

**A SIMPLE PROOF OF WERNER SCHULTE'S  
CONJECTURE****D. HIMANE** *Communicated by \*\*\*\*\**

**Abstract:** Lately, Werner Schulte has conjectured that for all positive  $n > 1$ ,  $n$  divides  $\frac{(n-2)!(n-1)!}{2^{n-3}} + 4$  if and only if  $n$  is prime. In this paper, We use elementary methods, to give a simple proof of this conjecture.

**Keywords:** Legendre's formula, p-adic valuation, Wilson Theorem, Fermat's Little Theorem.

**1 Introduction and statement of results**

Taking

$$A_n = 1(1+2)(1+2+3)\dots(1+2+3+\dots+(n-1)) \quad (1)$$

the question is: does this sequence of positive integers significant?

In an attempt to find an explanation for this sequence. The triangular number is  $T_s = 1 + 2 + \dots + s = \frac{s(s+1)}{2}$ , as shown

$$A_n = \prod_{j=1}^{n-1} T_j = \prod_{j=1}^{n-1} \frac{j(j+1)}{2} = \frac{(n-1)! n!}{2^{n-1}} \quad (2)$$

The first terms of this sequence are 1, 1, 3, 18, 180, 2700, 56700, ... (sequence [A006472](#) in the OEIS <sup>1</sup>), also (sequence M3052 [1]). In the [OEIS](#), we found a conjecture attributed to W. Schulte, concerning the previous sequence. for which we will present a simple proof.

The main result we seek to prove is

**Main Theorem** (*Werner Schulte's conjecture*).

For all positive  $n > 1$ ,  $n$  divides  $\frac{(n-2)!(n-1)!}{2^{n-3}} + 4$  if and only if  $n$  is prime.

## 2 Preliminaries

For all prime numbers  $p$  and non-negative integer  $n$ , let  $v_p(n)$  is the exponent of the greatest power of  $p$  dividing  $n$  (i.e. the  $p$ -adic valuation of  $n$ ). Then

$$v_p(n!) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor, \tag{3}$$

where  $\lfloor x \rfloor$  is the floor function. The formula of Legendre (3) gives an expression to the greatest power of a prime  $p$  the exponent of which divide the factorial  $n!$  [6]. Whereas the right-hand formula is an infinite sum, for every particular value of  $n$  and  $p$ , there are only a few nonzero terms: for each  $i$  sufficiently big as  $p^i > n$ , we have  $\left\lfloor \frac{n}{p^i} \right\rfloor = 0$ .

We also have the recurrence relation:

$$v_p(n!) = \lfloor n/p \rfloor + v_p(\lfloor n/p \rfloor!) \tag{4}$$

Legendre's formula can also be reformulated in terms of a base- $p$  expansion of  $n$ . Let  $s_p(n)$  denotes the sum of the digits in the base- $p$  expansion of  $n$ , then

$$v_p(n!) = \frac{n - s_p(n)}{p - 1}. \tag{5}$$

You can easily check the veracity of the following relationships. For any positive integers  $a, b$  and any prime  $p$  [7].

- $v_p(a \cdot b) = v_p(a) + v_p(b)$ .
- $v_p\left(\frac{a}{b}\right) = v_p(a) - v_p(b)$ .
- if  $a \mid b$ , then  $v_p(a) \leq v_p(b)$ , in particularly  $v_p(a) < a$ .

The proof of the following lemma is known and easy, but is included here for completeness as the proof is short.

**Lemma 1.** For all positive integers  $m > 4$ , we have  $m < 2^{m/2}$ .

*Proof.* For the induction step, note that instead of using that the left side increases by 1, we will use the increase in multiplication (since the right side

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<sup>1</sup>[A006472](#), The On-Line Encyclopedia of Integer Sequences.

increases by a multiplicative factor) to get

$$m < 2^{m/2}$$

equivalent to

$$m \left( \frac{m+1}{m} \right) < 2^{m/2} \left( \frac{m+1}{m} \right)$$

equivalent to

$$m+1 < 2^{m/2} \left( \frac{m+1}{m} \right). \quad (6)$$

Next, let

$$g(w) = \sqrt{2} - \frac{w+1}{w} = (\sqrt{2} - 1) - \frac{1}{w}.$$

This gives  $g(5) = (\sqrt{2} - 1) - \frac{1}{5} \approx 0.2142$ . Also,  $g'(w) = \frac{1}{w^2} > 0$ , which means  $g(w)$  is a strictly increasing function for  $w \geq 5$ , so it's always positive then,

$$\sqrt{2} - \frac{w+1}{w} > 0 \quad \text{implies} \quad \sqrt{2} > \frac{w+1}{w}$$

Thus, for  $m \geq 5$ ,

$$\frac{m+1}{m} < 2^{1/2} \quad \text{gives} \quad 2^{m/2} \left( \frac{m+1}{m} \right) < 2^{(m+1)/2}$$

Using this in (6) gives

$$m+1 < 2^{(m+1)/2}$$

which completes the induction step. As a result for  $m > 4$ , we get

$$m < 2^{m/2} \quad \text{implies} \quad v_2(m) < \frac{m}{2}.$$

□

**Lemma 2** (Wilson Theorem). *If  $p$  is prime then  $(p-1)! + 1$  is divisible by  $p$ .*

This theorem was affirmed by Alhazen (c. 965-1000 A.D.), [2] and, in the eighteenth century, from J. Wilson [3]. Proof of this came from J. L. Lagrange in 1771 [4]. An integer primality test derives from the Wilson theorem: a non-negative integer  $n > 1$  is prime, If and only if

$$(n-1)! + 1 \equiv 0 \pmod{n}. \quad (7)$$

**Lemma 3** (Fermat's Little Theorem). *For any positive integers  $a$  which cannot be divided by a prime number  $p$ , we have*

$$a^{p-1} \equiv 1 \pmod{p}. \quad (8)$$

Among one of the basic results of the theory of elemental numbers Little theorem of Fermat, This is the basis of Fermat's primality test. this theorem is called from Pierre de Fermat, who declared during 1640. It is referred to as the: «Little theorem» to set it apart from Last Theorem of Fermat. (see page 514, [5] ).

### 3 Proof of the Main Theorem

*Proof. First step.* It is shown that if  $n$  is prime, then  $n$  divides

$$\frac{(n-2)!(n-1)!}{2^{n-3}} + 4. \quad (9)$$

We assume  $n = p$  is odd prime, thus from Wilson's theorem 2, we get  $p$  divided both  $(p-1)!+1$  and  $(p-2)!-1$ . Hence  $p$  divided  $(p-1)!(p-2)!+1$ . Instead, from little theorem of Fermat 3, we get  $p$  divided  $2^{p-1}-1$ , hence from (2) we get

$$2 \times A_{p-1} = \frac{4(p-2)!(p-1)!}{2^{p-1}} \equiv 4(p-2)!(p-1)! \equiv -4 \pmod{p}.$$

So  $p$  divides the number

$$\frac{(p-2)!(p-1)!}{2^{p-3}} + 4.$$

Finally, for  $n = 2$ , the validity of the property can be easily checked.

**Second step.** We show that if  $n$  is a composite, then

$$\frac{(n-2)!(n-1)!}{2^{n-3}} + 4$$

It is not divisible by  $n$ .

We assume  $n = pq$  is a composite, where  $1 < p \leq q < n$  three sub-cases arise :

- **First case.** if  $n = 4$ , then  $\frac{(n-2)!(n-1)!}{2^{n-3}} + 4 = 10$ , It is not divisible by  $n$ .
- **Second case.** if  $n$  odd ( $n > 4$ ), then  $n \mid (n-2)!(n-1)!$  and  $\gcd(n, 2^{n-3}) = 1$  from Euclid's lemma, we have  $n \mid \frac{(n-2)!(n-1)!}{2^{n-3}}$ , Hence  $\frac{(n-2)!(n-1)!}{2^{n-3}} + 4 \equiv 4 \pmod{n}$ , So  $\frac{(n-2)!(n-1)!}{2^{n-3}} + 4$  it is not divisible by  $n$ .
- **Third case.** if  $n$  even ( $n = 2^{v_2(n)} \times k > 4$ ) where  $k$  odd, then  $n \mid (n-2)!(n-1)!$  from Euclid's lemma, we get  $k \mid 2 \times A_{n-1}$ , hence  $n \mid 2 \times A_{n-1}$  if and only if

$$v_2(n) \leq v_2(2 \times A_{n-1}).$$

Through the above properties of  $p$ -adic valuation, we get

$$v_2(2 \times A_{n-1}) = v_2((n-1)!) + v_2((n-2)!) - v_2(2^{n-3})$$

We also have the recurrence relation:

$$v_2((n-1)!) = v_2(n-1) + v_2((n-2)!) = v_2((n-2)!),$$

Because  $n-1$  odd, and

$$v_2((n-2)!) = \lfloor \frac{n-2}{2} \rfloor + \lfloor \frac{n-2}{4} \rfloor + v_2(\lfloor \frac{n-2}{4} \rfloor!) \geq \frac{n-2}{2} + \frac{n-2}{4}$$

Therefore

$$v_2(2 \times A_{n-1}) \geq \left( \frac{n-2}{2} + \frac{n-2}{4} \right) + \left( \frac{n-2}{2} + \frac{n-2}{4} \right) - (n-3)$$

As a result

$$v_2(2 \times A_{n-1}) \geq \frac{n}{2}.$$

From Lemma 1, we obtain

$$\frac{n}{2} > v_2(n).$$

Then

$$v_2(2 \times A_{n-1}) \geq v_2(n).$$

Hence

$$\frac{(n-2)!(n-1)!}{2^{n-3}} + 4 \equiv 4 \pmod{n},$$

So

$$\frac{(n-2)!(n-1)!}{2^{n-3}} + 4$$

is not divisible by  $n$ .

In summary, we have obtained to all positive integer  $n > 1$ , that  $n$  divides

$$\frac{(n-2)!(n-1)!}{2^{n-3}} + 4$$

if and only if  $n$  is prime. □

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DJAMEL HIMANE  
 LA3C LABORATORY,  
 FACULTY OF MATHEMATICS, USTHB,  
 PO. BOX 32, EL ALIA, 16111,  
 BAB EZZOUAR, ALGIERS, ALGERIA.  
 Email address: [dhimane@usthb.dz](mailto:dhimane@usthb.dz)