 A^0 a_0^0)$ form a bad collection of cycles, a contradiction. So, $a_q^0 \to a_1^1$. If $a_q^0 \to a_n^1$ then $(a_0^0, a_0^1, a_1^1, a_0^0)$ and $(a_q^0, a_n^1, a_m^1, a_p^0 A^0 a_q^0)$ form a bad collection of cycles, a contradiction. So, $a_n^1 \to a_q^0$.

Since $d_D^+(a_0^1) \geq 3$, there exists a_h^0 in A^0 with $h \notin \{0, p, q\}$ such that $a_0^1 \to a_h^0$.

First, we assume that $0 < h < p$. It is clear that $a_h^0 \to a_m^1$. If $a_h^0 \to a_1^1$ then $(a_p^0, a_n^1, a_m^1, a_p^0)$ and $(a_h^0, a_1^1, a_0^0, a_0^1, a_h^0)$ form a bad collection of cycles, a contradiction. So, $a_1^1 \to a_h^0$. If $a_n^1 \to a_h^0$ then $(a_0^0, a_0^1, a_1^1, a_0^0)$ and $(a_n^1, a_h^0, a_m^1, a_p^0, a_n^1)$ form a bad collection of cycles, a contradiction. So, $a_h^0 \to a_n^1$. If $a_1^1 \to a_p^0$ then $(a_1^1, a_p^0 A^0 a_q^0, a_1^1)$, $(a_0^1, a_1^1, a_p^0 A^0 a_q^0, a_0^1)$ and $(a_n^1, a_0^0 A^0 a_h^0, a_n^1)$ form a bad collection of cycles, a contradiction. So, $a_p^0 \to a_1^1$. First, we assume that $t = 4$. Let φ be the following mapping from $V(D)$ to $V(D_8^3)$: $a_0^0 \mapsto v_5$, $a_h^0 \mapsto v_1$, $a_p^0 \mapsto v_2$, $a_q^0 \mapsto v_4$, $a_0^1 \mapsto v_0$, $a_1^1 \mapsto w$, $a_n^1 \mapsto v_3$ and $a_m^1 \mapsto v$. Then it is not difficult to verify that φ is an isomorphism between D and D_8^3. Now, we assume that $t > 4$. Then, there exists a_l^1 in A^1 with $l \notin \{0, 1, n, m\}$ such that $a_l^1 \to a_0^0$. If $1 < l < n$ then $(a_p^0, a_n^1, a_m^1, a_p^0)$ and $(a_0^0, a_0^1, a_1^1 A^1 a_l^1, a_0^0)$ form a bad collection of cycles, a contradiction. So, $m < l < t$. If $a_h^0 \to a_l^1$ (resp., $a_l^1 \to a_h^0$), then $(a_p^0, a_n^1, a_m^1, a_p^0)$ and $(a_h^0, a_l^1, a_0^0, a_0^1, a_h^0)$ (resp., $(a_0^0, a_0^1, a_1^1, a_0^0)$ and $(a_l^1, a_h^0, a_n^1, a_m^1 A^1 a_l^1)$) form a bad collection of cycles, a contradiction.

Next, we assume that $p < h < q$. It is clear that $a_h^0 \to a_m^1$. Then $(a_0^0, a_0^1, a_1^1, a_0^0)$, $(a_0^0, a_0^1 A^1 a_n^1, a_0^0)$ and $(a_m^1, a_p^0 A^0 a_h^0, a_m^1)$ form a bad collection of cycles, a contradiction.

Finally, we assume that $q < h < t$. If $a_h^0 \to a_1^1$ then $(a_p^0, a_n^1, a_m^1, a_p^0)$ and $(a_h^0, a_1^1, a_0^0, a_0^1, a_h^0)$ form a bad collection of cycles, a contradiction. So, $a_1^1 \to a_h^0$. Then $(a_p^0, a_n^1, a_m^1, a_p^0)$ and $(a_1^1, a_h^0 A^0 a_0^0, a_0^1, a_1^1)$ form a bad collection of cycles, a contradiction.

Subcase 2.2. $m < n < t$.

If $a_1^1 \to a_q^0$ then $(a_1^1, a_q^0, a_0^1, a_1^1)$ and $(a_0^0, a_m^1, a_p^0, a_n^1, a_0^0)$ form a bad collection of cycles, a contradiction. So, $a_q^0 \to a_1^1$. Then $(a_q^0, a_1^1 A^1 a_m^1, a_q^0)$, $(a_q^0, a_0^1 A^1 a_m^1, a_q^0)$ and $(a_p^0, a_n^1, a_0^0 A^0 a_p^0)$ form a bad collection of cycles, a contradiction.

The proof of Theorem 1 is complete.

References

[1] M. Abreu, E. L. Aldred, M. Funk, B. Jackson, D. Labbate and J. Sheehan, Graphs and digraphs with all 2-factors isomorphic, *J. Combin. Theory Ser. B*, 92 (2004), 395 – 404.

[2] J. Bang-Jensen and G. Gutin, *Digraphs. Theory, Algorithms and Applications*, Springer, New York, 2001.

[3] J. Bang-Jensen and F. Havet, Tournaments and semicomplete digraphs, In: *Classes of Directed Graphs* (J. Bang-Jensen and G. Gutin, eds), pp. 35 – 124, Springer, New York, 2018.

[4] J. Bensmail, A. Harutyunyan, N. K. Le, B. Li and N. Lichiardopol, Disjoint cycles of different lengths in graphs and digraphs, *Electron. J. Combin.*, 24 (2017), Article P4.37.

[5] Y. Gao and D. Ma, Disjoint cycles with different length in 4-arc-dominated digraphs, *Oper. Res. Lett.*, 41 (2013), 650 – 653.

[6] Z. He, P. Cheng and Y. Gao, Disjoint cycles of different lengths in 3-regular digraphs, *Graphs Combin.*, 38 (3) (2022), Article 74.

[7] M. A. Henning and A. Yeo, Vertex disjoint cycles of different length in digraphs, *SIAM J. Discrete Math.*, 26 (2012), 687 – 694.

[8] L. X. Hung, D. D. Hieu and N. D. Tan, Vertex-disjoint cycles of different lengths in multipartite tournaments, *Discrete Math.*, 345 (2022), Article 112819.

[9] L. X. Hung and N. D. Tan, Vertex-Disjoint Cycles of Different Lengths in Local Tournaments. *Graphs and Combinatorics*, Volume 39, Issue 5, Article 92 (2023).

[10] N. Lichiardopol, Proof of a conjecture of Henning and Yeo on vertex-disjoint directed cycles, *SIAM J. Discrete Math.*, 28 (2014), 1618 – 1627.

[11] R. M. Steiner, Disjoint cycles with length constraints in digraphs of large connectivity or large minimum degree, *SIAM J. Discrete Math.*, 36 (2) (2022), 1343 – 1362.

[12] N. D. Tan, Vertex disjoint cycles of different lengths in d-arc-dominated digraphs, *Oper. Res. Lett.*, 42 (2014), 351 – 354.

[13] N. D. Tan, On vertex disjoint cycles of different lengths in 3-regular digraphs, *Discrete Math.*, 338 (2015), 2485 – 2491.

[14] N. D. Tan, On 3-regular digraphs without vertex disjoint cycles of different lengths, *Discrete Math.*, 340 (2017), 1933 – 1943.

[15] N. D. Tan, On 3-regular digraphs of girth 4, *Discrete Math.*, 342 (2020), Article 111632.

[16] N. D. Tan, Tournaments and bipartite tournaments without vertex disjoint cycles of different lengths, *SIAM J. Discrete Math.*, 35 (1) (2021), 485 – 494.

[17] N. D. Tan, A decomposition for digraphs with minimum outdegree 3 having no vertex disjoint cycles of different lengths, *Discussiones Math. Graph Theory*, 43 (2023), 573 – 581.

[18] C. Thomassen, Disjoint cycles in digraphs, *Combinatorica*, 3 (1983), 393 – 396.

[19] A. Yeo, Semicomplete multipartite digraphs, In: *Classes of Directed Graphs* (J. Bang-Jensen and G. Gutin, eds), pp. 297 – 340, Springer, New York, 2018.