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LOCALLY ADJOINTABLE OPERATORS ON HILBERT C^*-MODULES

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Abstract: In the theory of Hilbert C^*-modules over a C^*-algebra $\mathcal{A}$ (in contrast with the theory of Hilbert spaces) not each bounded operator ($\mathcal{A}$-homomorphism) admits an adjoint. The interplay between the sets of adjointable and non-adjointable operators plays a very important role in the theory. We study an intermediate notion of locally adjointable operator $F : \mathcal{M} \to \mathcal{N}$, i.e. such an operator that $F \circ \gamma$ is adjointable for any adjointable $\gamma : \mathcal{A} \to \mathcal{M}$. We have introduced this notion recently and it has demonstrated its usefulness in the context of theory of uniform structures on Hilbert C^*-modules. In the present paper we obtain an explicit description of locally adjointable operators in important cases.

Keywords: Hilbert C^*-module, dual module, multiplier, adjointable operator, locally adjointable operator.

Definition 1. A (right) pre-Hilbert C^*-module over a C^*-algebra $\mathcal{A}$ is an $\mathcal{A}$-module equipped with a sesquilinear form on the underlying linear space $\langle .,. \rangle : \mathcal{M} \times \mathcal{M} \to \mathcal{A}$ such that

(1) $\langle x, x \rangle \geq 0$ for any $x \in \mathcal{M}$;

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(2) $\langle x, x\rangle = 0$ if and only if $x = 0$;
(3) $\langle y, x\rangle = \langle x, y\rangle^*$ for any $x, y \in \mathcal{M}$;
(4) $\langle x, y \cdot a\rangle = \langle x, y\rangle a$ for any $x, y \in \mathcal{M}$, $a \in \mathcal{A}$.

A complete pre-Hilbert C^*-module w.r.t. its norm $\|x\| = \|\langle x, x\rangle\|^{1/2}$ is called a *Hilbert C^*-module*.

Any C^*-algebra $\mathcal{A}$ can be considered as a module over itself with a sesquilinear form $\langle a, b\rangle_{\mathcal{A}} = a^*b$.

If a Hilbert C^*-module $\mathcal{M}$ has a countable subset which C^*-linear span is dense in $\mathcal{M}$, then it is called *countably generated*.

By $\oplus$ we will denote the orthogonal direct sum of Hilbert C^*-modules.

We refer to [8, 11, 10] for the theory of Hilbert C^*-modules.

Definition 2. The *standard* Hilbert C^*-module $\ell^2(\mathcal{A})$ is a Hilbert sum of countably many copies of $\mathcal{A}$ with the inner product $\langle a, b\rangle = \sum_i a_i^* b_i$, where $a = (a_1, a_2, \dots)$, $b = (b_1, b_2, \dots)$ and the series is norm-convergent. Denote by π_k, $k \in \mathbb{N}$, the projection $\pi_k : \ell^2(\mathcal{A}) \to \mathcal{A}$, $a \mapsto a_k$.

If $\mathcal{A}$ is unital, then $\ell^2(\mathcal{A})$ is countably generated.

This example of Hilbert C^*-modules is especially important due to the Kasparov stabilization theorem: for any countably generated Hilbert C^*-module $\mathcal{M}$ over any algebra $\mathcal{A}$, there exists an isomorphism of Hilbert C^*-modules (preserving the inner product) $\mathcal{M} \oplus \ell^2(\mathcal{A}) \cong \ell^2(\mathcal{A})$ [7] (see [11, Theorem 1.4.2]).

Definition 3. A bounded $\mathcal{A}$-homomorphism $F : \mathcal{M} \to \mathcal{N}$ of Hilbert C^*-modules is called *operator*.

Definition 4. For an operator $F : \mathcal{M} \to \mathcal{N}$ on Hilbert C^*-modules over $\mathcal{A}$, we say that F is *adjointable* with (evidently unique) *adjoint operator* $F^* : \mathcal{N} \to \mathcal{M}$ if $\langle Fx, y\rangle_{\mathcal{N}} = \langle x, F^*y\rangle_{\mathcal{M}}$ for any $x \in \mathcal{M}$ and $y \in \mathcal{N}$.

The following notion was introduced in a particular case of functionals in [6] and turned out very useful in the description of $\mathcal{A}$-compact operators in terms of uniform structures there (see also [14] and [15] for the previous research).

Definition 5. A bounded $\mathcal{A}$-morphism $F : \mathcal{M} \to \mathcal{N}$ of Hilbert C^*-modules is called *locally adjointable* if, for any adjointable morphism $\gamma : \mathcal{A} \to \mathcal{M}$, the composition $F \circ \gamma : \mathcal{A} \to \mathcal{N}$ is adjointable.

All these definitions are applicable in the case $\mathcal{N} = \mathcal{A}$. In this case bounded $\mathcal{A}$-operators are called $(\mathcal{A})$-functionals, adjointable operators are called adjointable functionals and locally adjointable operators are called locally adjointable functionals. These sets are denoted by $\mathcal{M}'$, $\mathcal{M}^*$ and $\mathcal{M}'_{LA}$, respectively. Evidently

$$\mathcal{M}^* \subseteq \mathcal{M}'_{LA} \subseteq \mathcal{M}'.$$

They are right Banach modules (for the last set see Theorem 1 below) with respect to the action $(fa)(x) = a^*f(x)$, where $f \in \mathcal{M}'$, $x \in \mathcal{M}$, $a \in \mathcal{A}$.

Typically $\mathcal{M}'$ is not a Hilbert C^*-module (see [9, 12] for a recent progress in the field).

The following notion was introduced and studied in [3] and applied to the frame theory in [1] (with developments in [4] and [5]). In [2] explicit results for $\ell_2(\mathcal{A})$ were obtained. Denote by $LM(\mathcal{A})$, $RM(\mathcal{A})$, and $M(\mathcal{A})$ *left*, *right*, and (two-sided) *multipliers* of algebra $\mathcal{A}$, respectively (the usual reference is [13], see also [11]). For any Hilbert $\mathcal{A}$-module $\mathcal{N}$ a Hilbert $M(\mathcal{A})$-module $M(\mathcal{N})$ (which is called the *multiplier module* of $\mathcal{N}$) containing $\mathcal{N}$ as an ideal submodule associated with $\mathcal{A}$, i.e. $\mathcal{N} = M(\mathcal{N})\mathcal{A}$ was defined in [3]. Namely, $M(\mathcal{N})$ is the space of all adjointable maps from $\mathcal{A}$ to $\mathcal{N}$ being a Hilbert C^*-module over $M(\mathcal{A})$ with the inner product $\langle r_1, r_2\rangle = r_1^* r_2$. This is really a multiplier because $\langle r_1, r_2\rangle a = r_1^* r_2(a) \in \mathcal{A}$. This is an essential extension of $\mathcal{N}$ in sense of [3].

Any (modular) *multiplier* $m \in M(\mathcal{N})$ represents an $\mathcal{A}$-functional $\widehat{m}$ on $\mathcal{N}$ by the formula $\widehat{m}(x) = \langle m, x\rangle$. This functional is adjointable and its adjoint is given by the formula $\widehat{m}^*(a) = ma$. In fact this map gives rise to an identification of $M(\mathcal{N})$ and the module $\mathcal{N}^*$ of adjointable functionals on $\mathcal{N}$ (see, [3, 2]), in particular,

$$(\ell^2(\mathcal{A}))^* \cong M(\ell^2(\mathcal{A})). \tag{1}$$

In [2, Theorem 2.3] the following isomorphism was obtained (we write it keeping in mind the difference between left and right modules):

$$(\ell^2(\mathcal{A}))' \cong \ell^2_{strong}(RM(\mathcal{A})), \tag{2}$$

where the last module is formed by all sequences $\Gamma_i \in RM(\mathcal{A})$ such that the series $\sum_i \Gamma_i^*\Gamma_i$ is strongly convergent in $B(H)$ (assuming that $\mathcal{A}$ is faithfully and non-degenerately represented on Hilbert space H).

Below in Lemma 2 we will prove some "intermediate variant" of these isomorphisms (1) and (2):

$$(\ell^2(\mathcal{A}))'_{LA} \cong (M(\ell^2(\mathcal{A})))'. \tag{3}$$

Now we pass to results of the present paper.

Lemma 1. *A bounded $\mathcal{A}$-morphism $F : \mathcal{K} \to \mathcal{N}$ of Hilbert C^*-modules is adjointable if and only if, for any $y \in \mathcal{N}$, the morphism $F_y : \mathcal{K} \to \mathcal{A}$, $F_y(x) = \langle y, F(x)\rangle$ is adjointable.*

Proof. Suppose that F is adjointable. Then, for any $x \in \mathcal{K}$, $a \in \mathcal{A}$

$$\langle F_y(x), a\rangle_{\mathcal{A}} = \langle y, F(x)\rangle^* a = \langle F^*(y), x\rangle^* a = \langle x, F^*(y)a\rangle, \quad (F_y)^*(a) = F^*(y)a,$$

and $F_y(x)$ is adjointable.

Conversely, suppose that each F_y is adjointable. Then, for an approximate unit $\{u_\lambda\}$ in $\mathcal{A}$, one has

$$\langle F(x), y\rangle u_\lambda = \langle y, F(x)\rangle^* u_\lambda = F_y(x)^* u_\lambda = \langle F_y(x), u_\lambda\rangle_{\mathcal{A}} = \langle x, (F_y)^* u_\lambda\rangle_{\mathcal{K}}.$$

Since

$$\|(F_y)^*(u_\lambda - u_\mu)\| = \sup_{z\in\mathcal{K},\, \|z\|\le 1} |\langle z, (F_y)^*(u_\lambda - u_\mu)\rangle| =$$
$$= \sup_{z\in\mathcal{K},\, \|z\|\le 1} |\langle F_y(z), u_\lambda - u_\mu\rangle_{\mathcal{A}}| = \sup_{z\in\mathcal{K},\, \|z\|\le 1} |\langle y, F(z)\rangle^*(u_\lambda - u_\mu)| =$$
$$= \sup_{z\in\mathcal{K},\, \|z\|\le 1} |\langle F(z), y(u_\lambda - u_\mu)\rangle_{\mathcal{N}}| \le \|F\|\cdot\|y(u_\lambda - u_\mu)\|,$$

we obtain (see [11, Lemma 1.3.8]) that the net $\{(F_y)^* u_\lambda\}$ is a Cauchy net. So we can define an operator G by $G(y) = \lim_\lambda (F_y)^* u_\lambda$. The operator G is evidently bounded by its defining formula. Since the above limit is in norm topology, for any $x \in \mathcal{K}$, $y \in \mathcal{N}$, we have

$$\langle F(x), y\rangle = \lim_\lambda \langle F(x), y\rangle u_\lambda = \lim_\lambda (\langle y, F(x)\rangle)^* u_\lambda = \lim_\lambda \langle F_y(x), u_\lambda\rangle_{\mathcal{A}} =$$
$$= \lim_\lambda \langle x, (F_y)^* u_\lambda\rangle = \left\langle x, \lim_\lambda (F_y)^* u_\lambda \right\rangle = \langle x, G(y)\rangle,$$

so, F is adjointable. □

Corollary 1. *A bounded $\mathcal{A}$-morphism $F : \mathcal{M} \to \mathcal{N}$ of Hilbert C^*-modules is locally adjointable if and only if, for any adjointable morphism $\gamma : \mathcal{A} \to \mathcal{N}$ and any $y \in \mathcal{N}$, the morphism $F_{\gamma,y} : \mathcal{A} \to \mathcal{A}$, $F_{\gamma,y}(a) = \langle y, F\circ\gamma(a)\rangle$ is adjointable.*

Theorem 1. *Locally adjointable operators from $\mathcal{M}$ to $\mathcal{N}$ form a Banach subspace of the Banach space of all bounded $\mathcal{A}$-morphisms from $\mathcal{M}$ to $\mathcal{N}$.*

In particular, locally adjointable endomorphisms of $\mathcal{M}$ form a Banach subalgebra of the algebra $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$ of all bounded $\mathcal{A}$-endomorphisms.

Proof. Indeed, if $\{F_n\}$ is a sequence of locally adjointable morphisms and $F_n \to F$ in norm, then for any adjointable morphism γ we have that $\|F_n \circ \gamma - F\circ\gamma\| \le \|F_n - F\|\cdot\|\gamma\|$, so $F_n\circ\gamma \to F\circ\gamma$ in norm too and $F\circ\gamma$ is adjointable. □

Proposition 1. *The dual module $(M(\ell_2(\mathcal{A})))'$ of $M(\ell_2(\mathcal{A}))$ consists of all sequences $\alpha_i \in M(\mathcal{A})$ such that*

1) *the partial sums of $\sum_i \alpha_i^*\alpha_i$ are bounded, i.e. this series is strong convergent in $B(H)$;*
2) *the series $\sum_i \alpha_i^*\beta_i$ is left strict convergent for any $\beta = \{\beta_i\} \in M(\ell_2(\mathcal{A}))$;*
3) *its limit belongs to $M(\mathcal{A}) \subseteq LM(\mathcal{A})$.*

Proof. Suppose, $\alpha \in (M(\ell_2(\mathcal{A})))'$, $\alpha : M(\ell_2(\mathcal{A})) \to M(\mathcal{A})$. Then its restriction on the submodule $\ell_2(M(\mathcal{A}))$ defines (by [2, Theorem 2.3]) a sequence $\alpha_i \in M(\mathcal{A})$ which has to satisfy the property 1). It also can be restricted to

$\ell_2(\mathcal{A}) = M(\ell_2(\mathcal{A}))\mathcal{A}$, and also by [2, Theorem 2.3] the action is given by

$$\sum_{i=1}^{\infty} \alpha_i^* \beta_i a, \qquad \{\beta_i\} \in M(\ell_2(\mathcal{A})), \quad a \in \mathcal{A}, \text{ the series is norm-convergent.} \tag{4}$$

This gives 2).

Two left multipliers u and v coincide, if $ua = va$ for any $a \in \mathcal{A}$. Thus, the equality

$$\alpha(\beta)a = \alpha(\beta a) = \sum_{i=1}^{\infty} \alpha_i^* \beta_i a = \left(\sum_{i=1}^{\infty} \alpha_i^* \beta_i \right) a$$

implies

$$\alpha(\beta) = \sum_{i=1}^{\infty} \alpha_i^* \beta_i \tag{5}$$

and hence 3).

Also, (5) implies that the linear mapping $\alpha \mapsto \{\alpha_i\}$ is injective.

Conversely, if $\{\alpha_i\}$ satisfies 1)-3), then (5) defines an element of the module $(M(\ell_2(\mathcal{A})))'$. Indeed, everything is evident, one needs only to verify that this α is bounded. For any $m < n$ and $a \in \mathcal{A}$ we have by the Cauchy inequality ([11, 1.2.4])

$$\left\| \sum_{i=m}^{n} \alpha_i^* \beta_i \right\|^2 = \left\| \left(\sum_{i=m}^{n} \alpha_i^* \beta_i \right)^* \sum_{i=m}^{n} \alpha_i^* \beta_i \right\| \le \left\| \sum_{i=m}^{n} \alpha_i^* \alpha_i \right\| \cdot \left\| \sum_{i=m}^{n} \beta_i^* \beta_i \right\|.$$

Hence, α is bounded, and the mapping is surjective. □

Recall that an $\mathcal{A}$-functional $\Gamma : \ell_2(\mathcal{A}) \to \mathcal{A}$ is defined by a sequence $\{\Gamma_i\}_{i \in \mathbb{N}}$, $\Gamma_i \in RM(\mathcal{A})$, such that

$$\sum_i \Gamma_i^* \Gamma_i \text{ strongly converges in } B(H) \tag{6}$$

(see [2] and (2) above). The action on $\alpha = (\alpha_1, \alpha_2, \dots) \in \ell_2(\mathcal{A})$ is defined by $\Gamma(\alpha) = \sum_i \Gamma_i^* \alpha_i$ and the series is norm-convergent.

Lemma 2. *An $\mathcal{A}$-functional $\Gamma : \ell_2(\mathcal{A}) \to \mathcal{A}$ is locally adjointable if and only if its above defined collection of coefficients Γ_i determines an element of $(M(\ell_2(\mathcal{A})))'$.*

Proof. Suppose that Γ is locally adjointable. Consider an arbitrary adjointable morphism $\gamma : \mathcal{A} \to \ell_2(\mathcal{A})$. The set of these morphisms is isomorphic, on the one hand, to the space $(\ell_2(\mathcal{A}))^*$ of adjointable $\mathcal{A}$-functionals, and on the other hand, to the module $M(\ell_2(\mathcal{A}))$ (see [3, 2]). Namely there exist (by [3, Theorems 1.8 and 2.1]) $\gamma_i \in M(\mathcal{A})$ such that $\sum_i \gamma_i^* \gamma_i$ is strictly convergent and

$$\gamma(a) = (\gamma_1 a, \gamma_2 a, \dots), \qquad a \in \mathcal{A}. \tag{7}$$

Then

$$\Gamma \circ \gamma(a) = \sum_i \Gamma_i^* \gamma_i a,$$

where the series $\sum_i \Gamma_i^* \gamma_i = \mu$ is convergent in left strict topology and defines an element $\mu \in LM(\mathcal{A})$. This gives property 2) of Proposition 1. This morphism $\mathcal{A} \to \mathcal{A}$ has to be adjointable and hence we have $\sum_i \Gamma_i^* \gamma_i \in M(\mathcal{A})$. This gives 3) of Proposition 1. In particular, for $\gamma = (0, \dots, 0, 1_{M(\mathcal{A})}, 0, \dots)$, we have that Γ_i^* is an adjointable left multiplier, i.e. $\Gamma_i \in M(\mathcal{A})$. Together with (6) this gives 1) of Proposition 1.

The converse is similar. Indeed, from 1) it follows that the sequence $\{\Gamma_i\}$ defines an element of $(\ell^2(\mathcal{A}))'$ which acts by formula $\Gamma(x) = \sum\limits_{i=1}^{\infty} \Gamma_i^* x_i$, where series is norm-convergent. In particular, for any adjointable $\gamma : \mathcal{A} \to \ell^2(\mathcal{A})$ and any $a \in \mathcal{A}$ we have $\Gamma(\gamma(a)) = \sum\limits_{i=1}^{\infty} \Gamma_i^* \gamma_i a$. From 2) it follows that $\left(\sum\limits_{i=1}^{\infty} \Gamma_i^* \gamma_i\right) a = \sum\limits_{i=1}^{\infty} \Gamma_i^* \gamma_i a = \Gamma(\gamma(a))$, and from 3) it follows that $\left(\sum\limits_{i=1}^{\infty} \Gamma_i^* \gamma_i\right) \in M(\ell^2(\mathcal{A}))$, i.e. $\sum\limits_{i=1}^{\infty} \Gamma_i^* \gamma_i = \Gamma \circ \gamma$ is adjointable. □

The following statement will be used below and also seems to be of independent interest.

Theorem 2. *A bounded $\mathcal{A}$-morphism $F : \mathcal{M} \to \ell_2(\mathcal{A})$ is adjointable if and only if all projections $\pi_k \circ F$, $k \in \mathbb{N}$, are adjointable.*

Proof. If F is adjointable then $\pi_k \circ F$ is adjointable since the projections π_k are adjointable.

Suppose that for any projection π_k we have that $\pi_k \circ F$ is adjointable. Then, for any $y = (y_1, y_2, \dots) \in \ell^2(\mathcal{A})$,

$$\left\| \sum_{k=p}^{q} (\pi_k \circ F)^*(\pi_k(y)) \right\| = \sup_{z \in \mathcal{M},\, \|z\| \le 1} \left| \left\langle z, \sum_{k=p}^{q} (\pi_k \circ F)^*(\pi_k(y)) \right\rangle_{\mathcal{M}} \right| =$$

$$= \sup_{z \in \mathcal{M},\, \|z\| \le 1} \left| \left\langle \sum_{k=p}^{q} (\pi_k \circ F)(z), \pi_k(y) \right\rangle_{\mathcal{A}} \right| =$$

$$= \sup_{z \in \mathcal{M},\, \|z\| \le 1} \left| \left\langle F(z), \sum_{k=p}^{q} \pi_k^* \pi_k(y) \right\rangle \right| \le$$

$$\le \|F\| \cdot \left\| \sum_{k=p}^{q} \pi_k^* \pi_k(y) \right\| = \|F\| \cdot \left\| \sqrt{\sum_{k=p}^{q} y_k^* y_k} \right\|.$$

Since the series $\sum_{k=1}^{\infty} y_k^* y_k$ is norm-convergent, this implies that, for every $y \in \ell_2(\mathcal{A})$, the series $\sum_{k=1}^{\infty} (\pi_k \circ F)^*(\pi_k(y))$ is also norm-convergent in $\mathcal{M}$ and the equality $S(y) = \sum_{k=1}^{\infty} (\pi_k \circ F)^*(\pi_k(y))$ defines a bounded $\mathcal{A}$-operator, $S : \ell^2(\mathcal{A}) \to \mathcal{M}$. Also, for any $x \in \mathcal{M}$, $y \in \ell^2(\mathcal{A})$,

$$\langle F(x), y\rangle_{\ell^2(\mathcal{A})} = \sum_{k=1}^{\infty} \langle \pi_k \circ F(x), \pi_k(y)\rangle_{\mathcal{A}} =$$

$$= \sum_{k=1}^{\infty} \langle x, (\pi_k \circ F)^*(\pi_k(y))\rangle = \langle x, S(y)\rangle,$$

so F is adjointable with S being the adjoint operator. □

Corollary 2. 1). *A bounded $\mathcal{A}$-morphism $F : \mathcal{M} \to \ell_2(\mathcal{A})$ is locally adjointable if and only if all of its projections $\pi_k \circ F$, $k \in \mathbb{N}$, are locally adjointable.*

2). *An endomorphism F of the module $\ell_2(\mathcal{A})$ is locally adjointable if and only if its matrix rows belong to $M(\ell_2(\mathcal{A}))'$ ("i-th matrix row", $i \in \mathbb{N}$, is the functional $\pi_i \circ F$ defined by the sequence $\{\pi_i \circ F \circ \pi_j\}_{j\in\mathbb{N}}$ if we consider the operator F as an infinite matrix $\{F_{i,j} = \pi_i \circ F \circ \pi_j\}_{i,j\in\mathbb{N}}$).*

Corollary 3. $M(\ell_2(\mathcal{A})) \subset (\ell_2(\mathcal{A}))'_{LA}$.

Proof. Indeed, $M(\ell_2(\mathcal{A})) = (\ell_2(\mathcal{A}))^* \subset (\ell_2(\mathcal{A}))'_{LA}$. □

Corollary 4. *For a countably generated Hilbert $\mathcal{A}$-module $\mathcal{M}$ (for example, for an orthogonal direct summand of $\mathcal{A}^n$ or $\ell_2(\mathcal{A})$ when $\mathcal{A}$ is σ-unital) we have that $M(\mathcal{M}) \subset (\ell_2(\mathcal{A}))'_{LA}$.*

Proof. Indeed, by using Kasparov's stabiilization theorem and the fact that any adjointable functional can be extended from the summand to the sum we have that $M(\mathcal{M}) = \mathcal{M}^* \hookrightarrow (\ell_2(\mathcal{A}))^* \subset (\ell_2(\mathcal{A}))'_{LA}$.

Extension of $F \in \mathcal{M}^*$ to $\hat{F} \in (\mathcal{M} \oplus \mathcal{N})^*$ is defined by formula $\hat{F}(m, n) = F(m)$; its adjoint is defined by formula $\hat{F}^*(a) = (F^*(a), 0)$ since for any $a \in \mathcal{A}$, $m \in \mathcal{M}$, $n \in \mathcal{N}$

$$\langle \hat{F}^*(a), (m, n)\rangle_{\mathcal{M}\oplus\mathcal{N}} = \langle F^*(a), m\rangle_{\mathcal{M}} + 0 = \langle a, F(m)\rangle_{\mathcal{A}} = \langle a, \hat{F}(m, n)\rangle.$$

□

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