

LOCALLY ADJOINTABLE OPERATORS ON HILBERT C^* -MODULES

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Abstract: In the theory of Hilbert C^* -modules over a C^* -algebra \mathcal{A} (in contrast with the theory of Hilbert spaces) not each bounded operator (\mathcal{A} -homomorphism) admits an adjoint. The interplay between the sets of adjointable and non-adjointable operators plays a very important role in the theory. We study an intermediate notion of locally adjointable operator $F : \mathcal{M} \rightarrow \mathcal{N}$, i.e. such an operator that $F \circ \gamma$ is adjointable for any adjointable $\gamma : \mathcal{A} \rightarrow \mathcal{M}$. We have introduced this notion recently and it has demonstrated its usefulness in the context of theory of uniform structures on Hilbert C^* -modules. In the present paper we obtain an explicit description of locally adjointable operators in important cases.

Keywords: Hilbert C^* -module, dual module, multiplier, adjointable operator, locally adjointable operator.

Definition 1. A (right) pre-Hilbert C^* -module over a C^* -algebra \mathcal{A} is an \mathcal{A} -module equipped with a sesquilinear form on the underlying linear space $\langle \cdot, \cdot \rangle : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{A}$ such that

$$(1) \quad \langle x, x \rangle \geq 0 \text{ for any } x \in \mathcal{M};$$

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- (2) $\langle x, x \rangle = 0$ if and only if $x = 0$;
- (3) $\langle y, x \rangle = \langle x, y \rangle^*$ for any $x, y \in \mathcal{M}$;
- (4) $\langle x, y \cdot a \rangle = \langle x, y \rangle a$ for any $x, y \in \mathcal{M}, a \in \mathcal{A}$.

A complete pre-Hilbert C^* -module w.r.t. its norm $\|x\| = \|\langle x, x \rangle\|^{1/2}$ is called a *Hilbert C^* -module*.

If a Hilbert C^* -module \mathcal{M} has a countable subset which linear span is dense in \mathcal{M} , then it is called *countably generated*.

By \oplus we will denote the orthogonal direct sum of Hilbert C^* -modules.

We refer to [8, 11, 10] for the theory of Hilbert C^* -modules.

Definition 2. The *standard* Hilbert C^* -module $\ell^2(\mathcal{A})$ is a Hilbert sum of countably many copies of \mathcal{A} with the inner product $\langle a, b \rangle = \sum_i a_i^* b_i$, where $a = (a_1, a_2, \dots)$ and $b = (b_1, b_2, \dots)$. Denote by $\pi_k, k \in \mathbb{N}$, the projection $\pi_k : \ell^2(\mathcal{A}) \rightarrow \mathcal{A}, a \mapsto a_k$.

If \mathcal{A} is unital, then $\ell^2(\mathcal{A})$ is countably generated.

This example of Hilbert C^* -modules is especially important due to the Kasparov stabilization theorem: for any countably generated Hilbert C^* -module \mathcal{M} over any algebra \mathcal{A} , there exists an isomorphism of Hilbert C^* -modules (preserving the inner product) $\mathcal{M} \oplus \ell^2(\mathcal{A}) \cong \ell^2(\mathcal{A})$ [7] (see [11, Theorem 1.4.2]).

Definition 3. A bounded \mathcal{A} -homomorphism $F : \mathcal{M} \rightarrow \mathcal{N}$ of Hilbert C^* -modules is called *operator*.

Definition 4. For an operator $F : \mathcal{M} \rightarrow \mathcal{N}$ on Hilbert C^* -modules over \mathcal{A} , we say that F is *adjointable* with (evidently unique) *adjoint operator* $F^* : \mathcal{N} \rightarrow \mathcal{M}$ if $\langle Fx, y \rangle_{\mathcal{N}} = \langle x, F^*y \rangle_{\mathcal{M}}$ for any $x \in \mathcal{M}$ and $y \in \mathcal{N}$.

The following notion was introduced in a particular case of functionals in [6] and turned out very useful in the description of \mathcal{A} -compact operators in terms of uniform structures there (see also [14] and [15] for the previous research).

Definition 5. A bounded \mathcal{A} -morphism $F : \mathcal{M} \rightarrow \mathcal{N}$ of Hilbert C^* -modules is called *locally adjointable* if, for any adjointable morphism $\gamma : \mathcal{A} \rightarrow \mathcal{N}$, the composition $F \circ \gamma : \mathcal{A} \rightarrow \mathcal{N}$ is adjointable.

All these definitions are applicable in the case $\mathcal{N} = \mathcal{A}$. In this case bounded \mathcal{A} -operators are called (\mathcal{A})-functionals, adjointable operators are called adjointable functionals and locally adjointable operators are called locally adjointable functionals. These sets are denoted by $\mathcal{M}', \mathcal{M}^*$ and \mathcal{M}'_{LA} , respectively. Evidently

$$\mathcal{M}^* \subseteq \mathcal{M}'_{LA} \subseteq \mathcal{M}'.$$

They are right Banach modules (for the last set see Theorem 1 below) with respect to the action $(fa)(x) = a^*f(x)$, where $f \in \mathcal{M}', x \in \mathcal{M}, a \in \mathcal{A}$. Typically \mathcal{M}' is not a Hilbert C^* -module (see [9, 12] for a recent progress in the field).

The following notion was introduced and studied in [3] and applied to the frame theory in [1] (with developments in [4] and [5]). In [2] explicit results for $\ell^2(\mathcal{A})$ were obtained. Denote by $LM(\mathcal{A})$, $R(\mathcal{A})$, and $M(\mathcal{A})$ *left*, *right*, and (two-sided) *multipliers* of algebra \mathcal{A} , respectively (the usual reference is [13], see also [11]). For any Hilbert \mathcal{A} -module \mathcal{N} a Hilbert $M(\mathcal{A})$ -module $M(\mathcal{N})$ (which is called the *multiplier module* of \mathcal{N}) containing \mathcal{N} as an ideal submodule associated with \mathcal{A} , i.e. $\mathcal{N} = M(\mathcal{N})\mathcal{A}$ was defined in [3]. Namely, $M(\mathcal{N})$ is the space of all adjointable maps from \mathcal{A} to \mathcal{N} being a Hilbert C^* -module over $M(\mathcal{A})$ with the inner product $\langle r_1, r_2 \rangle = r_1^*r_2$. This is really a multiplier because $\langle r_1, r_2 \rangle a = r_1^*r_2(a) \in \mathcal{A}$. This is an essential extension of \mathcal{N} in sense of [3].

Any (modular) *multiplier* $m \in M(\mathcal{N})$ represents an \mathcal{A} -functional \widehat{m} on \mathcal{N} by the formula $\widehat{m}(x) = \langle m, x \rangle$. This functional is adjointable and its adjoint is given by the formula $\widehat{m}^*(a) = ma$. In fact this map gives rise to an identification of $M(\mathcal{N})$ and the module \mathcal{N}^* of adjointable functionals on \mathcal{N} (see, [3, 2]), in particular,

$$(\ell^2(\mathcal{A}))^* \cong M(\ell^2(\mathcal{A})). \quad (1)$$

In [2, Theorem 2.3] the following isomorphism was obtained (we write it keeping in mind the difference between left and right modules):

$$(\ell^2(\mathcal{A}))' \cong \ell^2_{strong}(RM(\mathcal{A})), \quad (2)$$

where the last module is formed by all sequences $\Gamma_i \in RM(\mathcal{A})$ such that the series $\sum_i \Gamma_i^* \Gamma_i$ is strongly convergent in $B(H)$ (assuming that \mathcal{A} is faithfully and non-degenerately represented on Hilbert space H).

Below in Lemma 2 we will prove some “intermediate variant” of these isomorphisms (1) and (2):

$$(\ell^2(\mathcal{A}))'_{LA} \cong (M(\ell^2(\mathcal{A})))'. \quad (3)$$

Now we pass to results of the present paper.

Lemma 1. *A bounded \mathcal{A} -morphism $F : \mathcal{K} \rightarrow \mathcal{N}$ of Hilbert C^* -modules is adjointable if and only if, for any $y \in \mathcal{N}$, the morphism $F_y : \mathcal{K} \rightarrow \mathcal{A}$, $F_y(x) = \langle y, F(x) \rangle$ is adjointable.*

Proof. Suppose that F is adjointable. Then, for $x \in \mathcal{K}$, $a \in \mathcal{A}$

$$\langle F_y(x), a \rangle = \langle y, F(x) \rangle^* a = \langle F^*(y), x \rangle^* a = \langle x, F^*(y) \rangle a, \quad (F_y)^*(a) = F^*(y)a,$$

and $F_y(x)$ is adjointable.

Conversely, suppose that each F_y is adjointable. Then, for an approximate unit $\{u_\lambda\}$ in \mathcal{A} , one has

$$\langle F(x), y \rangle u_\lambda = \langle y, F(x) \rangle^* u_\lambda = F_y(x)^* u_\lambda = \langle F_y(x), u_\lambda \rangle_{\mathcal{A}} = \langle x, (F_y)^* u_\lambda \rangle_{\mathcal{K}}.$$

Since

$$\begin{aligned} \|(F_y)^*(u_\lambda - u_\mu)\| &= \sup_{z \in \mathcal{K}, \|z\| \leq 1} |\langle z, (F_y)^*(u_\lambda - u_\mu) \rangle| = \\ &= \sup_{z \in \mathcal{K}, \|z\| \leq 1} |\langle F_y(z), u_\lambda - u_\mu \rangle_{\mathcal{A}}| = \sup_{z \in \mathcal{K}, \|z\| \leq 1} |\langle y, F(z) \rangle^*(u_\lambda - u_\mu)| = \\ &= \sup_{z \in \mathcal{K}, \|z\| \leq 1} |\langle F(z), y(u_\lambda - u_\mu) \rangle_{\mathcal{N}}| \leq \|F\| \cdot \|y(u_\lambda - u_\mu)\|, \end{aligned}$$

we obtain (see [11, Lemma 1.3.8]) that the net $(F_y)^*u_\lambda$ is a Cauchy net. So we can define an operator G by $G(y) = \lim_{\lambda} (F_y)^*u_\lambda$. The operator G is evidently bounded by its defining formula. Since the above limit is in norm topology, for any $x \in \mathcal{K}$, $y \in \mathcal{N}$, we have

$$\begin{aligned} \langle F(x), y \rangle &= \lim_{\lambda} \langle F(x), y \rangle u_\lambda = \lim_{\lambda} (\langle y, F(x) \rangle)^* u_\lambda = \lim_{\lambda} \langle F_y(x), u_\lambda \rangle = \\ &= \lim_{\lambda} \langle x, (F_y)^* u_\lambda \rangle = \left\langle x, \lim_{\lambda} (F_y)^* u_\lambda \right\rangle = \langle x, G(y) \rangle, \end{aligned}$$

so, F is adjointable. □

Corollary 1. *A bounded \mathcal{A} -morphism $F : \mathcal{M} \rightarrow \mathcal{N}$ of Hilbert C^* -modules is locally adjointable if and only if, for any adjointable morphism $\gamma : \mathcal{A} \rightarrow \mathcal{N}$ and any $y \in \mathcal{N}$, the morphism $F_{\gamma, y} : \mathcal{A} \rightarrow \mathcal{A}$, $F_{\gamma, y}(a) = \langle y, F \circ \gamma(a) \rangle$ is adjointable.*

Theorem 1. *Locally adjointable operators from \mathcal{M} to \mathcal{N} form a Banach subspace of the Banach space of all bounded \mathcal{A} -morphisms from \mathcal{M} to \mathcal{N} .*

In particular, locally adjointable endomorphisms of \mathcal{M} form a Banach subalgebra of the algebra $\text{End}_{\mathcal{A}}(\mathcal{M})$ of all bounded \mathcal{A} -endomorphisms.

Proof. Indeed, if $\{F_n\}$ is a sequence of locally adjointable morphisms and $F_n \rightarrow F$ in norm, then for any adjointable morphism γ we have that $\|F_n \circ \gamma - F \circ \gamma\| \leq \|F_n - F\| \cdot \|\gamma\|$, so $F_n \circ \gamma \rightarrow F \circ \gamma$ in norm too and $F \circ \gamma$ is adjointable. □

Proposition 1. *The dual module $(M(\ell_2(\mathcal{A})))'$ of $M(\ell_2(\mathcal{A}))$ consists of all sequences $\alpha_i \in M(\mathcal{A})$ such that*

- 1) *the partial sums of $\sum_i \alpha_i^* \alpha_i$ are bounded, i.e. this series is strong convergent in $B(H)$;*
- 2) *the series $\sum_i \alpha_i^* \beta_i$ is left strict convergent for any $\beta = \{\beta_i\} \in M(\ell_2(\mathcal{A}))$;*
- 3) *its limit belongs to $M(\mathcal{A}) \subseteq LM(\mathcal{A})$.*

Proof. Suppose, $\alpha \in (M(\ell_2(\mathcal{A})))'$, $\alpha : M(\ell_2(\mathcal{A})) \rightarrow M(\mathcal{A})$. Then its restriction on the submodule $\ell_2(M(\mathcal{A}))$ defines (by [2, Theorem 2.3]) a sequence $\alpha_i \in M(\mathcal{A})$ which must satisfy the property 1). It also can be restricted to

$\ell_2(\mathcal{A}) = M(\ell_2(\mathcal{A}))\mathcal{A}$, and also by [2, Theorem 2.3] the action is given by

$$\sum_{i=1}^{\infty} \alpha_i^* \beta_i a, \quad \{\beta_i\} \in M(\ell_2(\mathcal{A})), \quad a \in \mathcal{A}, \text{ the series is norm-convergent.} \quad (4)$$

This gives 2).

Two left multipliers u and v coincide, if $ua = va$ for any $a \in \mathcal{A}$. Thus, the equality

$$\alpha(\beta)a = \alpha(\beta a) = \sum_{i=1}^{\infty} \alpha_i^* \beta_i a = \left(\sum_{i=1}^{\infty} \alpha_i^* \beta_i \right) a$$

implies

$$\alpha(\beta) = \sum_{i=1}^{\infty} \alpha_i^* \beta_i \quad (5)$$

and hence 3).

Also, (5) implies that the linear mapping $\alpha \mapsto \{\alpha_i\}$ is injective.

Conversely, if $\{\alpha_i\}$ satisfies 1)-3), then (5) defines an element of the module $(M(\ell_2(\mathcal{A})))'$. Indeed, everything is evident, one needs only to verify that this α is bounded. For any $m < n$ and $a \in \mathcal{A}$ we have

$$\left(\sum_{i=m}^n \alpha_i^* \beta_i \right)^* \sum_{i=m}^n \alpha_i^* \beta_i \leq \left\| \sum_{i=m}^n \alpha_i^* \alpha_i \right\| \cdot \left\| \sum_{i=m}^n \beta_i^* \beta_i \right\|.$$

Hence, α is bounded, and the mapping is surjective. \square

Lemma 2. *An \mathcal{A} -functional $\Gamma : \ell_2(\mathcal{A}) \rightarrow \mathcal{A}$ is locally adjointable if and only if its collection of coefficients Γ_i determines an element of $(M(\ell_2(\mathcal{A})))'$.*

Proof. An \mathcal{A} -functional is defined by a sequence $\Gamma_i \in RM(\mathcal{A})$ such that

$$\sum_i \Gamma_i^* \Gamma_i \text{ strongly converges in } B(H) \quad (6)$$

(see [2] and (2) above). The action on $\alpha = (\alpha_1, \alpha_2, \dots) \in \ell_2(\mathcal{A})$ is defined by $\Gamma(\alpha) = \sum_i \Gamma_i^* \alpha_i$ and the series is norm-convergent. Suppose that Γ is locally adjointable. Consider an arbitrary adjointable morphism $\gamma : \mathcal{A} \rightarrow \ell_2(\mathcal{A})$. The set of these morphisms is isomorphic, on the one hand, to the space $(\ell_2(\mathcal{A}))^*$ of adjointable \mathcal{A} -functionals, and on the other hand, to the module $M(\ell_2(\mathcal{A}))$ (see [3, 2]). Namely there exist (by [3, Theorems 1.8 and 2.1]) $\gamma_i \in M(\mathcal{A})$ such that $\sum_i \gamma_i^* \gamma_i$ is strictly convergent and

$$\gamma(a) = (\gamma_1 a, \gamma_2 a, \dots), \quad a \in \mathcal{A}. \quad (7)$$

Then

$$\Gamma \circ \gamma(a) = \sum_i \Gamma_i^* \gamma_i a,$$

where the series $\sum_i \Gamma_i^* \gamma_i = \mu$ is convergent in left strict topology and defines an element $\mu \in LM(\mathcal{A})$. This gives property 2) of Proposition 1. This morphism $\mathcal{A} \rightarrow \mathcal{A}$ has to be adjointable and hence we have $\sum_i \Gamma_i^* \gamma_i \in M(\mathcal{A})$.

This gives 3) of Proposition 1. In particular, for $\gamma = (0, \dots, 0, 1_{M(\mathcal{A})}, 0, \dots)$, we have that Γ_i^* is an adjointable left multiplier, i.e. $\Gamma_i \in M(\mathcal{A})$. Together with (6) this gives 1) of Proposition 1.

The converse is similar. Indeed, from 1) it follows that the sequence $\{\Gamma_i\}$ defines an element of $(\ell^2(\mathcal{A}))'$ which acts by formula $\Gamma(x) = \sum_{i=1}^{\infty} \Gamma_i^* x_i$, where series is norm-convergent. In particular, for any adjointable $\gamma : \mathcal{A} \rightarrow \ell^2(\mathcal{A})$ and any $a \in \mathcal{A}$ we have $\Gamma(\gamma(a)) = \sum_{i=1}^{\infty} \Gamma_i^* \gamma_i a$. From 2) it follows that $\left(\sum_{i=1}^{\infty} \Gamma_i^* \gamma_i\right) a = \sum_{i=1}^{\infty} \Gamma_i^* \gamma_i a = \Gamma(\gamma(a))$, and from 3) it follows that $\left(\sum_{i=1}^{\infty} \Gamma_i^* \gamma_i\right) \in M(\ell^2(\mathcal{A}))$, i.e. $\sum_{i=1}^{\infty} \Gamma_i^* \gamma_i = \Gamma \circ \gamma$ is adjointable. \square

The following statement will be used below and also seems to be of independent interest.

Theorem 2. *A bounded \mathcal{A} -morphism $F : \mathcal{M} \rightarrow \ell_2(\mathcal{A})$ is adjointable if and only if all projections $\pi_k \circ F$, $k \in \mathbb{N}$, are adjointable.*

Proof. If F is adjointable then $\pi_k \circ F$ is adjointable since the projections π_k are adjointable.

Suppose that for any projection π_k we have that $\pi_k \circ F$ is adjointable. Then, for any $y = (y_1, y_2, \dots) \in \ell^2(\mathcal{A})$,

$$\begin{aligned} \left\| \sum_{k=p}^q (\pi_k \circ F)^*(\pi_k(y)) \right\| &= \sup_{z \in \mathcal{M}, \|z\| \leq 1} \left| \left\langle z, \sum_{k=p}^q (\pi_k \circ F)^*(\pi_k(y)) \right\rangle_{\mathcal{M}} \right| = \\ &= \sup_{z \in \mathcal{M}, \|z\| \leq 1} \left| \left\langle \sum_{k=p}^q (\pi_k \circ F)(z), \pi_k(y) \right\rangle_{\mathcal{A}} \right| = \\ &= \sup_{z \in \mathcal{M}, \|z\| \leq 1} \left| \left\langle F(z), \sum_{k=p}^q \pi_k^* \pi_k(y) \right\rangle \right| \leq \\ &\leq \|F\| \cdot \left\| \sum_{k=p}^q \pi_k^* \pi_k(y) \right\| = \|F\| \cdot \left\| \sqrt{\sum_{k=p}^q y_k^* y_k} \right\|. \end{aligned}$$

Since the series $\sum_{k=1}^{\infty} y_k^* y_k$ is norm-convergent, this implies that, for every $y \in \ell_2(\mathcal{A})$, the series $\sum_{k=1}^{\infty} (\pi_k \circ F)^*(\pi_k(y))$ is also norm-convergent in \mathcal{M} and the equality $S(y) = \sum_{k=1}^{\infty} (\pi_k \circ F)^*(\pi_k(y))$ defines a bounded \mathcal{A} -operator,

$S : \ell^2(\mathcal{A}) \rightarrow \mathcal{M}$. Also, for any $x \in \mathcal{M}$, $y \in \ell^2(\mathcal{A})$,

$$\begin{aligned} \langle F(x), y \rangle_{\ell^2(\mathcal{A})} &= \sum_{k=1}^{\infty} \langle \pi_k \circ F(x), \pi_k(y) \rangle_{\mathcal{A}} = \\ &= \sum_{k=1}^{\infty} \langle x, (\pi_k \circ F)^*(\pi_k(y)) \rangle = \langle x, S(y) \rangle, \end{aligned}$$

so F is adjointable with S being the adjoint operator. \square

Corollary 2. 1). *A bounded \mathcal{A} -morphism $F : \mathcal{M} \rightarrow \ell_2(\mathcal{A})$ is locally adjointable if and only if all its projections $\pi_k \circ F$, $k \in \mathbb{N}$, are locally adjointable.*

2). *An endomorphism F of the module $\ell_2(\mathcal{A})$ is locally adjointable if and only if its matrix rows belong to $M(\ell_2(\mathcal{A}))'$.*

Corollary 3. $M(\ell_2(\mathcal{A})) \subset (\ell_2(\mathcal{A}))'_{LA}$.

Proof. Indeed, $M(\ell_2(\mathcal{A})) = (\ell_2(\mathcal{A}))^* \subset (\ell_2(\mathcal{A}))'_{LA}$. \square

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