ENHANCED MODELLING OF SURVIVAL DATA: ESTIMATION AND SIMULATION WITH THE NEW-WEIBULL-G FAMILY IN ACCELERATED FAILURE TIME MODELS, VALIDITY BY THE NIKULIN-RAO-ROBSON TEST

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Abstract

This article explores the use of the Accelerated Failure Time (AFT) model in survival analysis, with a particular focus on the new generalization of the *Weibull* distribution, named New-*Weibull*-G family (NWG). The AFT model is commonly used method for analyzing survival data, as it provides a way to estimate the effect of covariates on the time to an event of interest. The New-*Weibull*-G family is a flexible and versatile distribution that can be used to model a wide range of survival data. The article begins with an overview of survival analysis and the AFT model, including an explanation of the New-*Weibull*-G distribution and its parameters. It then discusses the estimation of the AFT model, including maximum likelihood estimation and the statistic of the *Nikulin-Rao-Robson* to assess model fit. The article also provides examples of how AFT model with the NWG family can be used in practice, including the analysis of blood systolic pressure survival data and the prediction of time-to-failure for electrical components and Tumor size and emission of radioactivity. Overall, this article provides a comprehensive introduction to the use of the accelerated failure time model with the New generalization of classical *Weibull* distribution in survival analysis. It is intended for researchers, statisticians, and anyone interested in analysing survival data using advanced statistical methods.

Keywords: Accelerated Failure Time, AFT-models, covariates, Nikulin-Rao-Robson test, Weibull distribution.
MSC: 62F10, 62F03, 62P10, 62N03.

1 Introduction

Survival analysis plays a vital role in modelling and understanding the time-to-event data encountered in various fields such as medicine, engineering, and social sciences. Accurate estimation of survival distributions is crucial for making informed decisions and predictions. In this study, we focus on the new Weibull G family, a recently proposed distribution that shows promise in capturing complex survival patterns and overcoming the limitations of traditional distributions. The accelerated failure time (AFT) model is a widely used framework in survival analysis for relating covariates to the time-to-event outcome. Estimating the parameters of the AFT model is essential for understanding the underlying relationships and making reliable predictions. However, the choice of an appropriate distributional assumption for the survival times is critical, as it directly affects the accuracy of parameter estimation and subsequent inference. The New-*Weibull*-G family offers a flexible alternative to traditional distributions in the AFT model, with the ability to capture various hazard shapes and accommodate heterogeneity in the data. This family incorporates a shape parameter that governs the distribution's tail behaviour, providing greater flexibility than standard *Weibull*, exponential, or *Rayleigh* distributions. Despite its potential advantages, the New-*Weibull*-G (NWG) family lacks comprehensive research on estimation procedures, simulation techniques, and validation methods specific to the AFT model. Therefore, this study aims to bridge this gap by addressing the following objectives: (1) develop robust estimation procedures for the New-*Weibull*-G family in the AFT model, (2) assess the performance of the estimators through extensive simulation studies, and (3) propose a modified *Nikulin-Rao-Robson* (NRR) test to validate the adequacy of the New-*Weibull*-G family in capturing the observed survival data. By achieving these objectives, our research endeavours to provide practitioners and researchers with a better understanding of the estimation, simulation, and validation aspects related to the New-*Weibull*-G family in the AFT model. The findings have the potential to enhance the accuracy and reliability of survival analysis, leading to improved decision-making and predictions in various fields.

The subsequent sections of this paper are organized as follows: Section 2 provides a detailed overview of the AFT model for two special models of the New *Weibull*-G family: the New-*Weibull-Weibull* ($AFT-NWW$) and New-*Weibull-Raleigh* ($AFT-NWR$) models. Section 3 provides a detailed overview of the methodology, including the estimation procedures for the studied models, simulation techniques, and the proposed modified

Nikulin-Rao Robson test. Section 4 presents the results of our empirical studies and simulation experiments. In Section 5, we discuss the implications of our findings and their potential applications. Finally, Section 6 concludes the paper with a summary of the key contributions and avenues for future research.

2 Accelerated Failure time for New-Weibull-G family

Suppose that the n independent failure time variables $X = (X_1, ...X_n)^T$ are observed and consider that the hypothesis H_0 stating that the survival function given the vector of explanatory variables (covariates such as temperature, drug doses, stress, etc) :

$$z(t) = [z_0(x), z_1(x), ..., z_m(x)], \quad z_0(x) = 1,$$

has the form

$$S(x/z) = S_0\left(\int_0^x e^{-\beta^T z(u)} du, \zeta\right),$$

where $\beta = (\beta_0, \beta_1, ..., \beta_m)^T$ is the unknown m-dimensional regression vector parameters, the function S_0 is specified functional of time and does not depend on z_i. If explanatory variables are constant over time, the parametric accelerated failure time (AFT) model has form $S(x/z) = S_0\left(xe^{-\beta^T z}, \zeta\right)$.

Let us consider the New-*Weibull*-G family (NWG) as baseline distribution S_0:

$$H_0 : F(x) = F_{AFT-NWG}(x).$$

So, the cumulative distribution function (cdf) of the Accelerated failure New-*Weibull*-G family (AFT $-NWG$) is :

$$F_{AFT-NWG}(x; \alpha, \xi, \eta) = \exp\left(-\alpha\left\{-\log\left[G\left(xe^{-\beta^T z}, \eta\right)\right]\right\}^{\xi}\right).$$

Then, the corresponding probability density (pdf), hazard rate (hrf) and survival functions are given by :

$$f(x, \alpha, \xi, \eta) = \alpha\beta \frac{g\left(xe^{-\beta^T z}, \eta\right)}{G\left(xe^{-\beta^T z}\right)} \left\{-\log\left[G\left(xe^{-\beta^T z}, \eta\right)\right]\right\}^{\beta-1} \exp\left(-\alpha\left\{-\log\left[G\left(xe^{-\beta^T z}\eta\right)\right]\right\}^{\beta}\right),$$

$$\lambda(x, \alpha, \xi, \eta) = \frac{\alpha\beta \frac{g\left(xe^{-\beta^T z}, \eta\right)}{G\left(xe^{-\beta^T z}, \eta\right)} \left\{-\log\left[G\left(xe^{-\beta^T z}, \eta\right)\right]\right\}^{\beta-1} \exp\left(-\alpha\left\{-\log\left[G\left(xe^{-\beta^T z}, \eta\right)\right]\right\}^{\beta}\right)}{1 - \exp\left(-\alpha\left\{-\log\left[G\left(xe^{-\beta^T z}, \eta\right)\right]\right\}^{\beta}\right)},$$

$$S_{AFT-NWG}(x; \alpha, \xi, \eta) = 1 - \exp\left(-\alpha\left\{-\log\left[G\left(xe^{-\beta^T z}, \eta\right)\right]\right\}^{\xi}\right)$$

2.1 Accelerated Failure time for the New-Weibull-Weibull model (AFT-NWW)

We replace the parent distribution $G(x, \eta)$ by traditional *Weibull* distribution $G(x, \lambda, \gamma) = 1 - e^{-\lambda x^{\gamma}}$. We introduce the 4 parameters N-*Weibull*-*Weibull* (NWW) cumulative distribution function (cdf) by :

$$F_{NWW}(x; \alpha, \xi, \lambda, \gamma) = \exp\left\{-\alpha\left[-\log\left(1 - e^{-\lambda x^{\gamma}}\right)\right]^{\xi}\right\}, \quad x > 0, \quad \alpha, \xi, \lambda, \gamma > 0,$$

with the scale parameter represented by α, ξ and γ represent the shape parameters and λ the location parameter. Therefore, we can define the $AFT - NWW$ distribution by the cumulative distribution :

$$F_{NWW-AFT}(x; \vartheta) = \exp\left[-\alpha\left(-\log\left\{1 - \exp\left[-\lambda\left(xe^{-\beta^T z}\right)^{\gamma}\right]\right\}\right)^{\xi}\right], \quad \vartheta = (\alpha, \xi, \lambda, \gamma).$$

Then, the corresponding probability density function (pdf), survival and hazard rate (hrf) functions are expressed as :

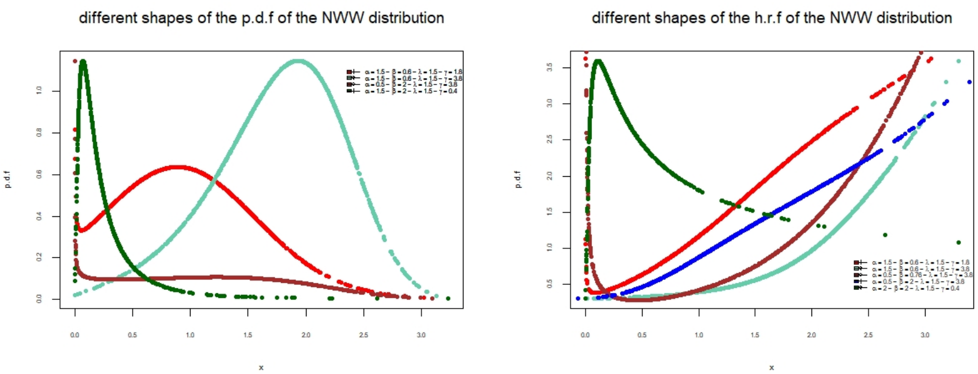
$$\begin{aligned} f(x; \vartheta) &= \alpha\xi\lambda\gamma \frac{\left(xe^{-\beta^T z}\right)^{\gamma-1} \exp\left[-\lambda\left(xe^{-\beta^T z}\right)^{\gamma}\right]}{1 - \exp\left[-\lambda\left(xe^{-\beta^T z}\right)^{\gamma}\right]} \left(-\log\left\{1 - \exp\left[-\lambda\left(xe^{-\beta^T z}\right)^{\gamma}\right]\right\}\right)^{(\xi-1)} \\ &\times \exp\left[-\alpha\left(-\log\left\{1 - \exp\left[-\lambda\left(xe^{-\beta^T z}\right)^{\gamma}\right]\right\}\right)^{\xi}\right] \end{aligned}$$

$$S\left(x;\vartheta\right) = S_0\left(xe^{-\beta^T z}\right) = 1-\exp\left[-\alpha\left(-\log\left\{1-\exp\left[-\lambda\left(xe^{-\beta^T z}\right)^{\gamma}\right]\right\}\right)^{\xi}\right],$$

$$\lambda\left(x;\vartheta\right) = \alpha\xi\lambda\gamma\frac{\left(xe^{-\beta^T z}\right)^{\gamma-1}\left(-\log\left\{1-\exp\left[-\lambda\left(xe^{-\beta^T z}\right)^{\gamma}\right]\right\}\right)^{(\xi-1)}}{\left\{\exp\left[\lambda\left(xe^{-\beta^T z}\right)^{\gamma}\right]-1\right\}\times\left\{\exp\left[\alpha\left(-\log\left\{1-\exp\left[-\lambda\left(xe^{-\beta^T z}\right)^{\gamma}\right]\right\}\right)^{\xi}\right]-1\right\}}.$$

Respectively. For $\gamma=1$, we obtain as special model the AFT N-*Weibull*-exponential (NWE) distribution.

To show the quality and flexibility of the $AFT-NWW$ distribution, we plot in Figure1 the probability density function and the hazard rate function. Figure1 gives some plots of the $NWW-AFT$ pdf and hrf for selected parameter values, respectively.



(a) *pdf* (b) hrf

Figure 1: Different shapes of the pdf and hrf of the $AFT-NWW$ distribution.

Figure 1 depicts the $AFT-NWW$ distribution's ability to produce a range of pdf shapes, including left- and right-skewed, bathtub, and bell shapes, as well as the family's capacity to produce increasing, bathtub, bell, and J-shaped hazard rates. The $AFT-NWW$ distribution is in fact quite useful for fitting a variety of data sets with different real application domains

2.2 Accelerated Failure time N-Weibull-Rayleigh (AFTNWR)

By replacing the parent distribution $G\left(x,\eta\right)$ of the NWG family by the classical *Rayleigh* distribution $G\left(x,\lambda\right)=1-e^{-\lambda x^2}$, we define the cumulative distribution function of the N-*Weibull-Rayleigh* distribution as :

$$F_{NWR}\left(x;\alpha,\xi,\lambda\right)=\exp\left\{-\alpha\left[-\log\left(1-e^{-\lambda x^2}\right)\right]^{\xi}\right\}.$$

Under the null hypothesis H_0, we can then define the $AFT-NWR$ distribution by the cdf given by :

$$F_{NWR-AFT}\left(x;\varsigma\right)=\exp\left[-\alpha\left(-\log\left\{1-\exp\left[-\lambda\left(xe^{-\beta^T z}\right)^{2}\right]\right\}\right)^{\xi}\right],\quad \vartheta=\left(\alpha,\xi,\lambda,\beta_0,\beta_1\right).$$

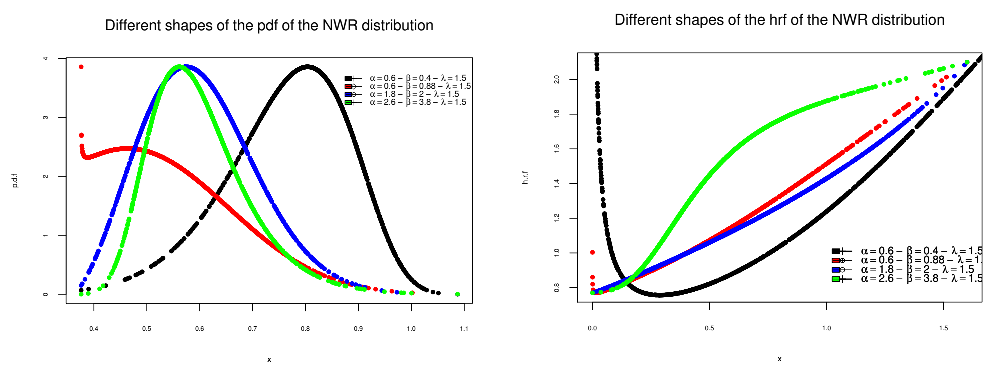
The corresponding probability density function (pdf), survival and hazard rate (hrf) functions of the $AFT-NWR$ distribution are expressed as :

$$f\left(x;\varsigma\right) = \alpha\xi\lambda\gamma\frac{\left(xe^{-\beta^T z}\right)\exp\left[-\lambda\left(xe^{-\beta^T z}\right)^{2}\right]}{1-\exp\left[-\lambda\left(xe^{-\beta^T z}\right)^{\gamma}\right]}\left(-\log\left\{1-\exp\left[-\lambda\left(xe^{-\beta^T z}\right)^{2}\right]\right\}\right)^{(\xi-1)}$$
$$\times\exp\left[-\alpha\left(-\log\left\{1-\exp\left[-\lambda\left(xe^{-\beta^T z}\right)^{2}\right]\right\}\right)^{\xi}\right]$$

$$\begin{aligned} S(x;\varsigma) &= 1-\exp\left[-\alpha\left(-\log\left\{1-\exp\left[-\lambda\left(xe^{-\beta^T z}\right)^2\right]\right\}\right)^{\xi}\right], \\ \lambda(x;\varsigma) &= 2\alpha\xi\lambda\frac{\left(xe^{-\beta^T z}\right)\left(-\log\left\{1-\exp\left[-\lambda\left(xe^{-\beta^T z}\right)^2\right]\right\}\right)^{(\xi-1)}}{\left\{\exp\left[\lambda\left(xe^{-\beta^T z}\right)^2\right]-1\right\}\times\left\{\exp\left[\alpha\left(-\log\left\{1-\exp\left[-\lambda\left(xe^{-\beta^T z}\right)^2\right]\right\}\right)^{\xi}\right]-1\right\}}, \end{aligned}$$

respectively.

We display in Figure 2 the pdf and hrf of the $AFT-NWR$ distribution in order to demonstrate the flexibility and variability of this distribution ($AFT-NWR$)



(a) *pdf* (b) hrf

Figure 2: Different shapes of the pdf and hrf of the $AFT-NWR$ distribution.

Figure 2 shows that the $AFT-NWR$ distribution generates a range of pdf shapes, including symmetric, left and right-skewed, decreasing and bell-shaped, as well as the family's ability to produce several hazard rate shapes, such as increasing, bathtub, bell, and S form. In fact the $AFT-NWR$ distribution can be highly helpful for fitting a range of data sets with variable forms in many practical applications.

3 Adequacy of AFT -NWG family

Nikulin and *Rao-Robson* (NRR) (1973, 1974) proposed a chi-squared type goodness-of-fit-test for completed data, the NRR test is based on maximum likelihood estimation for ungrouped data. For testing H_0 where $X=(X_1,X_2,...,X_n)^T$ is a random sample provided from a parametric distribution $F(x,\vartheta)$, we have :

$$H_0: P(X_i\le x)=F(x,\vartheta)\ ;\ x\in\mathbb{R};\theta=(\theta_1,...,\theta_s)^T\in\Theta\in\mathbb{R}^s.$$

Where $\theta=(\theta_1,...,\theta_s)^T\in\Theta\in\mathbb{R}^s$ is unknown s-dimensional vector. Let $X_1,X_2,...,X_n$ be grouped in r sub-intervals $I_1,I_2,...,I_r$ as :

$$I_j=\left]a_{j-1},a_j\right],\quad j=1,2,...,r,$$

$-\infty=a_0<a_1\le a_2\le ...<a_{r-1}=+\infty$, which are mutually disjoint. The limit $a_{(.)}$ of the intervals I_j are calculated such that :

$$a_j=F^{-1}\left(\frac{j}{r}\right),\quad j=\overline{1,r},\quad \widehat{a}_r=+\infty.$$

So, a_j are random data functions such as the r intervals chosen have equal expected numbers of failure p_j

$$p_j=\int_{a_{j-1}}^{a_j}F(x)\,dx,$$

where F^{-1} is the inverse of the cumulative distribution F.

Let the vector of empirical frequencies ν_j obtained by grouping the sample $X = (X_1, ..., X_n)^T$ in the intervals I_j be :

$$\nu_j = (\nu_1, \nu_2, ..., \nu_r)^T,$$

where :

$$\nu_j = \sum_{i=1}^{n} 1_{\{X_i \in I_j\}}, \qquad j = \overline{1, r}.$$

So, the NRR test is based on the vector :

$$X_n(\vartheta) = \left(\frac{\nu_1 - np_1(\vartheta)}{\sqrt{np_1(\vartheta)}}, ..., \frac{\nu_r - np_r(\vartheta)}{\sqrt{np_r(\vartheta)}} \right).$$

Under the null hypothesis H_0, the NRR statistic Y^2 can derived as :

$$\begin{aligned} Y^2 &= X_n^2\left(\widehat{\vartheta}\right) + Q_n\left(\widehat{\vartheta}\right) \\ &= X_n^2\left(\widehat{\vartheta}\right) + \frac{1}{n} L^T\left(\widehat{\vartheta}\right) \times \left[I\left(\widehat{\vartheta}\right) - J\left(\widehat{\vartheta}\right) \right]^{-1} \times L\left(\widehat{\vartheta}\right) \end{aligned}$$

Where $I\left(\widehat{\vartheta}\right)$ is the estimated *Fisher* information matrix, $J\left(\widehat{\vartheta}\right)$ is the information matrix for the grouped data which can be defined as : $J\left(\widehat{\vartheta}\right) = B^T\left(\widehat{\vartheta}\right) \times B\left(\widehat{\vartheta}\right)$, where :

$$B\left(\widehat{\vartheta}\right) = \left[\frac{1}{\sqrt{p_j(\vartheta)}} \frac{\partial p_j}{\partial \vartheta_s} \right]_{r \times s}, \quad j = 1, 2, ..., r.$$

and,

$$\begin{aligned} L &= \left[L_1\left(\widehat{\vartheta}\right), ..., L_s\left(\widehat{\vartheta}\right) \right]^T \quad ; \quad L_s\left(\widehat{\vartheta}\right) = \sum_{j=1}^{r} \frac{\nu_j}{p_j} \frac{\partial p_j}{\partial \vartheta_s}, \\ X_n^2\left(\widehat{\vartheta}\right) &= X_n^T(\vartheta) \times X_n(\vartheta) = \sum_{j=1}^{r} \frac{\left[\nu_j - np_j\left(\widehat{\vartheta}\right) \right]^2}{np_j}, \end{aligned}$$

The hypothesis is rejected with approximate significance level if $Y_n^2 > \chi_\epsilon(r-1)$ where χ_ϵ is the quantile of chi-square with $(r-1)$ degrees of freedom.

$$\lim \mathbf{P}\left[Y_n^2\left(\widehat{\vartheta}\right) \geq x \right] = \mathbf{P}\left(\chi_{r-1} \geq x\right).$$

For more details, see *Nikulin* (1973a, 1973b).

For testing the goodness-of-fit of a parametric family of survival distribution from right censored data, Habib and Thomas (1986), Hollander and Pena (1992) considered natural modifications of the *Nikulin-Rao-Robson* (NRR) statistic for data without covariates. Also, Hjort (1990), Pena (1998) considered goodness-of-fit for parametric Cox models, Gray and Pierce (1985), Akritas and Torbeyns (1997), and Zhang (1999) for linear regression models. In 2011, *Bagdonavicius* and *Nikulin* gave a modified chi-squared type goodness-of-fit-test for right-censored data with possibly time-dependent covariates and considered random grouping intervals as data functions. These tests are based on maximum likelihood estimation for ungrouped data, and random grouping intervals are considered as data functions. We adapt this test for an accelerated failure time model for the N-*Weibull*-G family. Suppose the null hypothesis H_0 satisfies :

$$H_0 : F(x) \in F_0 = F_0(x, \theta), x \in \mathbb{R}^1, \theta \in \Theta \subset \mathbb{R}^s,$$

where F_0 is a known cumulative distribution function and $\theta = (\theta_1, ..., \theta_s)^T$ i is s-dimensional unknown vector of parameters of F_0. Let us consider a finite time interval only say $[0, \tau]$ and divide it into$k > s$ smaller intervals

$$Ij = (a_{j-1}, a_j], 0 \leq a_0 < a_1 < ... < a_{k-1} < a_k = +\infty.$$

The estimated a_j in this case is given by :

$$\begin{aligned}\widehat{a}_j &= \Lambda^{-1}\left\{\frac{E_j-\sum_{l=1}^{i-1}\Lambda\left(X_{(l)},\widehat{\theta}\right)}{n-i+1},\widehat{\theta}\right\},\\ \widehat{a}_k &= \max\left(X_{(n)},\tau\right),\quad j=\overline{1,k}.\end{aligned}$$

Where $\widehat{\theta}$ is the Maximum likelihood estimator of the parameter θ, Λ^{-1} is the inverse of the cumulative hazard function Λ, $X_{(i)}$ is the *ith* element in the ordered statistics $(X_1,...,X_n)$ and

$$E_J=(n-i+1)\,\Lambda\left(\widehat{a}_j,\widehat{\theta}\right)+\sum_{l=1}^{i-1}\Lambda\left(X_{(l)},\widehat{\theta}\right),$$

a_j are random data functions such as the k intervals chosen have equal expected numbers of failures e_j. The modified NRR test for right censored data is based on the statistic :

$$Y_n^2=Z^T\widehat{\Sigma}^-Z,$$

where $\widehat{\Sigma}^-=\widehat{A}^{-1}+\widehat{A}^{-1}\widehat{C}^T\widehat{G}-\widehat{C}\widehat{A}^{-1}$. The vector Z is given by :

$$Z=(Z_1,...,Z_k)^T\ ;\ Z_j=\frac{1}{\sqrt{n}}\left(U_j-E_j\right)\ ;\ j=\overline{1,k}.$$

U_j represent the number of observed failures in the intervals I_j. Under the hypothesis H_0, the limit distribution of the test statistic :

$$Y_n^2=\sum_{i=1}^{n}\frac{\left(U_j-e_j\right)^2}{U_j}+Q,$$

where,

$$Q=W^T\widehat{G}^{-1}W,\quad W=\widehat{C}\widehat{A}^{-1}Z=(W_1,...,W_s)^T,$$

$$\widehat{G}=[\widehat{g}_{ll'}]_{s\times s},\quad \widehat{g}_{ll'}=\widehat{i}_{ll'}-\sum_{j=1}^{k}\widehat{C}_{lj}\widehat{C}_{l'j}\widehat{A}_j^{-1},\quad W_l=\sum_{j=1}^{k}\widehat{C}_{lj}\widehat{A}_j^{-1}Z_j,$$

$$\widehat{i}_{ll'}=\frac{1}{n}\sum_{j=1}^{n}\delta_i\frac{\partial\ln\lambda\left(X_i,\widehat{\theta}\right)}{\partial\theta_l}\frac{\partial\ln\lambda\left(X_i,\widehat{\theta}\right)}{\partial\theta_{l'}},\quad \widehat{\theta}=(\theta_1,...,\theta_s),$$

$$\widehat{C}_{lj}=\frac{1}{n}\sum_{i:X_i\in I_j}\delta_i\frac{\partial}{\partial\theta_l}\ln\lambda\left(X_i,\widehat{\theta}\right),\quad j=1,...k,\quad l,l'=1,...,s.$$

We reject the null hypothesis H_0 if $Y_n^2>\chi_\epsilon^2(r)$ with approximate significance level α. This test statistic was used to fit different distributions, for more details see *Bagdonavicius* and *Nikulin* (2011), *Bagdonavicius et al.* (2011), *Goual* and *Seddik-Ameur* (2014).

3.1 Adequacy of the $AFT-NWW$ model

3.1.1 The NRR test in completed case

The maximum likelihood estimators (MLEs) of the unknown parameters are necessary for the construction of the NRR statistic and other selection model criteria, so, first, we provide the score functions for the studied models NWW and NWR, for more details see B*alakrishnan* and *Kateri* (2008), *Nikulin* (1973*b*)

Let $X=(X_1,...,X_n)$ be a random sample from the $AFT-NWW$ with parameter vector $\vartheta=(\alpha,\xi,\lambda,\gamma,\beta_0,\beta_1)^T$. The log-likelihood function $\ell(x;\vartheta)$ for ϑ can given by :

$$\begin{aligned}\ell(x;\vartheta) =\ & n\log(\alpha\xi\lambda\gamma)+(\gamma-1)\sum_{i=1}^{n}\log\left(x_ie^{-\beta^Tz}\right)-\lambda\sum_{i=1}^{n}\left(x_ie^{-\beta^Tz}\right)^\gamma-\sum_{i=1}^{n}\log\left\{1-\exp\left[-\lambda\left(x_ie^{-\beta^Tz}\right)^\gamma\right]\right\}\\ &-\alpha\sum_{i=1}^{n}\left(-\log\left\{1-\exp\left[-\lambda\left(x_ie^{-\beta^Tz}\right)^\gamma\right]\right\}\right)^\xi+(\xi-1)\sum_{i=1}^{n}\log\left(-\log\left\{1-\exp\left[-\lambda\left(x_ie^{-\beta^Tz}\right)^\gamma\right]\right\}\right)\end{aligned}$$

The score functions are obtained by deriving the log-likelihood function relative to the unknown parameters $\widehat{\vartheta}=\left(\widehat{\alpha},\widehat{\xi},\widehat{\lambda},\widehat{\gamma},\widehat{\beta}_0,\widehat{\beta}_1\right)^T$ of the $AFT-NWW$ distribution :

$$\begin{aligned}
\frac{\partial \ell(x_i,\vartheta)}{\partial \alpha} &= \frac{n}{\alpha}-\sum_{i=1}^{n}\left[U(x_i,\vartheta)\right]^{\xi}\\
\frac{\partial \ell(x_i,\vartheta)}{\partial \xi} &= \frac{n}{\xi}+\sum_{i=1}^{n}\log\left[U(x_i,\vartheta)\right]\times\left[1-\alpha\left[U(x_i,\vartheta)\right]^{\xi}\right]\\
\frac{\partial \ell(x_i,\vartheta)}{\partial \lambda} &= \frac{n}{\lambda}-\sum_{i=1}^{n}\left(x_ie^{-\beta^Tz}\right)^{\gamma}-(\xi-1)\sum_{i=1}^{n}\frac{M(x_i,\vartheta)}{D(x_i,\vartheta)\times U(x_i,\vartheta)}-\sum_{i=1}^{n}\frac{M(x_i,\vartheta)}{D(x_i,\vartheta)}\left[1-\alpha\xi\left[U(x_i,\vartheta)\right]^{\xi-1}\right],\\
\frac{\partial \ell(x_i,\vartheta)}{\partial \gamma} &= \frac{n}{\gamma}+\sum_{i=1}^{n}\log\left(x_ie^{-\beta^Tz}\right)\left[1-\lambda\left(x_ie^{-\beta^Tz}\right)^{\gamma}\right]-\lambda(\xi-1)\sum_{i=1}^{n}\frac{M(x_i,\vartheta)\log\left(xe^{-\beta^Tz}\right)}{D(x_i,\vartheta)\times U(x_i,\vartheta)}\\
&\quad -\lambda\sum_{i=1}^{n}\frac{M(x_i,\vartheta)\log\left(x_ie^{-\beta^Tz}\right)}{D(x_i,\vartheta)}\left[1-\alpha\xi\left[U(x_i,\vartheta)\right]^{\xi-1}\right]\\
\frac{\partial \ell(x_i,\vartheta)}{\partial \beta_0} &= -1+\lambda(\xi-1)\gamma\sum_{i=1}^{n}\frac{M(x_i,\vartheta)}{D(x_i,\vartheta)\times U(x_i,\vartheta)}+\lambda\gamma\sum_{i=1}^{n}\frac{M(x_i,\vartheta)}{D(x_i,\vartheta)}\left[1-\alpha\xi\left[U(x_i,\vartheta)\right]^{\xi-1}\right],\\
\frac{\partial \ell(x_i,\vartheta)}{\partial \beta_1} &= -\sum_{i=1}^{n}z_i-+\lambda(\xi-1)\gamma\sum_{i=1}^{n}z_i\frac{M(x_i,\vartheta)}{D(x_i,\vartheta)\times U(x_i,\vartheta)}+\lambda\gamma\sum_{i=1}^{n}z_i\frac{M(x_i,\vartheta)}{D(x_i,\vartheta)}\left[1-\alpha\xi\left[U(x_i,\vartheta)\right]^{\xi-1}\right].
\end{aligned}$$

Where

$$\begin{aligned}
U\left(x_i,\vartheta\right) &= -\log\left\{1-\exp\left[-\lambda\left(xe^{-\beta^Tz}\right)^{\gamma}\right]\right\},\\
M\left(x_i,\vartheta\right) &= \left(xe^{-\beta^Tz}\right)^{\gamma}\exp\left[-\lambda\left(xe^{-\beta^Tz}\right)^{\gamma}\right],\\
D\left(x_i,\vartheta\right) &= 1-\exp\left[-\lambda\left(xe^{-\beta^Tz}\right)^{\gamma}\right],\quad \vartheta=(\alpha,\xi,\lambda,\gamma).
\end{aligned}$$

So, the maximum likelihood estimators of ϑ are obtained by equalling to zero the first derivatives obtained above. To solve these equations, iterative methods are required using different software, R, MATLAB, SPSS and others. The elements of the estimated Information Fisher Matrix (IFM) are also derived and given in appendix 1.

Now, Suppose that a sample $X=(X_1,...,X_n)$, is distributed by a $AFT-NWW$ distribution, the problem is to test the null hypothesis H_0 such :

$$H_0: P(x_i\le X)=F_{AFT-NWW}(x,\vartheta),x>0,\vartheta=(\alpha,\beta,\lambda,\gamma)^T.$$

The observations are grouped into r intervals I_j, ($I_j=]a_{j-1},a_j]$. AFT er computing the MLEs of the unknown parameters vector ϑ, we calculated the estimated limits $\widehat{a}_j$ such that the grouped intervals are equiprobable, we obtain :

$$\widehat{a}_j=e^{-\beta^Tz}\left[-\frac{\log\left(1-\exp\left\{-\left[-\frac{\log\left(\frac{j}{r}\right)}{\widehat{\alpha}}\right]^{\frac{1}{\widehat{\beta}}}\right\}\right)}{\widehat{\lambda}}\right]^{\frac{1}{\widehat{\gamma}}}.$$

The elements of the Y_n^2 statistic test are calculated and expressed in appendix 1.

3.1.2 The modified NRR test in right-censored case

Let $X_{i_{\left(i=\overline{1,n}\right)}}$, be a random variable distributed with the vector of parameters Θ. The data encountered in survival analysis and reliability studies are often censored. A very simple random censoring mechanism that is often realistic is one in which each individual i is assumed to have a life time X_i and a censoring time C_i, where X_i and C_i are independent random variables, for more details see.... Suppose that the data consist of

n independent observations $x_i = min(X_i, C_i)$, for $i = 1, ..., n$. The distribution of C_i does not depend on any of the unknown parameters of X_i. The likelihood function can be given as follow :

$$L(\theta) = \Pi_{i=1}^{n} \lambda^{\delta_i}(X_i, \theta) S(X_i, \theta) = 1\{X_i \leq C_i\}, \theta \in \Theta.$$

Then the log-likelihood function

$$l(\theta) = \sum_{i=1}^{n} \delta_i \ln \lambda(X_i, \theta) + \sum_{i=1}^{n} S(X_i, \theta), \quad \theta \in \Theta.$$

We obtain the likelihood of $AFT-NWW$ model by substituting the hazard rate and survival functions with those of the $AFT-NWW$ model given in section 2.

$$\begin{aligned} l_{AFT-NWW}(x, \vartheta) &= r\log(\alpha\beta\lambda\gamma) + (\gamma-1)\sum_{i\in F}\log\left(x_i e^{-\beta^T z}\right) - \lambda\sum_{i\in F} x_i^{\gamma} e^{-\beta^T z} + \sum_{i\in C}\log\left[1-\exp\left(-\alpha\left\{-\log\left[1-\exp\left(-\lambda x_i^{\gamma} e^{-\beta^T z}\right)\right]\right\}^{\beta}\right)\right] \\ &\quad -\alpha\sum_{i\in F}\left\{-\log\left[1-\exp\left(-\lambda x_i^{\gamma} e^{-\beta^T z}\right)\right]\right\}^{\beta} - \sum_{i\in F}\log\left[1-\exp\left(-\lambda x_i^{\gamma} e^{-\beta^T z}\right)\right] \\ &\quad +(\beta-1)\sum_{i\in F}\log\left\{-\log\left[1-\exp\left(-\lambda x_i^{\gamma} e^{-\beta^T z}\right)\right]\right\}, \end{aligned}$$

where r is the number of failures and F and C denote the sets of uncensored and censored observations, respectively. The score functions of the unknown vector of parameters ϑ and the elements of Information Fisher Matrix of the $AFT-NWW$ distribution are calculated and given in appendix 1.

The choice of $\widehat{a_j}$ when the baseline distribution is a New-*Weibull*-*Weibull* (NWW), is obtained as follow :

$$\widehat{a}_j = e^{-\beta^T z}\left\{-\frac{1}{\widehat{\lambda}}\log\left[1-\exp\left(-\left\{-\frac{1}{\widehat{\alpha}}\log\left[1-\exp\left(\frac{\sum_{l=1}^{i-1}\Lambda\left(X_{(l)}, \widehat{\vartheta}\right)-E_j}{N-i+1}\right)\right]\right\}^{\frac{1}{\widehat{\beta}}}\right)\right]\right\}^{\frac{1}{\widehat{\gamma}}}, \qquad \widehat{a}_k = \max\left(X_{(n)}, \tau\right)$$

The elements of the Y_n^2 statistic test of the $AFT-NWW$ model are calculated and expressed in appendix 1.

3.2 Adequacy of the $AFT-NWR$ model

3.2.1 The NRR test in completed case

A random sample from the $AFT-NWR$ with a parameter vector $\varsigma = (\alpha, \xi, \lambda, \beta_0, \beta_1)^T$ is denoted by $X = (X_1, ..., X_n)$. The likelihood function of the $AFT-NWR$ model can given by :

$$\begin{aligned} L(x;\varsigma) &= \prod_{i=1}^{n} f(x_i;\varsigma) \\ &= \prod_{i=1}^{n}\{2\alpha\xi\lambda\frac{\left(x_i e^{-\beta^T z}\right)\exp\left[-\lambda\left(x_i e^{-\beta^T z}\right)^2\right]}{1-\exp\left[-\lambda\left(x_i e^{-\beta^T z}\right)^2\right]}\left(-\log\left\{1-\exp\left[-\lambda\left(x_i e^{-\beta^T z}\right)^2\right]\right\}\right)^{(\xi-1)} \\ &\quad \times\exp\left[-\alpha\left(-\log\left\{1-\exp\left[-\lambda\left(x_i e^{-\beta^T z}\right)^2\right]\right\}\right)^{\xi}\right]\}. \end{aligned}$$

The log-likelihood function is :

$$\begin{aligned} \ell(x;\varsigma) &= n\log(2\alpha\xi\lambda) + \sum_{i=1}^{n}\log\left(x_i e^{-\beta^T z}\right) - \lambda\sum_{i=1}^{n}\left(x_i e^{-\beta^T z}\right)^2 - \sum_{i=1}^{n}\log\left\{1-\exp\left[-\lambda\left(x_i e^{-\beta^T z}\right)^2\right]\right\} \\ &\quad -\alpha\sum_{i=1}^{n}\left(-\log\left\{1-\exp\left[-\lambda\left(x_i e^{-\beta^T z}\right)^2\right]\right\}\right)^{\xi} + (\xi-1)\sum_{i=1}^{n}\log\left(-\log\left\{1-\exp\left[-\lambda\left(x_i e^{-\beta^T z}\right)^2\right]\right\}\right). \end{aligned}$$

The score functions for the parameters α, ξ, λ, β_0 and β_1 are given by :

$$
\begin{aligned}
\frac{\partial \ell\left(x_i,\varsigma\right)}{\partial \alpha} &= \frac{n}{\alpha}-\sum_{i=1}^{n}\left[U\left(x_i,\vartheta\right)\right]^{\xi}, \\
\frac{\partial \ell\left(x_i,\varsigma\right)}{\partial \xi} &= \frac{n}{\xi}+\sum_{i=1}^{n}\log\left[U\left(x_i,\vartheta\right)\right]\times\left[1-\alpha\left[U\left(x_i,\vartheta\right)\right]^{\xi}\right], \\
\frac{\partial \ell\left(x_i,\varsigma\right)}{\partial \lambda} &= \frac{n}{\lambda}-\sum_{i=1}^{n}\left(x_i e^{-\beta^T z}\right)^2-(\xi-1)\sum_{i=1}^{n}\frac{M\left(x_i,\vartheta\right)}{D\left(x_i,\vartheta\right)\times U\left(x_i,\vartheta\right)}-\sum_{i=1}^{n}\frac{M\left(x_i,\vartheta\right)}{D\left(x_i,\vartheta\right)}\left[1-\alpha\xi\left[U\left(x_i,\vartheta\right)\right]^{\xi-1}\right], \\
\frac{\partial \ell\left(x_i,\varsigma\right)}{\partial \beta_0} &= -1+2\lambda(\xi-1)\sum_{i=1}^{n}\frac{M\left(x_i,\vartheta\right)}{D\left(x_i,\vartheta\right)\times U\left(x_i,\vartheta\right)}+2\lambda\sum_{i=1}^{n}\frac{M\left(x_i,\vartheta\right)}{D\left(x_i,\vartheta\right)}\left[1-\alpha\xi\left[U\left(x_i,\vartheta\right)\right]^{\xi-1}\right], \\
\frac{\partial \ell\left(x_i,\varsigma\right)}{\partial \beta_1} &= -\sum_{i=1}^{n}z_i+2\lambda\sum_{i=1}^{n}z_i\frac{M\left(x_i,\vartheta\right)}{D\left(x_i,\vartheta\right)}\left\{\frac{\xi-1}{U\left(x_i,\vartheta\right)}+1-\alpha\xi\left[U\left(x_i,\vartheta\right)\right]^{\xi-1}\right\}.
\end{aligned}
$$

Where,

$$
\begin{aligned}
U(x,\varsigma) &= -\log\left\{1-\exp\left[-\lambda\left(xe^{-\beta^T z}\right)^2\right]\right\}, \\
M(x,\varsigma) &= \left(xe^{-\beta^T z}\right)^2\exp\left[-\lambda\left(xe^{-\beta^T z}\right)^2\right], \\
D(x,\varsigma) &= 1-\exp\left[-\lambda\left(xe^{-\beta^T z}\right)^2\right], \quad \vartheta=(\alpha,\xi,\lambda).
\end{aligned}
$$

The maximum likelihood estimators $\widehat{\upsilon}=\left(\widehat{\alpha},\widehat{\xi},\widehat{\lambda},\widehat{\beta}_0,\widehat{\beta}_1\right)^T$ of the $AFT\ -NWR$ distribution can be obtained by equalling to zero the first derivatives calculated above. To solve these equations, iterative methods are required using different software R, Matlab, SPSS and others. The elements of the estimated Information Fisher Matrix (IFM) a are derived and given in appendix 2.

Assuming a sample $X=(X_1,...,X_n)$ follows $AFT-NWW$ distribution, the issue is to test the null hypothesis H_0 such :

$$
H_0: P\left(x_i \le X\right)=F_{AFT-NWW}(x,\vartheta), x>0, \vartheta=(\alpha,\beta,\lambda,\gamma)^T.
$$

The observations are grouped into r intervals I_j, ($I_j =]a_{j-1},a_j]$. AFT er computing the MLEs of the unknown parameters vector ϑ, we calculated the estimated limits $\widehat{a}_j$ such that the grouped intervals are equiprobable, we obtain :

$$
\widehat{a}_j = e^{-\beta^T z}\left[-\frac{\log\left(1-\exp\left\{-\left[-\frac{\log\left(\frac{j}{r}\right)}{\widehat{\alpha}}\right]^{\frac{1}{\widehat{\beta}}}\right\}\right)}{\widehat{\lambda}}\right]^{\frac{1}{\widehat{\gamma}}}.
$$

The elements of the Y_n^2 statistic test are calculated and expressed in appendix 2.

3.2.2 The modified NRR test in censored case

We suppose the failure rate $X_i \rightsquigarrow AFT\ -NWR(x,\varsigma)$. We calculate the maximum likelihood estimators of the vector of parameters $\varsigma=(\alpha,\beta,\lambda,\beta_0,\beta_1)$. The censored log-likelihood function of the $AFT\ -NWR$ distribution is :

$$
\begin{aligned}
l(x,\varsigma) &= \sum_{i-1}^{n}\delta_i[\log(2\alpha\beta\lambda)+\log\left(x_i\right)-\lambda x_i^2+(\beta-1)\log\left\{-\log\left[1-\exp\left(-\lambda x_i^2\right)\right]\right\} \\
&= r\log(2\alpha\beta\lambda)+\sum_{i\in F}\log\left(x_i\right)-\lambda\sum_{i\in F}x_i^2+\sum_{i\in C}\log\left[1-\exp\left(-\alpha\left\{-\log\left[1-\exp\left(-\lambda x_i^2\right)\right]\right\}^{\beta}\right)\right] \\
&\quad -\alpha\sum_{i\in F}\left\{-\log\left[1-\exp\left(-\lambda x_i^2\right)\right]\right\}^{\beta}-\sum_{i\in F}\log\left[1-\exp\left(-\lambda x_i^2\right)\right]+(\beta-1)\sum_{i\in F}\log\left\{-\log\left[1-\exp\left(-\lambda x_i^2\right)\right]\right\}.
\end{aligned}
$$

The score functions are obtained by deriving the log-likelihood function relative to the unknown parameters ς of the $AFT\ -NWR$ distribution. The formula of the score functions and the elements of the estimated Information Fisher Matrix (IFM) of the $AFT\ -NWR$ in the right-censored case are derived and given in appendix 2.

The choice of the limits $\widehat{a_j}$ of the intervals I_j is obtained as follow :

$$\widehat{a}_j = e^{-\beta^T z} \left\{ -\frac{1}{\widehat{\lambda}} \log \left[1 - \exp \left(- \left\{ -\frac{1}{\widehat{\alpha}} \log \left[1 - \exp \left(\frac{\sum_{l=1}^{i-1} \Lambda \left(X_{(l)}, \widehat{\vartheta} \right) - E_j}{N - i + 1} \right) \right] \right\}^{\frac{1}{\widehat{\beta}}} \right) \right] \right\}^{\frac{1}{2}}, \quad \widehat{a}_k = \max \left(X_{(n)}, \tau \right).$$

TheNRR test statistic's Y_n^2 components of the $AFT-NWR$ have been calculated and are listed appendix 2.

4 Performance by simulation study

4.1 the performance of the maximum likelihood estimation

In this step, we suppose the data provided from the studied models $AFT - NWW$ and $AFT - NWR$ where simulated $N = 12,000$ times for different sample sizes are : $n = 15$, $n = 50$, $n = 100$, $n = 150$ and $n = 250$. The initial values of parameters used in the simulation study for the : $AFT\ -NWW$ model : $(\alpha = 1.5, \xi = 0.61, \lambda = 2.4, \gamma = 2.5, \beta_0 = 1.6, \beta_1 = 2)$ and for the $AFT\ -NWR$ model : $(\alpha = 1.3, \xi = 0.7, \lambda = 2.5, \beta_0 = 1.1, \beta_1 = 3.7)$. Using Barzilai-Borwein (BB) algorithm (see Ravi (2009)) in R software, we calculate in Tables follow the averages of the simulated values of the estimators $\widehat{\alpha}$, $\widehat{\xi}$, $\widehat{\lambda}$, $\widehat{\gamma}$, $\widehat{\beta}_0$, $\widehat{\beta}_1$ and their corresponding mean squared errors (MSE) presented in Table 1

n = 15

Parameters	$AFT-NWW$	$AFT-NWE$	$AFT-NWR$
α	1.5124	2.8926	1.3267
MSE	1.8231×10^{-4}	3.9115×10^{-4}	7.8940×10^{-3}
ξ	0.6322	0.6562	0.7394
MSE	3.5001×10^{-2}	8.8898×10^{-3}	2.2957×10^{-2}
λ	2.4182	1.2621	2.5209
MSE	8.7570×10^{-3}	1.6610×10^{-2}	8.9076×10^{-3}
γ	2.5699	/	/
MSE	1.9885×10^{-3}	/	/
β_0	1.6637	4.7128	1.3316
MSE	1.7970×10^{-2}	1.0113×10^{-2}	6.2974×10^{-3}
β_1	2.2505	5.4265	3.7444
MSE	3.1489×10^{-3}	3.6549×10^{-3}	5.1312×10^{-3}

n = 50

	$AFT-NWW$	$AFT-NWE$	$AFT-NWR$
α	1.5084	2.8896	1.3129
MSE	1.0332×10^{-4}	2.5479×10^{-4}	6.4421×10^{-4}
ξ	0.6228	0.6161	0.7122
MSE	3.7413×10^{-3}	6.6436×10^{-3}	7.2998×10^{-4}
λ	2.4101	1.2533	2.5126
MSE	1.7579×10^{-3}	8.9670×10^{-3}	4.7196×10^{-4}
γ	2.5304	/	/
MSE	1.0053×10^{-3}	/	/
β_0	1.6273	4.7064	1.1677
MSE	5.7994×10^{-3}	9.5286×10^{-3}	8.0344×10^{-4}
β_1	2.1932	5.4108	3.7129
MSE	8.3304×10^{-4}	9.5401	1.9550×10^{-3}

n = 100

	$AFT-NWW$	$AFT-NWE$	$AFT-NWR$
α	1.5001	2.8744	1.3077
MSE	8.7699×10^{-5}	1.5154×10^{-4}	7.7210×10^{-5}
ξ	0.6173	0.6079	0.7086
MSE	7.0339×10^{-4}	8.1445×10^{-4}	1.1374×10^{-4}
λ	2.4071	1.2498	2.5069
MSE	6.3347×10^{-4}	7.4458×10^{-4}	1.9087×10^{-4}
γ	2.5203	/	/
MSE	8.5596×10^{-4}	/	/
β_0	1.6198	4.6986	1.1249
MSE	5.4776×10^{-4}	9.6877×10^{-4}	2.2633×10^{-4}
β_1	2.1337	5.4033	3.7029
MSE	6.0342×10^{-4}	2.7059×10^{-4}	3.8665×10^{-4}

n = 150			
	$AFT\ -NWW$	$AFT\ -NWE$	$AFT\ -NWR$
α	1.4986	2.8645	1.2996
MSE	5.4344×10^{-5}	6.4582×10^{-5}	4.5517×10^{-5}
ξ	0.6111	0.5977	0.6987
MSE	3.1334×10^{-4}	5.0813×10^{-4}	8.2839×10^{-5}
λ	2.4001	1.2403	2.4936
MSE	1.4706×10^{-4}	1.8752×10^{-4}	7.8661×10^{-5}
γ	2.5123	/	/
MSE	1.3396×10^{-4}	/	/
β_0	1.6077	4.69006	1.1002
MSE	1.6690×10^{-4}	1.5040×10^{-4}	1.0339×10^{-4}
β_1	2.0870	5.4005	3.6928
MSE	2.7761×10^{-4}	8.6945×10^{-5}	2.5023×10^{-4}

n = 250			
	$AFT - NWW$	$AFT - NWE$	$AFT - NWR$
α	1.4966	2.8599	1.2977
(MSE)	4.3312×10^{-5}	1.22550×10^{-5}	2.7639×10^{-5}
ξ	0.6093	0.5906	0.6901
MSE	1.1775×10^{-4}	3.8879×10^{-4}	6.8397×10^{-5}
λ	2.3998	1.2389	2.4902
MSE	7.7533×10^{-5}	8.1691×10^{-5}	6.0148×10^{-5}
γ	2.5088	/	/
MSE	5.4498×10^{-5}	/	/
β_0	1.5913	4.6879	1.0959
(MSE)	7.7966×10^{-5}	7.3290×10^{-5}	9.2134×10^{-5}
β_1	2.0061	5.3984	3.6905
MSE	3.6617×10^{-5}	4.6117×10^{-5}	9.7026×10^{-5}

Table 1: Maximum likelihood estimators $\widehat{\alpha}$, $\widehat{\zeta}$, $\widehat{\lambda}$, $\widehat{\beta_0}$, $\widehat{\beta_1}$ of the parameters and their mean squared error (MSE) for the $AFT - NWW$, $AFT - NWE$ and $AFT - NWR$. models.

Table 1 indicates that when n increases the MSEs decrease and decay to zero. Which leads us to say that the maximum likelihood estimators of the parameters of the $AFT\ -NWW$ (respectively $AFT - NWE$) and $AFT - NWR$ distributions converge towards the initial values.

To confirm the results obtained in Table1 and the convergence of the maximum likelihood estimators of the parameters of the $AFT\ -NWW$ and $AFT - NWR$ models, we plot in Figure 3 the mean absolute error of the parameters estimated by the maximum likelihood method with respect to their actual values as a function of the corresponding sample size n used in the simulation study above.

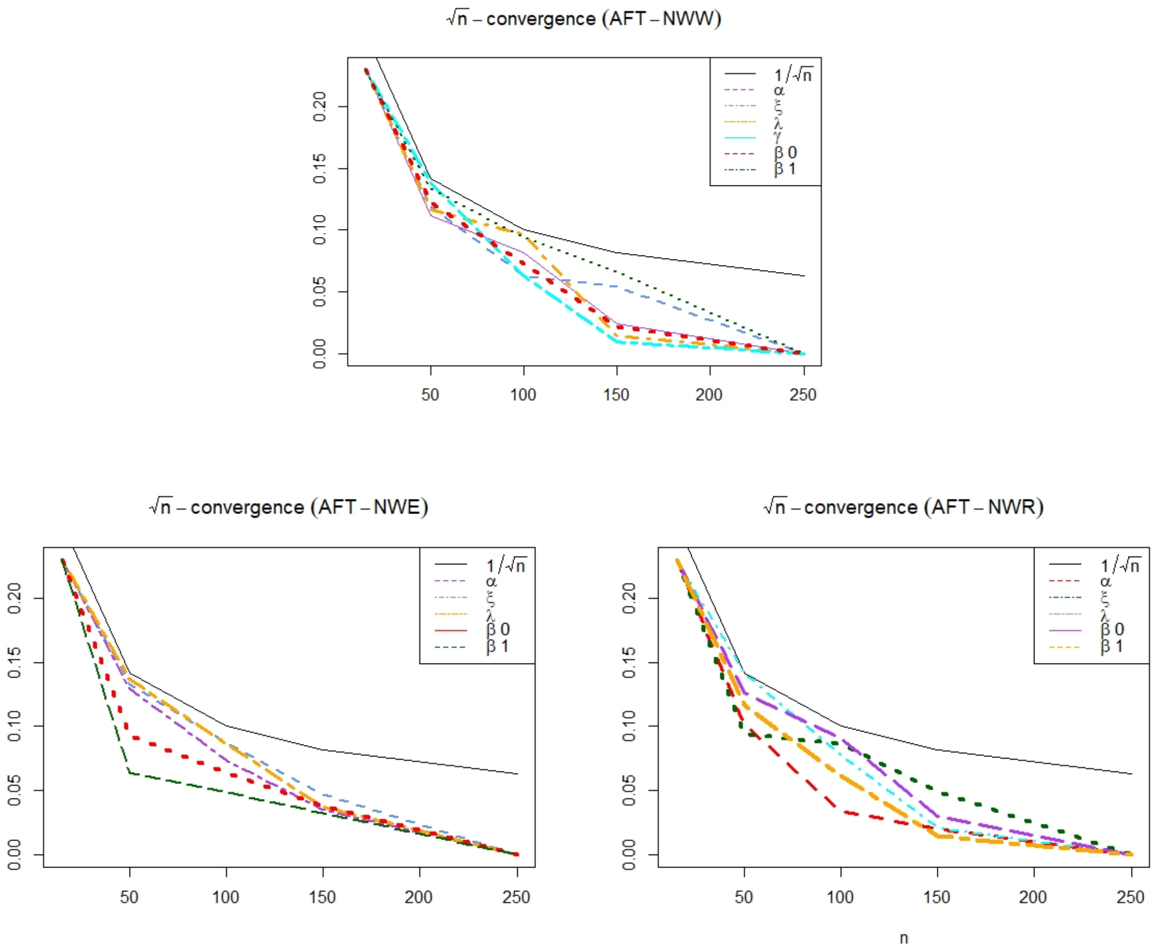


Figure 3: the Mean absolute error of MLEs VS. their true values with corresponding sample size n of the $AFT-NWW$, $AFT-NWE$ and $AFT-NWR$ models.

From Figure 3 and the simulation results Table 1, we notice that all the estimators converge faster than $n^{-0.5}$ affirm that the MLEs of the $AFT-NWW$, $AFT-NWE$ and $AFT-NWR$ models are $\sqrt{n}$ convergent.

4.2 Simulation of modified NRR test for the $AFT-NWW$ and $AFT-NWR$ models

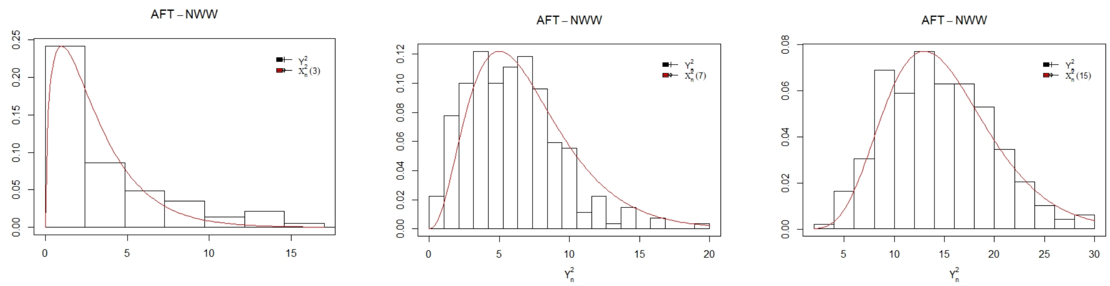
For computing the test statistic Y^2, we must calculate the elements of the matrix $\widehat{J}$, symmetric Fisher matrix $I=(i_{ll'})_{s\times s}$ and the vector $\widehat{L}$ for the $AFT-NWW$ and $AFT-NWR$ distributions. The results are presented in Appendix1. To test the null hypothesis H_0 such that a n-sample belongs to the $AFT-NWW$ (respectively $AFT-NWE$) and $AFT-NWR$ models, we calculate the values of criteria statistics Y^2 of $N=12,000$ simulated complete data samples with different sample sizes : $n=15$, $n=50$, $n=100$, $n=150$ and $n=250$. For dfferent significance levels : $\epsilon=1\%, 5\%, 10\%$, the average numbers of the rejection's cases of the null hypothesis H_0 is calculated when $Y^2>Y^2_\epsilon(r-1)$. In Table 2 we note that the simulated levels of significance for Y^2 are close to theoretical values which mean that the proposed GOF test (NRR) is suited to the $AFT-NWW$, $AFT-NWE$ and $AFT-NWR$ models.

$\mathbf{n = 15}$	$AFT-NWW$	$AFT-NWE$	$AFT-NWR$
$\alpha = 1\%$	0.0081	0.0079	0.0069
$\alpha = 5\%$	0.0361	0.0450	0.0340
$\alpha = 10\%$	0.0832	0.0918	0.0918
$\mathbf{n = 50}$	$AFT-NWW$	$AFT-NWE$	$AFT-NWR$
$\alpha = 1\%$	0.0097	0.0106	0.0074
$\alpha = 5\%$	0.0442	0.0441	0.0403
$\alpha = 10\%$	0.0906	0.0921	0.0933
$\mathbf{n = 100}$	$AFT-NWW$	$AFT-NWE$	$AFT-NWR$
$\alpha = 1\%$	0.0083	0.0095	0.0088
$\alpha = 5\%$	0.0439	0.0505	0.0467
$\alpha = 10\%$	0.0977	0.1019	0.0997
$\mathbf{n = 150}$	$AFT-NWW$	$AFT-NWE$	$AFT-NWR$
$\alpha = 1\%$	0.0133	0.0092	0.0112
$\alpha = 5\%$	0.0507	0.0487	0.0517
$\alpha = 10\%$	0.1034	0.0955	0.0955
$\mathbf{n = 250}$	$AFT-NWW$	$AFT-NWE$	$AFT-NWR$
$\alpha = 1\%$	0.0139	0.0102	0.0174
$\alpha = 5\%$	0.0521	0.0492	0.0521
$\alpha = 10\%$	0.1065	0.1007	0.1221

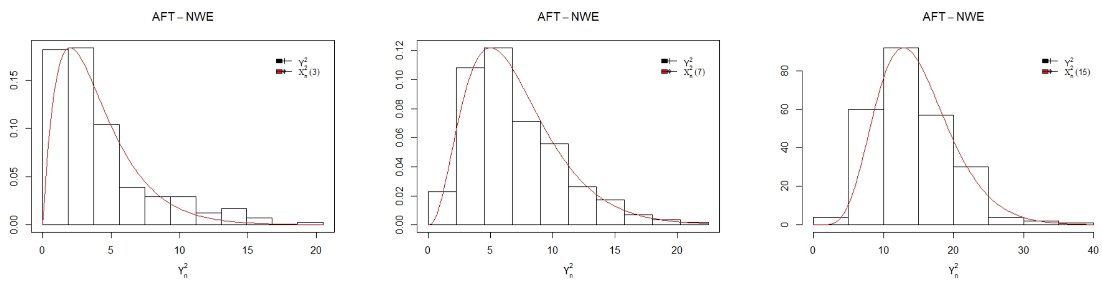
Table 2: Rejected cases of the censored Y_n^2 of the $AFT-NWW$, $AFT-NW$ and $AFT-NWR$ distributions.

Table 2 reveals that the rejected Y_n^2 values are quite near to the corresponding theoretical values ϵ, allowing us to conclude that the statistic test Y_n^2 is a chi-square with k degrees of freedom.

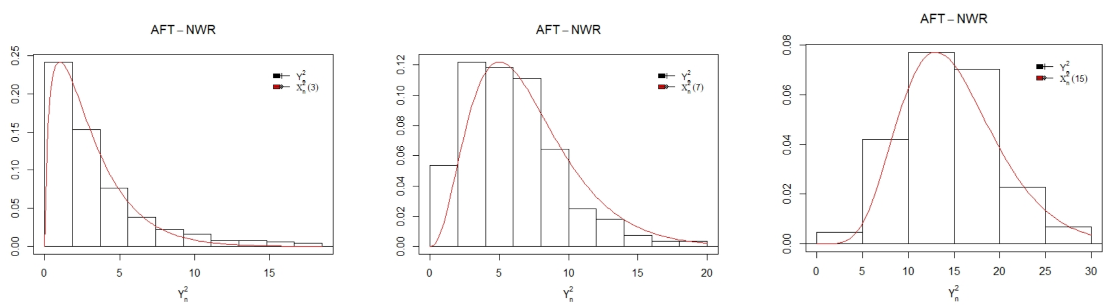
We represent in Figure 4 the histograms of the test statistics calculated previously of the studied models NWW and NWR, compared to the chi-square distribution with the corresponding degree of freedom.



(a) $AFT - NWW$



(b) $AFT - NWE$



(c) $AFT - NWR$

Figure 4: Histograms of the simulated Y_n^2 versus the chi-squared distribution with the corresponding degree of freedom, of the $AFT - NWW$, $AFT - NWE$ and $AFT - NWR$ models.

5 Application to real data

To demonstrate the viability and utility of the goodness-of-fit test proposed by *Nikulin* for accelerated data, we consider two real data applied to the models studied (presented) in this paper ($AFT - NWW$ and $AFT - NWR$)

5.1 Data 1

Let us consider the dataset of He, et al. (2007) available on the 'simex AFT package in statistical software R which gives the systolic blood pressure (SBP) of 100 persons of Busselton in Western Australia, measured at different values of body mass index (BMI). We fit these data by the $NWW - AFT$ model. To compare the theoretical quantiles, theoretical probabilities of the $AFT - NWW$ distribution to the empirical distribution of the SBP of 100 persons, we present in Figure 5 the QQ-plot and PP-plot :

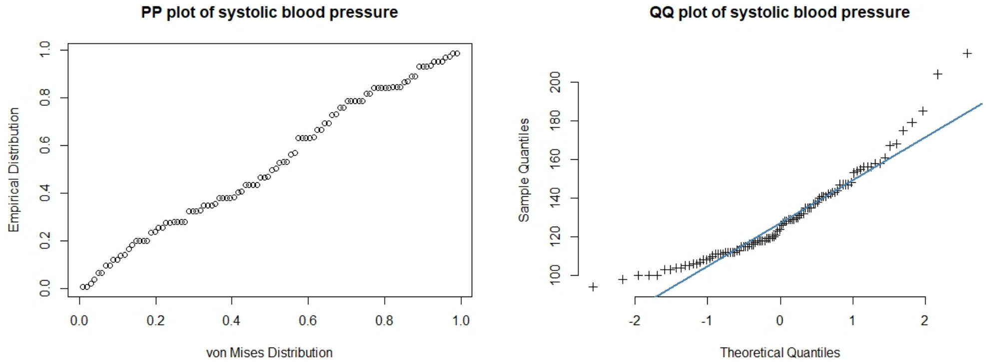


Figure 5: Q-Q plot and P-P plot of the systolic blood pressure of 100 persons.

As shown Figure 5, the distribution of the systolic blood pressure (SBP) of 100 persons measured at different values of body mass index (BMI) has almost the same form as theoretical quantiles and theoretical probabilities of the $AFT-NWW$ distribution.

We suppose the null hypothesis H_0 that the data used in this example are adjusted by the $AFT-NWW$ distribution. For that, we calculate the vector of MLEs $\widehat{varsigma}$ of the $AFT-NWW$ model :

$$\widehat{\vartheta}=\left(\widehat{\alpha}=1.4740,\widehat{\xi}=1.9953,\widehat{\lambda}=0.7936,\widehat{\gamma}=0.4239,\widehat{\beta}_0=0.2267,\widehat{\beta}_1=0.0383\right),$$

Under $H0$, we opt for $r=10$ grouping intervals, using the modified NRR statistic test obtained previously we can calculate the estimated classes limits $\$widehata_i$ of the grouping intervals, the corresponding observed and expected frequencies v_i and p_j. The results are illustrated in Table **??** :

Table 3: Values of $\widehat{a}_j$, v_j and p_j.

$\widehat{a}_j$	1.6160	2.9170	4.4715	6.4460	9.0764	1.2780	1.8421	2.8205	5.0576	147.49
v_j	38	7	13	17	4	9	2	6	3	1
p_j	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1

and the estimated Fisher matrix $\widehat{I}\left(\widehat{\vartheta}\right)$ expressed as :

$$I\left(\widehat{\vartheta}\right)=\begin{pmatrix} 0.4602 & 0.4938 & -2.6307 & 4.6336 & 0.8851 & 120.3079 \\ 0.4938 & 0.9791 & -2.0718 & 4.5572 & 0.6971 & 105.4088 \\ -2.6307 & -2.0718 & 6.8873 & -4.1526 & 0.1011 & -47.9199 \\ 4.6336 & 4.5572 & -4.1526 & 1.6492 & -0.2904 & -12840.1082 \\ 0.8851 & 0.6971 & 0.1011 & -0.2904 & 0.3853 & -5582.3523 \\ 120.3079 & 105.4088 & -47.9199 & -12840.1082 & -5582.3523 & 51.0477 \end{pmatrix}.$$

Consequently, we get the value of NRR statistic test criterion : $Y^2=8.9740$ and we compare it to the $(r-1)$ χ^2_ϵ critical value for different level of significance $\epsilon=1\%$, $\epsilon=5\%$ and $\epsilon=10\%$:

$$\begin{aligned} Y^2 &< \chi^2_{0.01}(10-1)=21.6659, \\ Y^2 &< \chi^2_{0.05}(10-1)=16.9189, \\ Y^2 &< \chi^2_{0.1}(10-1)=14.6836. \end{aligned}$$

The results obtained prove that the systolic blood pressure data can be modeled by the $NWW-AFT$ model. Therefore, we can conclude that the $AFT-NWW$ distribution effectively models medical data.

5.2 Data 2

We use the following data taken from Lawless (2003) which describe the failure time (hours) to electrical breakdown of 41 insulating fluids tested at a voltage ranging from 26 to 38 kilovolts (kV). We suppose H_0 satisfies that these observations are modeled by the $AFT-NWR$ distribution. Data are given in Table 4

Voltage Level (z_i)	n_i	Breakdown time x_i
26	3	$5.79, 1579.52, 2323.7,$
30	11	$7.74, 17.05, 20.46, 21.02, 22.66, 43.40,$
		$47.30, 139.07, 144.12, 175.88, 194.90,$
34	19	$0.19, 0.78, 0.96, 1.31, 2.78, 3.16,$
		$4.15, 4.67, 4.85, 6.50, 7.35, 8.01, 8.27$
		$12.06, 31.75, 32.52, 33.91, 36.71, 72.89,$
38	8	$0.09, 0.39, 0.47, 0.73, 0.74, 1.13, 1.40, 2.38.$

Table 4: the failure time (hours) of 41 insulating fluids tested at a voltage ranging from 26 to 38 kilovolts

We plot in Figure 6 the QQ-plot and PP-plot graphs, which QQ-plot allows us to compare the theoretical quantiles calculated from the $AFT-NWR$ distribution and the empirical distribution of the real data used. While at the PP plot gives us a visual comparison of the theoretical probabilities of the $AFT-NWR$ distribution and the empirical distribution of the actual data used in this example :

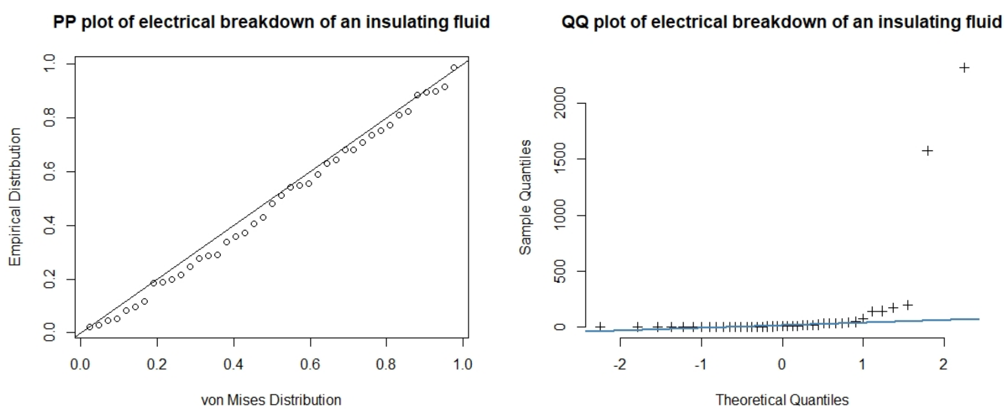


Figure 6: Q-Q plot and P-P plot of the he failure time (hours) of 41 insulating fluids.

As we can see in Figure 6 the values of the failure times of 41 insulating fluids has the same shape as the $AFT-NWR$ distribution we have supposed. Using R statistical software, we obtain the values of the maximum likelihood estimators (MLEs) of $AFT-NWR$ model :

$$\widehat{\varsigma} = \left(\widehat{\alpha} = 0.4076, \widehat{\xi} = 2.1102, \widehat{\lambda} = 0.5601, \widehat{\beta}_0 = 6.2333, \widehat{\beta}_1 = -0.089\right).$$

we grouped these 41 data into $r = 5$ classes and we calculate the estimated classes limits $\widehat{a_j}$ and the corresponding observed and expected frequencies v_j and p_j, the results are shown in Table**??** :

Table 5: Values of $\widehat{a}_j$, v_j and p_j.

$\widehat{a}_j$	0.4331	0.5462	0.6556	0.7836	0.9709	2423.70
v_j	3	1	3	3	1	30
p_j	0.1666	0.1666	0.1666	0.1666	0.1669	0.1669

The estimated Fisher matrix is :

$$\widehat{I}(\widehat{\varsigma}) = \begin{pmatrix} 173.1301 & 30.4650 & -2.2136 & 2.6563 & 0.0801 \\ 30.4650 & 3.1417 & -0.1747 & 0.2097 & 0.0042 \\ -2.2136 & -0.1747 & -0.4301 & -1.8571 & 1.8415 \\ 2.6563 & 0.2097 & -1.8571 & 0.1610 & -0.4823 \\ 0.0801 & 0.0042 & 1.8415 & -0.4823 & 0.0061 \end{pmatrix}.$$

Therefore, we obtain the NRR statistic of the $AFT-NWR$ model value $Y^2 = 2.7659$, then we compare it to the $(r-1)$ chi-square critical values χ^2 for different level of significance $\epsilon = 1\%$; 5% an 10% :

$$Y^2 < \chi^2_{1\%}(6-1) = 15.0862 \text{ - } Y^2 < \chi^2_{5\%}(6-1) = 11.0705 \text{ - } Y^2 < \chi^2_{10\%}(6-1) = 9.2363.$$

These obtained results indicate the $AFT - NWR$ distribution can be used to accurately fit the failure time to electrical breakdown of 41 insulating fluids tested at different voltage ranging.

5.3 Data 3

A study was carried out on the weight (x_i) of 18 tumors exposed to different doses of radioactivity (z_i) obtained with a special medical technique (scintigraphic images). The data reported by *Shin et al.* (2005). We assume the null hypothesis H_0 that we can fit these medical data by the $AFT - NWR$ distribution, and we display in Figure 7 the PP-plot and QQ-plot the weight of 18 tumors exposed to different doses of radioactivity :

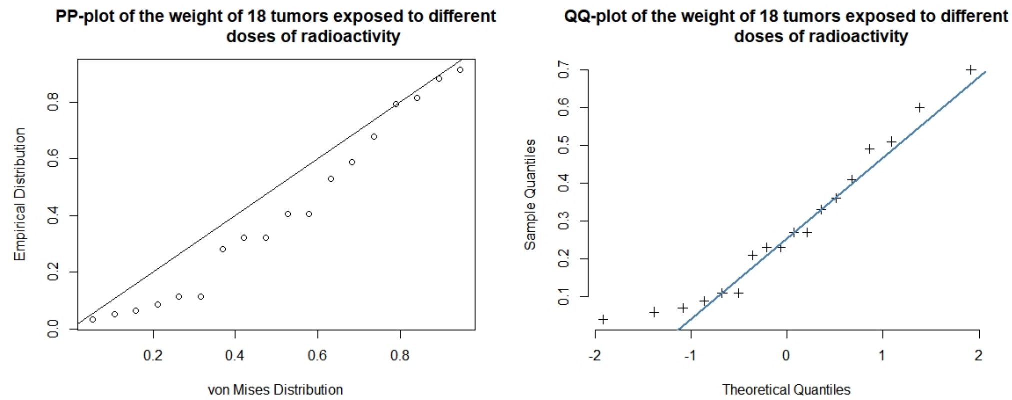


Figure 7: QQ-plot and PP-plot of 18 tumors exposed to different doses of radioactivity.

As shown Figure 7, the distribution of weight of 18 tumors exposed to different doses of radioactivity has almost the same form as theoretical quantiles and theoretical probabilities of the $AFT - NWR$ model. Under the null hypothesis H_0, we calculate the vector of MLEs $\widehat{varsigma}$ of the $AFT - NWR$ model :

$$\widehat{\varsigma} = \left(\widehat{\alpha} = 3.5474, \widehat{\xi} = 0.5292\widehat{\lambda} = 0.1217, \widehat{\beta}_0 = -3.0324, \widehat{\beta}_1 = 0.18873\right),$$

We opt for $r = 4$ grouping intervals, using the NRR statistic test obtained previously we can calculate the estimated classes limits $\$widehata_i$ of the grouping intervals, the corresponding observed and expected frequencies v_i and p_j. The results are shown in Table **??** :

Table 6: Values of $\widehat{a}_j$, v_j and p_j.

$\widehat{a}_j$	0.1626	0.2102	0.2599	100.70	
v_j	6	1	2	9	
p_j	0.25	0.25	0.25	0.25	

and the estimated Fisher matrix $\widehat{I}(\widehat{\varsigma})$ expressed as :

$$\widehat{I}(\widehat{\varsigma}) = \begin{pmatrix} 6.25 & 10.0875 & -11.2254 & 11.2254 & 51.8609 \\ 10.0875 & 4.2617 & -6.0116 & 6.0116 & 21.2706 \\ -11.2254 & -6.0116 & 9.8023 & -5.6423 & -6.3847 \\ 11.2254 & 6.0116 & -5.6423 & 3.0340 & -870.4102 \\ 51.8609 & 21.2706 & -6.3847 & -870.4102 & 15.8255 \end{pmatrix}.$$

As a result, we obtain the NRR statistic test criterion value : $Y^2 = 4.7651$ and we compare it to the $(r-1)$ χ^2_ϵ critical value for different level of significance $\epsilon = 1\%$, $\epsilon = 5\%$ and $\epsilon = 10\%$:

$$\begin{aligned} Y^2 &< \chi^2_{1\%}(4-1) = 11.3448, \\ Y^2 &< \chi^2_{5\%}(3) = 7.8147, \\ Y^2 &< \chi^2_{10\%}(3) = 6.2513. \end{aligned}$$

The results obtained prove that the distribution of weight of 18 tumors exposed to different doses of radioactivity can be modelled by the $AFT-NWR$ model. Therefore, we can conclude that the $AFT-NWR$ distribution effectively models medical data.

6 Conclusion

The accelerated Failure time (AFT) model with the New-*Weibull*-*Weibull* ($AFT-NWW$) and the New-*Weibull*-*Rayleigh* ($AFT-NWR$) distributions are powerful tool for analyzing survival data. It allows for the estimation of the effect covariates on the time to an event of interest, and flexibility of the studied models ($AFT-NWW$, $AFT-NWR$) makes it suitable for a wide range of applications and its ability to model a wide range of hazard functions. The use of the AFT model with the New-*Weibull*-G family has been demonstrated in various fields including medical research and the prediction of electrical component failure. In these applications, the AFT model provides valuable insights into the factors that influence survival and can be used to inform decision-making.

however, it is important to assess the goodness-of-fit of the AFT model to the data, as an incorrect specification of the model can lead to biased estimates and incorrect inferences. The maximum likelihood estimation and the *Nikulin-Rao-Robson* test provide an essential and useful tool for assessing studied models ($AFT-NWW$, $AFT-NWR$) fit. overall, the AFT model with the New-*Weibull*-G family is a valuable tool for survival analysis and helps identify important predictors of survival time, but it should be used the caution and accompanied by appropriate diagnostic procedures to ensure the validity of the result.

Appendix 1
$AFT-NWW$ model

Fisher Information Matrix $\hat{I}$ (AFT-NWW)

The components of the information matrix $\widehat{I}\left(\widehat{\vartheta}\right)=\left(\widehat{i}_{ll\prime}\right)_{6\times6}$ are given as follows

$$
\begin{aligned}
\widehat{i}_{11} &= -\frac{n}{\alpha^2},\\
\widehat{i}_{22} &= -\frac{n}{\xi^2}-\alpha\sum_{i=1}^{n}U\left(x_i,\vartheta\right)^{\xi}\log\left[U\left(x_i,\vartheta\right)\right],\\
\widehat{i}_{33} &= -\frac{n}{\lambda^2}+\left(\xi-1\right)\sum_{i=1}^{n}\frac{M\left(x_i,\vartheta\right)^2}{D\left(x_i,\vartheta\right)^2U\left(x_i,\vartheta\right)^2}\left[1+U\left(x_i,\vartheta\right)-\alpha\xi U\left(x_i,\vartheta\right)^{\xi}\right]\\
&\quad+\sum_{i=1}^{n}\frac{M\left(x_i,\vartheta\right)^2}{D\left(x_i,\vartheta\right)^2}\left[1-\alpha\xi U\left(x_i,\vartheta\right)^{\xi-1}\right]+\sum_{i=1}^{n}\frac{M\left(x_i,\vartheta\right)}{D\left(x_i,\vartheta\right)U\left(x_i,\vartheta\right)}\left[1+U\left(x_i,\vartheta\right)-\alpha\xi U\left(x_i,\vartheta\right)^{\xi}\right],\\
\widehat{i}_{44} &= -\frac{n}{\gamma^2}-\lambda\sum_{i=1}^{n}\left(x_ie^{-\beta^Tz}\right)\log\left(x_ie^{-\beta^Tz}\right)^2+\lambda^2\sum_{i=1}^{n}\frac{E\left(x_i,\vartheta\right)M\left(x_i,\vartheta\right)}{D\left(x_i,\vartheta\right)^2}\left[1-\alpha\xi U\left(x_i,\vartheta\right)^{\xi-1}\right]\\
&\quad+\left(\xi-1\right)\lambda^2\sum_{i=1}^{n}\frac{E\left(x_i,\vartheta\right)M\left(x_i,\vartheta\right)}{D\left(x_i,\vartheta\right)^2U\left(x_i,\vartheta\right)^2}\left[1+U\left(x_i,\vartheta\right)-\alpha\xi U\left(x_i,\vartheta\right)^{\xi}\right]\\
&\quad-\lambda\sum_{i=1}^{n}\frac{E\left(x_i,\vartheta\right)M\left(x_i,\vartheta\right)}{D\left(x_i,\vartheta\right)}\left[1-\lambda\left(x_ie^{-\beta^Tz}\right)^{\gamma}\right]\times\left[\frac{\left(\xi-1\right)}{U\left(x_i,\vartheta\right)}+1-\alpha\xi U\left(x_i,\vartheta\right)^{\xi-1}\right],
\end{aligned}
$$

$$
\begin{aligned}
\widehat{i}_{55} &= \left(\xi-1\right)\lambda^2\gamma^2\sum_{i=1}^{n}\frac{M\left(x_i,\vartheta\right)^2}{D\left(x_i,\vartheta\right)^2}\left[1-\alpha\xi U\left(x_i,\vartheta\right)^{\xi}\right]-\lambda\gamma^2\sum_{i=1}^{n}\left(x_ie^{-\beta^Tz}\right)^{\gamma}\\
&\quad-\lambda\gamma^2\sum_{i=1}^{n}\frac{M\left(x_i,\vartheta\right)\times\left[1-\lambda\left(x_ie^{-\beta^Tz}\right)^{\gamma}\right]}{D\left(x_i,\vartheta\right)}\left[1-\alpha\xi U\left(x_i,\vartheta\right)^{\xi-1}+\frac{\left(\xi-1\right)}{U\left(x_i,\vartheta\right)}\right]\\
&\quad+\lambda^2\gamma^2\sum_{i=1}^{n}\frac{M\left(x_i,\vartheta\right)^2}{D\left(x_i,\vartheta\right)^2}\left[1-\alpha\xi U\left(x_i,\vartheta\right)^{\xi-1}+\frac{\left(\xi-1\right)}{U\left(x_i,\vartheta\right)}\right],\\
\widehat{i}_{66} &= \left(\xi-1\right)\lambda^2\gamma^2\sum_{i=1}^{n}z_i\frac{M\left(x_i,\vartheta\right)^2}{D\left(x_i,\vartheta\right)^2}\left[1-\alpha\xi U\left(x_i,\vartheta\right)^{\xi}\right]-\lambda\gamma^2\sum_{i=1}^{n}z_i\left(x_ie^{-\beta^Tz}\right)^{\gamma}\\
&\quad-\lambda\gamma^2\sum_{i=1}^{n}z_i\frac{M\left(x_i,\vartheta\right)\times\left[1-\lambda\left(x_ie^{-\beta^Tz}\right)^{\gamma}\right]}{D\left(x_i,\vartheta\right)}\left[1-\alpha\xi U\left(x_i,\vartheta\right)^{\xi-1}+\frac{\left(\xi-1\right)}{U\left(x_i,\vartheta\right)}\right]\\
&\quad+\lambda^2\gamma^2\sum_{i=1}^{n}z_i\frac{M\left(x_i,\vartheta\right)^2}{D\left(x_i,\vartheta\right)^2}\left[1-\alpha\xi U\left(x_i,\vartheta\right)^{\xi-1}+\frac{\left(\xi-1\right)}{U\left(x_i,\vartheta\right)}\right],
\end{aligned}
$$

$$
\begin{aligned}
\widehat{i}_{12} &= -\sum_{i=1}^{n}U\left(x_i,\vartheta\right)^{\xi}\log\left[U\left(x_i,\vartheta\right)\right],\\
\widehat{i}_{13} &= \xi\sum_{i=1}^{n}\frac{M\left(x_i,\vartheta\right)\times U\left(x_i,\vartheta\right)^{\xi-1}}{D\left(x_i,\vartheta\right)},\\
\widehat{i}_{14} &= \lambda\xi\sum_{i=1}^{n}\frac{M\left(x_i,\vartheta\right)\times U\left(x_i,\vartheta\right)^{\xi-1}}{D\left(x_i,\vartheta\right)}\log\left(x_ie^{-\beta^Tz}\right),\\
\widehat{i}_{15} &= -\lambda\gamma\xi\sum_{i=1}^{n}\frac{M\left(x_i,\vartheta\right)\times U\left(x_i,\vartheta\right)^{\xi-1}}{D\left(x_i,\vartheta\right)},\\
\widehat{i}_{16} &= -\lambda\gamma\xi\sum_{i=1}^{n}z_i\frac{M\left(x_i,\vartheta\right)\times U\left(x_i,\vartheta\right)^{\xi-1}}{D\left(x_i,\vartheta\right)}
\end{aligned}
$$

$$\begin{aligned}
\widehat{i}_{23} &= \alpha\sum_{i=1}^{n}\frac{M(x_i,\vartheta)\times U(x_i,\vartheta)^{\xi-1}}{D(x_i,\vartheta)}\left\{1+\xi\log\left[U(x_i,\vartheta)\right]\right\}-\sum_{i=1}^{n}\frac{M(x_i,\vartheta)}{D(x_i,\vartheta)\times U(x_i,\vartheta)},\\
\widehat{i}_{24} &= \alpha\lambda\sum_{i=1}^{n}\frac{M(x_i,\vartheta)\times U(x_i,\vartheta)^{\xi-1}\log\left(x_ie^{-\beta^Tz}\right)}{D(x_i,\vartheta)}\left\{1+\xi\log\left[U(x_i,\vartheta)\right]\right\}\\
&\quad -\lambda\sum_{i=1}^{n}\frac{M(x_i,\vartheta)\log\left(x_ie^{-\beta^Tz}\right)}{D(x_i,\vartheta)\times U(x_i,\vartheta)},\\
\widehat{i}_{25} &= -\alpha\lambda\gamma\sum_{i=1}^{n}\frac{M(x_i,\vartheta)\times U(x_i,\vartheta)^{\xi-1}}{D(x_i,\vartheta)}\left\{1+\xi\log\left[U(x_i,\vartheta)\right]\right\}+\lambda\gamma\sum_{i=1}^{n}\frac{M(x_i,\vartheta)}{D(x_i,\vartheta)\times U(x_i,\vartheta)},\\
\widehat{i}_{26} &= -\alpha\lambda\gamma\sum_{i=1}^{n}z_i\frac{M(x_i,\vartheta)\times U(x_i,\vartheta)^{\xi-1}}{D(x_i,\vartheta)}\left\{1+\xi\log\left[U(x_i,\vartheta)\right]\right\}+\lambda\gamma\sum_{i=1}^{n}z_i\frac{M(x_i,\vartheta)}{D(x_i,\vartheta)\times U(x_i,\vartheta)},
\end{aligned}$$

$$\begin{aligned}
\widehat{i}_{34} &= (\xi-1)\sum_{i=1}^{n}\frac{M(x_i,\vartheta)^2\log\left(x_ie^{-\beta^Tz}\right)}{D(x_i,\vartheta)^2\,U(x_i,\vartheta)^2}\left[1-\alpha\xi U(x_i,\vartheta)\right]-\sum_{i=1}^{n}\left(x_ie^{-\beta^Tz}\right)^{\gamma}\log\left(x_ie^{-\beta^Tz}\right)\\
&\quad -\sum_{i=1}^{n}\frac{M(x_i,\vartheta)\log\left(x_ie^{-\beta^Tz}\right)}{D(x_i,\vartheta)}\left\{\frac{D(x_i,\vartheta)-M(x_i,\vartheta)}{D(x_i,\vartheta)}-\left(x_ie^{-\beta^Tz}\right)^{\gamma}\right\}\times\\
&\quad \left\{1-\alpha\xi U(x_i,\vartheta)+\frac{\xi-1}{U(x_i,\vartheta)}\right\},\\
\widehat{i}_{35} &= -\lambda(\xi-1)\gamma\sum_{i=1}^{n}\frac{M(x_i,\vartheta)^2}{D(x_i,\vartheta)^2\,U(x_i,\vartheta)^2}\left[1-\alpha\xi U(x_i,\vartheta)\right]+\gamma\sum_{i=1}^{n}\left(x_ie^{-\beta^Tz}\right)^{\gamma}\\
&\quad +\gamma\sum_{i=1}^{n}\frac{M(x_i,\vartheta)}{D(x_i,\vartheta)}\left\{1+(\xi-1)U(x_i,\vartheta)^{-1}-\alpha\xi U(x_i,\vartheta)^{\xi-1}\right\}\times\left\{1-\lambda\left(x_ie^{-\beta^Tz}\right)^{\gamma}-\lambda\frac{M(x_i,\vartheta)}{D(x_i,\vartheta)}\right\},\\
\widehat{i}_{36} &= -\lambda(\xi-1)\gamma\sum_{i=1}^{n}z_i\frac{M(x_i,\vartheta)^2}{D(x_i,\vartheta)^2\,U(x_i,\vartheta)^2}\left[1-\alpha\xi U(x_i,\vartheta)\right]+\gamma\sum_{i=1}^{n}z_i\left(xe^{-\beta^Tz}\right)^{\gamma}\\
&\quad +\gamma\sum_{i=1}^{n}z_i\frac{M(x_i,\vartheta)}{D(x_i,\vartheta)}\left\{1+(\xi-1)U(x_i,\vartheta)^{-1}-\alpha\xi U(x_i,\vartheta)^{\xi-1}\right\}\left[1-\lambda\left(x_ie^{-\beta^Tz}\right)^{\gamma}-\lambda\frac{M(x_i,\vartheta)}{D(x_i,\vartheta)}\right],
\end{aligned}$$

$$\begin{aligned}
\widehat{i}_{45} &= -1-\lambda^2(\xi-1)\gamma\sum_{i=1}^{n}\frac{M(x_i,\vartheta)^2\log\left(xe^{-\beta^Tz}\right)}{D(x_i,\vartheta)^2\times U(x_i,\vartheta)^2}\left[1+U(x_i,\vartheta)\right]\\
&\quad -\lambda^2\gamma\sum_{i=1}^{n}\frac{M(x_i,\vartheta)^2\log\left(x_ie^{-\beta^Tz}\right)}{D(x_i,\vartheta)}\left\{1-\alpha\xi U(x_i,\vartheta)^{\xi-1}\left[1+\frac{\xi-1}{U(x_i,\vartheta)}\right]\right\}\\
&\quad +\lambda\sum_{i=1}^{n}\frac{M(x_i,\vartheta)}{D(x_i,\vartheta)}\left\{1+\gamma\log\left(x_ie^{-\beta^Tz}\right)\times\left[1-\lambda\left(x_ie^{-\beta^Tz}\right)^{\gamma}\right]\right\}\times\left[1+\frac{\xi-1}{U(x_i,\vartheta)}-\alpha\xi U(x_i,\vartheta)^{\xi-1}\right],\\
\widehat{i}_{46} &= -\sum_{i=1}^{n}z_i-\lambda^2(\xi-1)\gamma\sum_{i=1}^{n}z_i\frac{M(x_i,\vartheta)^2\log\left(x_ie^{-\beta^Tz}\right)}{D(x_i,\vartheta)^2\times U(x_i,\vartheta)^2}\left[1+U(x_i,\vartheta)\right]\\
&\quad -\lambda^2\gamma\sum_{i=1}^{n}z_i\frac{M(x_i,\vartheta)^2\log\left(x_ie^{-\beta^Tz}\right)}{D(x_i,\vartheta)}\left\{1-\alpha\xi U(x_i,\vartheta)^{\xi-1}\left[1+\frac{\xi-1}{U(x_i,\vartheta)}\right]\right\}\\
&\quad +\lambda\sum_{i=1}^{n}z_i\frac{M(x_i,\vartheta)}{D(x_i,\vartheta)}\left\{1+\gamma\log\left(x_ie^{-\beta^Tz}\right)\times\left[1-\lambda\left(x_ie^{-\beta^Tz}\right)^{\gamma}\right]\right\}\times\left[1+\frac{\xi-1}{U(x_i,\vartheta)}-\alpha\xi U(x_i,\vartheta)^{\xi-1}\right],
\end{aligned}$$

$$\begin{aligned}
\widehat{i}_{56} &= -\lambda\gamma^2 \sum_{i=1}^{n} z_i \frac{M(x_i,\vartheta)\left[\lambda\left(xe^{-\beta^T z}\right)^{\gamma}-1\right]}{D(x_i,\vartheta)} \left\{1+(\xi-1)U(x_i,\vartheta)^{-1}-\alpha\xi U(x_i,\vartheta)^{\xi-1}\right\} \\
&-\alpha\xi\lambda^2\gamma^2 \sum_{i=1}^{n} z_i \frac{M(x_i,\vartheta)^2 U(x_i,\vartheta)^{\xi-1}}{D(x_i,\vartheta)^2 U(x_i,\vartheta)} \left[U(x_i,\vartheta)+\xi-1\right] - \lambda\gamma^2 \sum_{i=1}^{n} z_i \left(x_i e^{-\beta^T z}\right)^{\gamma} \\
&(\xi-1)\lambda^2\gamma \sum_{i=1}^{n} z_i \frac{M(x_i,\vartheta)^2}{D(x_i,\vartheta)^2 U(x_i,\vartheta)^2}\left[1+U(x_i,\vartheta)\right] + \lambda^2\gamma^2 \sum_{i=1}^{n} z_i \frac{M(x_i,\vartheta)^2}{D(x_i,\vartheta)^2}.
\end{aligned}$$

Where

$$E(x,\vartheta) = \left(xe^{-\beta^T z}\right)^{\gamma} \exp\left[-\lambda\left(xe^{-\beta^T z}\right)^{\gamma}\right] \log\left(xe^{-\beta^T z}\right)^2, \quad \vartheta = (\alpha,\xi,\lambda,\gamma).$$

Elements of Y^2

The elements of the matrix $J\left(\widehat{\vartheta}\right)_{s\times r}$ are obtained as :

$$\begin{aligned}
J\left(\widehat{\vartheta}\right) &= B^T\left(\widehat{\vartheta}\right) \times B\left(\widehat{\vartheta}\right), \\
B\left(\widehat{\vartheta}\right) &= \left[\frac{1}{\sqrt{p_j(\vartheta)}} \frac{\partial p_j}{\partial \vartheta_s}\right]_{r\times s}, \quad j=1,2,\ldots,r.
\end{aligned}$$

Where :

$$p_j(\vartheta) = F(a_j) - F(a_{j-1}).$$

So,

$$\begin{aligned}
\frac{\partial p_j}{\partial \alpha} &= \sum_{j=1}^{r} U(\widehat{a}_{j-1},\vartheta)^{\xi} F(\widehat{a}_{j-1},\vartheta) - U(\widehat{a}_j,\vartheta)^{\xi} F(\widehat{a}_j,\vartheta), \\
\frac{\partial p_j}{\partial \xi} &= \alpha \sum_{j=1}^{r} U(\widehat{a}_{j-1},\vartheta)^{\xi} \log\left[U(\widehat{a}_{j-1},\vartheta)\right] F(\widehat{a}_{j-1},\vartheta) - \alpha \sum_{j=1}^{r} U(\widehat{a}_j,\vartheta)^{\xi} \log\left[U(\widehat{a}_j,\vartheta)\right] F(\widehat{a}_j,\vartheta), \\
\frac{\partial p_j}{\partial \lambda} &= \alpha\xi \sum_{j=1}^{r} \frac{M(\widehat{a}_j,\vartheta) U(\widehat{a}_j,\vartheta)^{\xi-1} F(\widehat{a}_j,\vartheta)}{D(\widehat{a}_j,\vartheta)} - \alpha\xi \sum_{j=1}^{r} \frac{M(\widehat{a}_{j-1},\vartheta) U(\widehat{a}_{j-1},\vartheta)^{\xi-1} F(\widehat{a}_{j-1},\vartheta)}{D(\widehat{a}_{j-1},\vartheta)}, \\
\frac{\partial p_j}{\partial \gamma} &= \alpha\xi\lambda \sum_{j=1}^{r} \frac{M(\widehat{a}_j,\vartheta) U(\widehat{a}_j,\vartheta)^{\xi-1} F(\widehat{a}_j,\vartheta)}{D(\widehat{a}_j,\vartheta)} \log\left(\widehat{a}_j e^{-\beta^T z}\right) - \alpha\xi\lambda \sum_{j=1}^{r} \frac{M(\widehat{a}_{j-1},\vartheta) U(\widehat{a}_{j-1},\vartheta)^{\xi-1} F(\widehat{a}_{j-1},\vartheta)}{D(\widehat{a}_{j-1},\vartheta)} \log\left(\widehat{a}_{j-1} e^{-\right. \\
\frac{\partial p_j}{\partial \beta_0} &= \alpha\xi\lambda\gamma \sum_{j=1}^{r} \frac{M(\widehat{a}_{j-1},\vartheta) U(\widehat{a}_{j-1},\vartheta)^{\xi-1} F(\widehat{a}_{j-1},\vartheta)}{D(\widehat{a}_{j-1},\vartheta)} - \alpha\xi\lambda\gamma \sum_{j=1}^{r} \frac{M(\widehat{a}_j,\vartheta) U(\widehat{a}_j,\vartheta)^{\xi-1} F(\widehat{a}_j,\vartheta)}{D(\widehat{a}_j,\vartheta)}, \\
\frac{\partial p_j}{\partial \beta_1} &= \alpha\xi\lambda\gamma \sum_{j=1}^{r} z_i \frac{M(\widehat{a}_{j-1},\vartheta) U(\widehat{a}_{j-1},\vartheta)^{\xi-1} F(\widehat{a}_{j-1},\vartheta)}{D(\widehat{a}_{j-1},\vartheta)} - \alpha\xi\lambda\gamma \sum_{j=1}^{r} z_i \frac{M(\widehat{a}_j,\vartheta) U(\widehat{a}_j,\vartheta)^{\xi-1} F(\widehat{a}_j,\vartheta)}{D(\widehat{a}_j,\vartheta)}.
\end{aligned}$$

Appendix 2

$AFT-NWR$ model

element of Fisher Information Matrix $\widehat{I}$

The components of the information matrix $\widehat{I}\left(\widehat{\vartheta}\right)=\left(\widehat{i}_{ll\prime}\right)_{5\times 5}$ are given as follows

$$
\begin{aligned}
\widehat{i}_{11} &= -\frac{n}{\alpha^2},\\
\widehat{i}_{22} &= -\frac{n}{\xi^2}-\alpha\sum_{i=1}^{n}U\left(x_i,\vartheta\right)^{\xi}\log\left[U\left(x_i,\vartheta\right)\right],\\
\widehat{i}_{33} &= -\frac{n}{\lambda^2}+\left(\xi-1\right)\sum_{i=1}^{n}\frac{M\left(x_i,\vartheta\right)^2}{D\left(x_i,\vartheta\right)^2U\left(x_i,\vartheta\right)^2}\left[1+U\left(x_i,\vartheta\right)-\alpha\xi U\left(x_i,\vartheta\right)^{\xi}\right]\\
&\quad+\sum_{i=1}^{n}\frac{M\left(x_i,\vartheta\right)^2}{D\left(x_i,\vartheta\right)^2}\left[1-\alpha\xi U\left(x_i,\vartheta\right)^{\xi-1}\right]+\sum_{i=1}^{n}\frac{M\left(x_i,\vartheta\right)}{D\left(x_i,\vartheta\right)U\left(x_i,\vartheta\right)}\left[1+U\left(x_i,\vartheta\right)-\alpha\xi U\left(x_i,\vartheta\right)^{\xi}\right],
\end{aligned}
$$

$$
\begin{aligned}
\widehat{i}_{44} &= 4\left(\xi-1\right)\lambda^2\sum_{i=1}^{n}\frac{M\left(x_i,\vartheta\right)^2}{D\left(x_i,\vartheta\right)^2}\left[1-\alpha\xi U\left(x_i,\vartheta\right)^{\xi}\right]-4\lambda\sum_{i=1}^{n}\left(x_ie^{-\beta^Tz}\right)^2\\
&\quad-4\lambda\sum_{i=1}^{n}\frac{M\left(x_i,\vartheta\right)\times\left[1-\lambda\left(x_ie^{-\beta^Tz}\right)^2\right]}{D\left(x_i,\vartheta\right)}\left[1-\alpha\xi U\left(x_i,\vartheta\right)^{\xi-1}+\frac{\left(\xi-1\right)}{U\left(x_i,\vartheta\right)}\right]\\
&\quad+4\lambda^2\sum_{i=1}^{n}\frac{M\left(x_i,\vartheta\right)^2}{D\left(x_i,\vartheta\right)^2}\left[1-\alpha\xi U\left(x_i,\vartheta\right)^{\xi-1}+\frac{\left(\xi-1\right)}{U\left(x_i,\vartheta\right)}\right],\\
\widehat{i}_{55} &= 4\left(\xi-1\right)\lambda^2\sum_{i=1}^{n}z_i\frac{M\left(x_i,\vartheta\right)^2}{D\left(x_i,\vartheta\right)^2}\left[1-\alpha\xi U\left(x_i,\vartheta\right)^{\xi}\right]-4\lambda\sum_{i=1}^{n}z_i\left(x_ie^{-\beta^Tz}\right)^2\\
&\quad-4\lambda\sum_{i=1}^{n}z_i\frac{M\left(x_i,\vartheta\right)\times\left[1-\lambda\left(x_ie^{-\beta^Tz}\right)^2\right]}{D\left(x_i,\vartheta\right)}\left[1-\alpha\xi U\left(x_i,\vartheta\right)^{\xi-1}+\frac{\left(\xi-1\right)}{U\left(x_i,\vartheta\right)}\right]\\
&\quad+4\lambda^2\sum_{i=1}^{n}z_i\frac{M\left(x_i,\vartheta\right)^2}{D\left(x_i,\vartheta\right)^2}\left[1-\alpha\xi U\left(x_i,\vartheta\right)^{\xi-1}+\frac{\left(\xi-1\right)}{U\left(x_i,\vartheta\right)}\right],
\end{aligned}
$$

$$
\begin{aligned}
\widehat{i}_{12} &= -\sum_{i=1}^{n}U\left(x_i,\vartheta\right)^{\xi}\log\left[U\left(x_i,\vartheta\right)\right],\\
\widehat{i}_{13} &= \xi\sum_{i=1}^{n}\frac{M\left(x_i,\vartheta\right)\times U\left(x_i,\vartheta\right)^{\xi-1}}{D\left(x_i,\vartheta\right)},\\
\widehat{i}_{14} &= -2\lambda\xi\sum_{i=1}^{n}\frac{M\left(x_i,\vartheta\right)\times U\left(x_i,\vartheta\right)^{\xi-1}}{D\left(x_i,\vartheta\right)},\\
\widehat{i}_{15} &= -2\lambda\xi\sum_{i=1}^{n}z_i\frac{M\left(x_i,\vartheta\right)\times U\left(x_i,\vartheta\right)^{\xi-1}}{D\left(x_i,\vartheta\right)}
\end{aligned}
$$

$$
\begin{aligned}
\widehat{i}_{23} &= \alpha\sum_{i=1}^{n}\frac{M\left(x_i,\vartheta\right)\times U\left(x_i,\vartheta\right)^{\xi-1}}{D\left(x_i,\vartheta\right)}\left\{1+\xi\log\left[U\left(x_i,\vartheta\right)\right]\right\}-\sum_{i=1}^{n}\frac{M\left(x_i,\vartheta\right)}{D\left(x_i,\vartheta\right)\times U\left(x_i,\vartheta\right)},\\
\widehat{i}_{24} &= -2\alpha\lambda\sum_{i=1}^{n}\frac{M\left(x_i,\vartheta\right)\times U\left(x_i,\vartheta\right)^{\xi-1}}{D\left(x_i,\vartheta\right)}\left\{1+\xi\log\left[U\left(x_i,\vartheta\right)\right]\right\}+2\lambda\sum_{i=1}^{n}\frac{M\left(x_i,\vartheta\right)}{D\left(x_i,\vartheta\right)\times U\left(x_i,\vartheta\right)},\\
\widehat{i}_{25} &= -2\alpha\lambda\sum_{i=1}^{n}z_i\frac{M\left(x_i,\vartheta\right)\times U\left(x_i,\vartheta\right)^{\xi-1}}{D\left(x_i,\vartheta\right)}\left\{1+\xi\log\left[U\left(x_i,\vartheta\right)\right]\right\}+2\lambda\sum_{i=1}^{n}z_i\frac{M\left(x_i,\vartheta\right)}{D\left(x_i,\vartheta\right)\times U\left(x_i,\vartheta\right)},
\end{aligned}
$$

$$
\begin{aligned}
\widehat{i}_{34} &= -2\lambda(\xi-1)\sum_{i=1}^{n}\frac{M(x_i,\vartheta)^2}{D(x_i,\vartheta)^2 U(x_i,\vartheta)^2}\left[1-\alpha\xi U(x_i,\vartheta)\right]+2\sum_{i=1}^{n}\left(x_i e^{-\beta^T z}\right)^2 \\
&+2\sum_{i=1}^{n}\frac{M(x_i,\vartheta)}{D(x_i,\vartheta)}\left[1+(\xi-1)U(x_i,\vartheta)^{-1}-\alpha\xi U(x_i,\vartheta)^{\xi-1}\right]\left[1-\lambda\left(x_i e^{-\beta^T z}\right)^2-\lambda\frac{M(x_i,\vartheta)}{D(x_i,\vartheta)}\right], \\
\widehat{i}_{35} &= -2\lambda(\xi-1)\sum_{i=1}^{n}z_i\frac{M(x_i,\vartheta)^2}{D(x_i,\vartheta)^2 U(x_i,\vartheta)^2}\left[1-\alpha\xi U(x_i,\vartheta)\right]+2\sum_{i=1}^{n}z_i\left(x_i e^{-\beta^T z}\right)^2 \\
&+2\sum_{i=1}^{n}z_i\frac{M(x_i,\vartheta)}{D(x_i,\vartheta)}\left[1+(\xi-1)U(x_i,\vartheta)^{-1}-\alpha\xi U(x_i,\vartheta)^{\xi-1}\right]\left[1-\lambda\left(x_i e^{-\beta^T z}\right)^2-\lambda\frac{M(x_i,\vartheta)}{D(x_i,\vartheta)}\right],
\end{aligned}
$$

$$
\begin{aligned}
\widehat{i}_{45} &= -4\lambda\sum_{i=1}^{n}z_i\frac{M(x_i,\vartheta)\left[\lambda\left(x_i e^{-\beta^T z}\right)^2-1\right]}{D(x_i,\vartheta)}\left[1+(\xi-1)U(x_i,\vartheta)^{-1}-\alpha\xi U(x_i,\vartheta)^{\xi-1}\right] \\
&-4\alpha\xi\lambda^2\sum_{i=1}^{n}z_i\frac{M(x_i,\vartheta)^2 U(x_i,\vartheta)^{\xi-1}}{D(x_i,\vartheta)^2 U(x_i,\vartheta)}\left[U(x_i,\vartheta)+\xi-1\right]-4\lambda\sum_{i=1}^{n}z_i\left(x_i e^{-\beta^T z}\right)^2 \\
&+2(\xi-1)\lambda^2\sum_{i=1}^{n}z_i\frac{M(x_i,\vartheta)^2}{D(x_i,\vartheta)^2 U(x_i,\vartheta)^2}\left[1+U(x_i,\vartheta)\right]+4\lambda^2\sum_{i=1}^{n}z_i\frac{M(x_i,\vartheta)^2}{D(x_i,\vartheta)^2}.
\end{aligned}
$$

Elements of Y_n^2

The elements of the matrix $J\left(\widehat{\vartheta}\right)_{s\times r}$ are obtained as :

$$
\begin{aligned}
J\left(\widehat{\vartheta}\right) &= B^T\left(\widehat{\vartheta}\right)\times B\left(\widehat{\vartheta}\right), \\
B\left(\widehat{\vartheta}\right) &= \left[\frac{1}{\sqrt{p_j(\vartheta)}}\frac{\partial p_j}{\partial \vartheta_s}\right]_{r\times s}, \quad j=1,2,\ldots,r.
\end{aligned}
$$

Where :

$$
p_j(\vartheta)=F(a_j)-F(a_{j-1}).
$$

So,

$$
\begin{aligned}
\frac{\partial p_j}{\partial \alpha} &= \sum_{j=1}^{r}U(\widehat{a}_{j-1},\vartheta)^{\xi}F(\widehat{a}_{j-1},\vartheta)-U(\widehat{a}_j,\vartheta)^{\xi}F(\widehat{a}_j,\vartheta), \\
\frac{\partial p_j}{\partial \xi} &= \alpha\sum_{j=1}^{r}U(\widehat{a}_{j-1},\vartheta)^{\xi}\log\left[U(\widehat{a}_{j-1},\vartheta)\right]F(\widehat{a}_{j-1},\vartheta)-\alpha\sum_{j=1}^{r}U(\widehat{a}_j,\vartheta)^{\xi}\log\left[U(\widehat{a}_j,\vartheta)\right]F(\widehat{a}_j,\vartheta), \\
\frac{\partial p_j}{\partial \lambda} &= \alpha\xi\sum_{j=1}^{r}\frac{M(\widehat{a}_j,\vartheta)U(\widehat{a}_j,\vartheta)^{\xi-1}F(\widehat{a}_j,\vartheta)}{D(\widehat{a}_j,\vartheta)}-\alpha\xi\sum_{j=1}^{r}\frac{M(\widehat{a}_{j-1},\vartheta)U(\widehat{a}_{j-1},\vartheta)^{\xi-1}F(\widehat{a}_{j-1},\vartheta)}{D(\widehat{a}_{j-1},\vartheta)}, \\
\frac{\partial p_j}{\partial \beta_0} &= 2\alpha\xi\lambda\sum_{j=1}^{r}\frac{M(\widehat{a}_{j-1},\vartheta)U(\widehat{a}_{j-1},\vartheta)^{\xi-1}F(\widehat{a}_{j-1},\vartheta)}{D(\widehat{a}_{j-1},\vartheta)}-2\alpha\xi\lambda\sum_{j=1}^{r}\frac{M(\widehat{a}_j,\vartheta)U(\widehat{a}_j,\vartheta)^{\xi-1}F(\widehat{a}_j,\vartheta)}{D(\widehat{a}_j,\vartheta)}, \\
\frac{\partial p_j}{\partial \beta_1} &= 2\alpha\xi\lambda\sum_{j=1}^{r}z_i\frac{M(\widehat{a}_{j-1},\vartheta)U(\widehat{a}_{j-1},\vartheta)^{\xi-1}F(\widehat{a}_{j-1},\vartheta)}{D(\widehat{a}_{j-1},\vartheta)}-2\alpha\xi\lambda\sum_{j=1}^{r}z_i\frac{M(\widehat{a}_j,\vartheta)U(\widehat{a}_j,\vartheta)^{\xi-1}F(\widehat{a}_j,\vartheta)}{D(\widehat{a}_j,\vartheta)}.
\end{aligned}
$$

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