ON PROJECTIVELY FLAT FINSLER SPACE WITH GENERALIZED (α, β)-METRIC

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Abstract. In 1991, projective flatness and flat-parallelness condition for (α, β)-metrics was studied by M. Matsumoto. The present paper is devoted to studying the condition for a Finsler space with p-power (α, β)-metrics to be projectively flat on the basis of Matsumoto's results. Further, we have shown that the Finsler space with p-power (α, β)-metrics is flat-parallel Minkowski space and obtained the condition under which the β-change for aforementioned metric is projective.

1. Introduction

As an n-dimensional differentiable manifold M^n equipped with a basic function $L(x, y)$, let $F^n = (M^n, L)$ be an n-dimensional Finsler space. M. Matsumoto [9] in 1972 first proposed the idea of a (α, β)-metric $L(\alpha, \beta)$. If L is a positively homogeneous function of α and β of degree one, where $\alpha^2 = a_{ij}(x)y^iy^j$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a one form on M^n then a Finsler metric $L(x, y)$ is termed as (α, β)-metric $L(\alpha, \beta)$. The Randers, Kropina, and Matsumoto metrics are the three most intriguing examples of (α, β)-metrics [15].

A Finsler space $F^n = (M^n, L)$ is called projectively flat, or with rectilinear geodesic, if the space space is covered by coordinate neighborhoods in which the geodesics can be represented by $(n-1)$ linear equations of the coordinates. Such a coordinate system is called rectilinear. The condition for a Finsler space to be projectively flat was studied by L. Berwald [2] in tensorial form and completed by M. Matsumoto [6]. Hashiguchi and Ichijyo's paper [5] gives interesting results on projective flatness of the Randers spaces.

We have two important projective invariant tensors: the Weyl tensor W and the Douglas tensor D. A Finsler space with both of these tensors vanishes as a projectively flat space that can be projectively mapped to a locally Minkowski space. Randers spaces, Kropina spaces and generalized Kropina spaces with $L = \frac{\beta^2}{\alpha}$ are examples of Finsler spaces with (α, β)-metric $L(\alpha, \beta)$. M. Matsumoto [7] demonstrated the criteria for the above spaces to be projectively flat.

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Furthermore, M. Matsumoto ([10, 11]) defined a flat parallel Minkowski space and β-change with (α, β)-metrics in 1991 and 1988 respectively. He dealt flat parallelness of Randers metric, Kropina metric and their generalized form in his paper [12]. The flat parallelness of Matsumoto metric was studied by T. Aikou, M. Hashiguchi and K. Yamauchi [13].

In the present paper, we have considered p-power (α, β)-metrics [3]. The main aim of this research paper is to investigate the condition that a Finsler space with the p-power (α, β)-metrics to be projectively flat on the basis of Matsumoto's results. Further, we proved that Finsler space with p-power (α, β)-metrics is a flat-parallel Minkowski space and obtained the condition for a change $\varphi : \alpha \to L = \alpha\left(1 + \frac{\beta}{\alpha}\right)^p$ to be projective.

We reference Matsumoto's monograph's [8] terminology and notations throughout the present paper.

2. Preliminaries

Let $F^n = (M^n, L(\alpha, \beta))$ represent an n-dimensional Finsler space and $R^n = (M^n, \alpha)$ represent the corresponding Riemannian space, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$. Let γ^i_{jk} be the Christoffel symbols with respect to α, and the covariant differentiation with respect to γ^i_{jk} be indicated by (;). From the differential 1-form $\beta(x, y) = b_i(x)y^i$ we define

$$2r_{ij} = b_{i;j} + b_{j;i}, \quad 2s_{ij} = b_{i;j} - b_{j;i} = (\partial_j b_i - \partial_i b_j),$$
$$s^i_j = a^{ir} s_{rj}, \quad b^i = a^{ir} b_r, \quad b^2 = a^{rs} b_r b_s.$$

Next, we consider the Berwald connection $B\Gamma = (G^i_{jk}, G^i_j, 0)$ of the Finsler space with the (α, β)-metric $L(\alpha, \beta)$. As is well-known, we have

$$G^i_{jk} = \dot{\partial}_k G^i_j, \quad 2G^i = G^i_0 = (G^i_r y^r), \quad G^i_j = \dot{\partial}_j G^i.$$

Also then the previous paper ([16, 11]) gives the equation to find the difference $B^i_{jk} = G^i_{jk} - \gamma^i_{jk}$:

$$L_\alpha B^k_{ji} y^j y_k = \alpha L_\beta (b_{j:i} - B^k_{ji} b_k) y^j, \tag{2.1}$$

We consider a locally Minkowski space $F^n = (M^n, L)$, that is, M^n admits a covering by coordinate neighborhoods in each of which the fundamental function L is a function of y^i alone. We denote by R^r_{hijk} a Riemannian curvature tensor with respect to γ^i_{jk}.

Definition 2.1. [12] A locally Minkowski space with (α, β)-metric is called flat parallel, if α is locally flat($R^r_{hijk} = 0$) and β is parallel with respect to α ($b_{i;j} = 0$).

Theorem 2.2. [11] *A $F^n = (M^n, (\alpha, \beta))$ is a locally Minkowski if and only if B^k_{ji} are functins of x alone and R^r_{hijk} of the Riemannian α is written as:*

$$R^r_{hijk} = -\upsilon(jk)\{B^i_{hj;k} + B^r_{hj} B^i_{rk}\}, \tag{2.2}$$

where $\upsilon(jk)$ denotes the terms obtained from the preceding terms by interchanging indices j and k.

Let $F^n = (M^n, L)$ and $\bar{F}^n = (M^n, \bar{L})$ be two Finsler spaces on the same underlying manifold M^n. If any geodesic of F^n is a geodesic of $\bar{F}^n$ and vice versa, then F^n is called projective to $\bar{F}^n$ and change $\sigma : L \to \bar{L}$ of metric is called projective. It is well known that σ is projective if and only if there exists a positively homogeneous function $P(x,y)$ of degree 1 in y^i satisfying $\bar{G} = G^i + Py^i$.

Furthermore, In the following we denote by the subscript 0 the contraction by y^i and by subscripts α and β of L the partial differentiation by α and β, respectively. Then by Theorem 1 of [7] a Finsler space F^n with an (α,β)-metric is projectively flat if and only if the space is covered by coordinate neighborhoods on which $\gamma^i_{jk}(x)$ satisfies

$$(\gamma^i_{00} - \gamma_{000}y^i/\alpha^2)/2 + (\alpha L_\beta/L_\alpha)s^i_0 + (L_{\alpha\alpha}/L_\alpha)(C + \alpha r_{00}/2\beta)(\alpha^2 b^i/\beta - y^i) = 0 \quad (2.3)$$

where C is given by

$$C + (\alpha^2 L_\beta/\beta L_\alpha)s_0 + (\alpha L_{\alpha\alpha}/\beta^2 L_\alpha)(\alpha^2 b^2 - \beta^2)(C + \alpha r_{00}/2\beta) = 0 \quad (2.4)$$

By the homogeneity of L, we know that $\alpha^2 L_{\alpha\alpha} = \beta^2 L_{\beta\beta}$, therefore equation (2.4) can be rewritten as :

$$\left\{1 + (L_{\beta\beta}/\alpha L_\alpha)(\alpha^2 b^2 - \beta^2)\right\}(C + \alpha r_{00}/2\beta) = (\alpha/2\beta)\left\{r_{00} - (2\alpha L_\beta/L_\alpha)s_0\right\} \quad (2.5)$$

If $1 + (L_{\beta\beta}/\alpha L_\alpha)(\alpha^2 b^2 - \beta^2) \neq 0$, then we can eliminate $(C + \alpha r_{00}/2\beta)$ in (2.3) and it is written as the form :

$$\begin{aligned} &\left\{1 + L_{\beta\beta}(\alpha^2 b^2 - \beta^2)/\alpha L_\alpha\right\}\left\{(\gamma^i_{00} - \gamma_{000}y^i/\alpha^2)/2 + (\alpha L_\beta/L_\alpha)s^i_0\right\} \\ &\quad + (L_{\alpha\alpha}/L_\alpha)(\alpha/2\beta)\left\{r_{00} - (2\alpha L_\beta/L_\alpha)s_0\right\}(\alpha^2 b^i/\beta - y^i) = 0. \end{aligned} \quad (2.6)$$

Thus we have [4]

Theorem 2.3. *If $1 + (L_{\beta\beta}/\alpha L_\alpha)(\alpha^2 b^2 - \beta^2) \neq 0$, then a Finsler space F^n with an (α,β)-metric is projectively flat if and only if (2.6) is satisfied.*

Remark 2.4. According to [14], if α^2 contains β as a factor, then $b^2 = 0$ and the dimension is equal to two. Throughout this paper, we assume that the dimension is more than two and $b^2 \neq 0$, i.e., $\alpha^2 \not\equiv 0(mod\beta)$.

3. Projectively flat Finsler space with p-power (α,β)-metrics

Let F^n be a Finsler space with an p-power (α,β)-metrics given by

$$L = \alpha\left(1 + \frac{\beta}{\alpha}\right)^p, \quad p \neq 0, 1, 2 \quad (3.1)$$

In this section, we shall find the condition for F^n with metric (3.1) to be projectively flat. The partial derivative with respect to α and β of (3.1) are given by

$$\begin{aligned} L_\alpha &= \frac{(\alpha+\beta)^{p-1}(\alpha+\beta)(1-p)}{\alpha^p}, \qquad L_\beta = p\alpha^{1-p}(\alpha+\beta)^{p-1}, \\ L_{\alpha\alpha} &= \frac{p(1-p)(-\beta^2)(\alpha+\beta)^{p-2}}{\alpha^{p+1}}, \qquad L_{\beta\beta} = p(p-1)\alpha^{1-p}(\alpha+\beta)^{p-2}. \end{aligned} \quad (3.2)$$

If $1+(L_{\beta\beta}/\alpha L_\alpha)(\alpha^2b^2-\beta^2)=0$, then we have $\alpha^2(1-b^2p+b^2p^2)+\alpha\beta(2-p)+\beta^2(1-p^2)=0$ which leads a contradiction. Therefore we can apply Theorem 2.3.

Substituting (3.2) into (2.6), we have

$$\begin{aligned}\left\{\alpha^2(1-b^2p+b^2p^2)+\alpha\beta(2-p)+\beta^2(1-p^2)\right\}\left\{(\alpha^2\gamma^i_{00}-\gamma_{000}y^i)(\alpha+\beta(1-p))\right.\\ \left.+2\alpha^4ps^i_0\right\}-p(1-p)\alpha^2\left\{r_{00}(\alpha+\beta(1-p))-2\alpha^2ps_0\right\}(\alpha^2b^i-y^i\beta)=0\end{aligned} \tag{3.3}$$

The above equation (3.3) is rewritten as a polynomial of sixth degree in α as follows:

$$p_6\alpha^6+p_4\alpha^4+p_2\alpha^2+p_0+\alpha(p_5\alpha^4+p_3\alpha^2+p_1)=0 \tag{3.4}$$

where

$$\begin{aligned}
p_6 &= 2p\left\{(1-b^2p+b^2p^2)s^i_0+p(1-p)b^is_0\right\},\\
p_5 &= p(1-p)b^ir_{00}+(1-b^2p+b^2p^2)\gamma^i_{00}+2\beta p(2-p)s^i_0,\\
p_4 &= \beta(1-b^2p+b^2p^2)(1-p)\gamma^i_{00}+\beta(2-p)\gamma^i_{00}+2\beta^2(1-p^2)ps^i_0\\
&\quad -\beta p(1-p)^2b^ir_{00}+2\beta p^2(1-p)s_0y^i,\\
p_3 &= -(1-b^2p+b^2p^2)\gamma_{000}y^i+\beta^2(2-p)(1-p)\gamma^i_{00}\\
&\quad +\beta^2(1-p^2)\gamma^i_{00}-\beta p(1-p)y^ir_{00},\\
p_2 &= -\beta(1-p)(1-b^2p+b^2p^2)\gamma_{000}y^i-\beta(2-p)\gamma_{000}y^i\\
&\quad +\beta^2(1-p)\gamma^i_{00}-\beta^2(1-p^2)y^ir_{00},\\
p_1 &= \beta^2(2-p)(1-p)\gamma_{000}y^i-\beta^2(1-p^2)\gamma_{000}y^i,\\
p_0 &= \beta^3(1-p^2)(1-p)\gamma_{000}y^i.
\end{aligned}$$

Since $p_6\alpha^6+p_4\alpha^4+p_2\alpha^2+p_0$ and $p_5\alpha^4+p_3\alpha^2+p_1$ are rational and α is irrational in y^i, we have

$$p_6\alpha^6+p_4\alpha^4+p_2\alpha^2+p_0=0, \tag{3.5}$$

$$p_5\alpha^4+p_3\alpha^2+p_1=0. \tag{3.6}$$

The term which does not contain β in (3.5) is $p_6\alpha^6$. Therefore there exists a homogeneous polynomial W_6 of degree six in y^i such that

$$2p\left\{(1-b^2p+b^2p^2)s^i_0+p(1-p)b^is_0\right\}\alpha^6=\beta W_6.$$

Since $\alpha^2\not\equiv 0(mod\beta)$, we must have a function $u^i=u^i(x)$ satisfying

$$2p\left\{(1-b^2p+b^2p^2)s^i_0+p(1-p)b^is_0\right\}=u^i\beta. \tag{3.7}$$

Contracting (3.7) by b_i, we have

$$2ps_0=u^i\beta b_i. \tag{3.8}$$

i.e., $2ps_j=u^ib_ib_j$. Furthermore contracting this equation by b^j, we have $u^ib_ib^2=0$. Substituting this equation into (3.8), we have $s_0=0$. Therefore from (3.7), we get

$$2p(1-b^2p+b^2p^2)s_{ij}=u_ib_j. \tag{3.9}$$

which gives $u_ib_j + u_jb_i = 0$. Contracting this equation by b^j, we have $u^ib^2 = 0$ by virtue of $u_jb^j = 0$. Thus we get $u_i = 0$. Hence from (3.9), we have $s_{ij} = 0$, provided $2p(1 - b^2p + b^2p^2) \neq 0$.

Further, from (3.6) we have 1-form $v_0 = v_i(x)y^i$ such that

$$\gamma_{000} = v_0\alpha^2. \tag{3.10}$$

Upon replacing $s_0 = 0$, $s^i_0 = 0$ and (3.10) into (3.3), we have

$$\begin{aligned}\left\{\alpha^2(1 - b^2p + b^2p^2) + \alpha\beta(2 - p) + \beta^2(1 - p^2)\right\}(\gamma^i_{00} - v_0y^i)\\ -p(1 - p)r_{00}(\alpha^2b^i - \beta y^i) = 0.\end{aligned} \tag{3.11}$$

by virtue of $(\alpha + \beta(1 - p)) \neq 0$. Then the equation (3.11) is written in the form $A\alpha + B = 0$, where

$$\begin{aligned}A &= \beta(2 - p)(\gamma^i_{00} - v_0y^i),\\ B &= \left\{\alpha^2(1 - b^2p + b^2p^2) + \beta^2(1 - p^2)\right\}(\gamma^i_{00} - v_0y^i)\\ &\quad - p(1 - p)r_{00}(\alpha^2b^i - \beta y^i).\end{aligned}$$

Since A and B are rational and α is irrational in y^i, we have $A = 0$ and $B = 0$. First, it follows from $A = 0$ that

$$\gamma^i_{00} - v_0y^i = 0, \quad 2 - p \neq 0, \tag{3.12}$$

i.e.,

$$2\gamma^i_{jk} = v_j\delta^i_k + v_k\delta^i_j.$$

which shows that the associated Riemannian space (M^n, α) is projectively flat.

Next, from $B = 0$ and (3.12), we have

$$-p(1 - p)r_{00}(\alpha^2b^i - \beta y^i) = 0. \tag{3.13}$$

Contracting (3.13) by b_i, we have $-p(1-p)r_{00}(\alpha^2b^2-\beta^2) = 0$, from which $r_{00} = 0$, provided $-p(1 - p) \neq 0$, i.e., $r_{ij} = 0$. From $s_{ij} = 0$ and $r_{ij} = 0$, we have $b_{i;j} = 0$.

On the other hand, it is easily verified that (3.3) is a consequence of (3.12) and $b_{i;j} = 0$. Consequently, we have

Theorem 3.1. *A Finsler space F^n with an p-power (α, β)-metrics (3.1) is projectively flat if and only if the associated Riemannian space (M^n, α) is projectively flat and $b_{i;j} = 0$.*

4. Flat parallel Minkowski space with p-power metrics

In [1], Kim And Choi had defined a procedure to show (α, β)-metrics are flat parallel Minkowski space. i.e.,

Substituting $B^k_{ji}y^iy_k = P_{i00}$ and $(b_{j:i} - B^k_{ji}b_k)y^i = Q_{i0}$ in the equation (2.1) we have

$$L_\alpha P_{i00} = \alpha L_\beta Q_{i0}. \tag{4.1}$$

where the index zero means as usual contraction by y^i. It is remarked that for a locally Minkowski space P_{i00} and Q_{i0} are polynomials in y^i of degree 2 and 1 respectively. If (4.1) gives $P_{i00} = Q_{i0} = 0$ necessarily, then we have $B^k_{ji} = 0$ and $b_{j:i} = 0$, and (2.2) shows $R^r_{hijk} = 0$. Consequently the Finsler space with (α, β)-metrics defines flat parallel Minkowski space.

In this section, we shall apply this procedure to the p-power (α, β)-metrics (3.1).

Substituting (3.2) into (4.1), we have

$$(\alpha + \beta(1-p))P_{i00} - \alpha^2 p Q_{i0} = 0 \tag{4.2}$$

Since α is irrational in y^i, (4.2) leads us to

$$\begin{aligned} P_{i00} &= 0, \\ -\alpha^2 p Q_{i0} + \beta(1-p)P_{i00} &= 0. \end{aligned} \tag{4.3}$$

Thus from equation (4.3), we have $P_{i00} = Q_{i0} = 0$. Hence we conclude the following.

Theorem 4.1. *A Finsler space with p-power (α, β)-metrics (3.1) is a flat-parallel Minkowski space.*

5. β-change with p-power metrics

we shall introduce β-change [12] as follows:

Definition 5.1. Let $L(\alpha, \beta)$ be an (α, β)-metric. The change $\phi : \alpha \to L(\alpha, \beta)$ of metric is called β-change.

If we denote by R^n the associated Riemannian space with a Finsler space F^n with (α, β)-metric, then the β-change is the change from R^n to F^n. There is a theorem between projective change and β-change as follows:

Theorem 5.2. [12] *A β-change is projective, if and only if we have*

$$L_\beta \psi_{ij} + L_{\beta\beta} \Omega_{ij} = 0, \tag{5.1}$$

where $\psi_{ij} = (b_{i;j} - b_{j;i})/2$, $\Omega = (p_i \beta_{;j} - p_j \beta_{;i})/2$ and $\beta_{;j} = b_{i;j} y^j$.

Now, we consider a change $\varphi : \alpha \to L = \alpha\left(1 + \frac{\beta}{\alpha}\right)^p$. Then, from Theorem 5.2 we can obtain the condition for a change φ to be projective.

Substituting (3.2) into (5.1), we have

$$\alpha \psi_{ij} + \beta \psi_{ij} + (p-1)\Omega_{ij} = 0 \tag{5.2}$$

Since α is an irrational polynomial of y^i, (5.2) lead us to $\psi_{ij} = 0$. Substituting this into (5.2), by virtue of $(p-1) \neq 0$ we have $\Omega_{ij} = 0$. Further, contracting this by y^i and using $p_i y^i = 0$ and $\psi_{ij} = 0$, we have $b_{i;j} = 0$. Conversely, if $b_{i;j} = 0$, it satisfies (5.2). Thus we have the following.

Theorem 5.3. *A change $\varphi \to L = \alpha\left(1 + \frac{\beta}{\alpha}\right)^p$ is projective if and only if we have $b_{i;j} = 0$.*

From the above theorem and (5.2), we have discuss three cases as follows: (i) $p = 1$, (ii) $p = 2$ and (iii) $p = 1/2$.

Case (i): $p = 1$

Let $p = 1$ in (5.2), we have $(\alpha + \beta)\psi_{ij} = 0$. Since $(\alpha + \beta) \neq 0$, we have $\psi_{ij} = 0$. From $\psi_{ij} = 0$, we have $b_{i;j} = 0$. Conversely if $b_{i;j} = 0$, it satisfies $(\alpha + \beta)\psi_{ij} = 0$. Thus we have

Corollary 5.4. [12] *A Randers change $\alpha \to \alpha + \beta$ is projective if and only if we have $b_{i;j} = 0$.*

Case (ii): $p = 2$
Let $p = 2$ in (5.2), we have

$$(\alpha + \beta)\psi_{ij} + \Omega_{ij} = 0. \tag{5.3}$$

Since α is irrational in y^i we have $\psi_{ij} = 0$. Substituting $\psi_{ij} = 0$ in (5.3), we have $\Omega_{ij} = 0$. Further, contracting this by y^i and using $p_i y^i = 0$ and $\psi_{ij} = 0$, we have $b_{i;j} = 0$. Conversely, if $b_{i;j} = 0$, it satisfies (5.3) . Thus we have

Corollary 5.5. *A Berwald change $\alpha \to \frac{(\alpha+\beta)^2}{\alpha}$ is projective, if and only if we have $b_{i;j} = 0$.*

Case (iii): $p = 1/2$
Let $p = 1/2$ in (5.2), we have

$$2(\alpha + \beta)\psi_{ij} - \Omega_{ij} = 0. \tag{5.4}$$

Since α is irrational in y^i we have $\psi_{ij} = 0$. Substituting $\psi_{ij} = 0$ in (5.4), we have $\Omega_{ij} = 0$. Further, contracting this by y^i and using $p_i y^i = 0$ and $\psi_{ij} = 0$, we have $b_{i;j} = 0$. Conversely, if $b_{i;j} = 0$, it satisfies (5.4). Thus we have

Corollary 5.6. *A square root metric change $\alpha \to \sqrt{\alpha(\alpha + \beta)}$ is projective, if and only if we have $b_{i;j} = 0$.*

On the other hand, we dealt with the condition that a Finsler space with p-power (α, β)-metrics to be a projectively flat in Theorem 3.1. Combining Theorem 3.1 and Theorem 5.3, we can give more geometrical meaning like a Matsumoto's Theorem ([7], Theorem 2). Thus we have

Corollary 5.7. *A Finsler space with the p-power (α, β)-metrics (3.1) is projectively flat if and only if a change $\varphi : \alpha \to L = \alpha \left(1 + \frac{\beta}{\alpha}\right)^p$ is projective and the associated Riemannian space with metric α is projectively flat.*

6. Conclusion

The infinitesimal transformations in Finsler geometry such as conformal, projective, semi-projective, β-changes, conformal β-changes play an important role not only in differential geometry but also in application to other branches of science, especially in the process of geometrization of physical theories. In this paper we have obtained results concerning projective flatness and flat-parallelness of p-power (α, β)-metrics (3.1). Further, we have shown that the β-change of aforementioned metric (3.1) $\varphi \to L = \alpha \left(1 + \frac{\beta}{\alpha}\right)^p$ is projective. Also we have discussed β-change is projective for some important Finsler metrics arising from p-power (α, β)-metrics (3.1).

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